Research Article

Existence and Upper Semicontinuity of Attractors for Nonautonomous Stochastic Sine-Gordon Lattice Systems with Random Coupled Coefficients

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Received 20 May 2016; Accepted 29 June 2016

Academic Editor: Bixiang Wang

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We study nonautonomous stochastic sine-Gordon lattice systems with random coupled coefficients and multiplicative white noise. We first consider the existence of random attractors in a weighted space for this system and then establish the upper semicontinuity of random attractors as the intensity of noise approaches zero.

1. Introduction

Stochastic lattice differential equations arise naturally in a wide variety of applications where the spatial structure has a discrete character and uncertainties are taken into account. In recent years, some works have been done regarding the existence of random attractors for stochastic lattice differential equations (see, e.g., [1–11]). Of those works, Bates et al. [2] considered the existence of random attractors for first-order nonautonomous stochastic lattice system driven by multiplicative white noise. Zhou and Wei [11] considered the existence of random attractors for second-order lattice system with random coupled coefficients and multiplicative white noise.

Motivated by [2, 11], we will study the asymptotic behavior of solutions of the following nonautonomous stochastic sine-Gordon lattice systems with random coupled coefficients and multiplicative white noise: for every $\tau \in \mathbb{R}$ and $t > \tau$,

$$\begin{aligned} \frac{d^2 u_i}{dt^2} + \alpha \frac{du_i}{dt} = & -\lambda u_i - \beta \sin u_i + g_i(t) \\ & + \sum_{j=-q}^q \eta_{i,j}(\theta_t \omega) u_{i+j} + c u_i \circ \frac{dw(t)}{dt}, \end{aligned} \quad (1)$$

$$i \in \mathbb{Z},$$

with the initial data

$$\begin{aligned} u_i(\tau) &= u_{\tau,i}, \\ \dot{u}_i(\tau) &= u_{1\tau,i}, \end{aligned} \quad (2)$$

$$i \in \mathbb{Z},$$

where λ, α, β , and c are positive constants; $u_i \in \mathbb{R}$; $g_i(t) \in \mathbb{R}$; $\circ$ denotes the Stratonovich sense of the stochastic term; $\eta_{i,j}(\omega)$ ($j \in \{-q, \dots, 0, \dots, q\}$, $q \in \mathbb{N}$) are random variables; $w(t)$ is a Brownian motion (Wiener process) on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ -algebra induced by the compact open topology of Ω , and $\mathcal{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$ (see [12]); and $\{\theta_t\}_{t \in \mathbb{R}}$ is a metric dynamical system defined on $(\Omega, \mathcal{F}, \mathcal{P})$, where here $\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(\cdot)$ for all $t \in \mathbb{R}$, $\omega \in \Omega$. Then $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$ is an ergodic metric dynamical system.

In practice, the coupled mode between two nodes (say, adjacent nodes) is usually random. It is then of great importance to investigate stochastic lattice system with random coupled coefficients. To the best of our knowledge, there are no results on nonautonomous stochastic sine-Gordon lattice systems with random coupled coefficients and multiplicative white noise.

In this paper, we first consider the existence of a tempered random attractor in a weighted space $l^2_\rho \times l^2_\rho$ for stochastic

sine-Gordon lattice systems (1), which attracts the random tempered bounded sets in pullback sense. Then, we consider the dependence of attractors on the parameters c of the system (1) and establish the upper semicontinuity of the random attractor as the intensity c of noise approaches zero.

The rest of this paper is organized as follows. In the next section, we present some mathematical setting for system (1). In Section 3, we mainly consider the existence of a tempered random attractor in a weighted space of infinite sequences for system (1). Then in Section 4, we consider the upper semicontinuity of the tempered random attractor for system (1) as $c \rightarrow 0$.

2. Mathematical Settings

We first introduce a weighted space l_ρ^2 of infinite sequences. Let ρ be a positive-valued function from $\mathbb{Z}$ into $(0, M_0] \subset \mathbb{R}^+$ which satisfies

$$\begin{aligned} 0 < \rho(i) &\leq M_0, \\ \rho(i) &\leq c_0 \rho(i \pm 1), \\ \forall i &\in \mathbb{Z}, \end{aligned} \quad (3)$$

where M_0 and c_0 are positive constants. For example, for $i \in \mathbb{Z}$, $\rho(i) = 1/(1 + \gamma^2 i^2)^q$ ($q > 1/2$) [13] and $\rho(i) = e^{-\gamma|i|}$ satisfy condition (3), where $\gamma > 0$. Define $\rho_i = \rho(i)$, $\forall i \in \mathbb{Z}$ and

$$l_\rho^2 = \left\{ u = (u_i)_{i \in \mathbb{Z}} : \sum_{i \in \mathbb{Z}} \rho_i |u_i|^2 < \infty, u_i \in \mathbb{R} \right\} \quad (4)$$

with norm $\|u\|_\rho = (\sum_{i \in \mathbb{Z}} \rho_i |u_i|^2)^{1/2}$ and inner product $(u, v)_\rho = \sum_{i \in \mathbb{Z}} \rho_i u_i v_i$ for $u = (u_i)_{i \in \mathbb{Z}}$, $v = (v_i)_{i \in \mathbb{Z}} \in l_\rho^2$. We write $\|\cdot\|_\rho$ as $\|\cdot\|$ if $\rho(i) \equiv 1$. Then l_ρ^2 is a separable Hilbert space with the norm $\|\cdot\|_\rho$.

For any $u = (u_i)_{i \in \mathbb{Z}}$, $v = (v_i)_{i \in \mathbb{Z}} \in l_\rho^2$, define an inner product on l_ρ^2 by

$$(u, v)_{\lambda, \rho} = \lambda \sum_{i \in \mathbb{Z}} \rho_i u_i v_i; \quad (5)$$

then the norm $\|\cdot\|_{\lambda, \rho}$ induced by inner product $(\cdot, \cdot)_{\lambda, \rho}$ is equivalent to the norm $\|\cdot\|_\rho$ induced by the inner product $(\cdot, \cdot)_\rho$. We write $l_{\lambda, \rho}^2 = (l_\rho^2, (\cdot, \cdot)_{\lambda, \rho}, \|\cdot\|_{\lambda, \rho})$; then $l_\rho^2 \times l_\rho^2$ and $l_{\lambda, \rho}^2 \times l_\rho^2$ are both separable Hilbert spaces.

Note that system (1) can be rewritten as

$$\begin{aligned} \frac{d^2 u}{dt^2} + \alpha \frac{du}{dt} &= -\lambda u - \beta \sin u + g(t) + \mathbb{B}(\theta_t \omega) u + cu \\ &\circ \frac{dw(t)}{dt}, \quad i \in \mathbb{Z}, \end{aligned} \quad (6)$$

$$u(\tau) = u_\tau,$$

$$\dot{u}(\tau) = u_{1\tau},$$

where $u = (u_i)_{i \in \mathbb{Z}}$, $g(t) = (g_i(t))_{i \in \mathbb{Z}}$ is a given time dependent sequence, and $\mathbb{B}(\omega)$ is a linear operator defined by

$$(\mathbb{B}(\omega)u)_i = \sum_{j=-q}^q \eta_{i,j}(\omega) u_{i+j}. \quad (7)$$

To convert problem (6) into a random differential equation, let $z(\theta_t \omega) := -\lambda \int_{-\infty}^0 e^{\lambda s} (\theta_t \omega)(s) ds$, $t \in \mathbb{R}$ and $\omega \in \Omega$, which is an Ornstein-Uhlenbeck process on $(\Omega, \mathcal{F}, \mathcal{P})$ and solve the Ornstein-Uhlenbeck equation $dz + \alpha z dt = dw(t, \omega)$, where $w(t, \omega) = \omega(t)$ for $\omega \in \Omega$. From [12, 14, 15], it is known that the random variables $|z(\omega)|$ are tempered, and there is a θ_t -invariant set $\tilde{\Omega} \subset \Omega$ of full $\mathcal{P}$ measure such that $t \mapsto z(\theta_t \omega)$ is continuous in t for every $\omega \in \tilde{\Omega}$.

Let $v(t, \omega) = \dot{u}(t, \omega) + \sigma u(t, \omega) - cu z(\theta_t \omega)$, $\omega \in \Omega$, $t \in \mathbb{R}$; then (6) can be written as the following equivalent random system with random coefficients: for every $\tau \in \mathbb{R}$ and $t > \tau$,

$$\begin{aligned} \frac{du}{dt} &= -\sigma u + v + cu z(\theta_t \omega), \\ \frac{dv}{dt} &= (\sigma(\alpha - \sigma) - \lambda)u + (\sigma - \alpha)v \\ &\quad + (\mathbb{B}(\theta_t \omega) + 2c\sigma z(\theta_t \omega) - c^2 z^2(\theta_t \omega))u \\ &\quad - cz(\theta_t \omega)v - \beta \sin u + g(t), \end{aligned} \quad (8)$$

$$u(\tau) = u_\tau,$$

$$v(\tau) = v_\tau = u_{1\tau} + \sigma u_\tau - cu_\tau z(\omega).$$

We will consider (8) for $\omega \in \tilde{\Omega}$ and write $\tilde{\Omega}$ as Ω from now on. In order to obtain the existence and uniqueness of solutions to problem (8), we make the following assumptions on g_i and the coefficients $\eta_{i,j}(\omega)$, $j \in -q, \dots, 0, \dots, q$, for $i \in \mathbb{Z}$:

(A1) Letting $\eta(\omega) = \sup\{|\eta_{i,j}(\omega)| : j \in \{-q, \dots, 0, \dots, q\}, q \in \mathbb{N} \text{ and } i \in \mathbb{Z}\} \geq 0$, $\eta(\theta_t \omega)$ belongs to $L_{\text{loc}}^1(\mathbb{R})$ with respect to $t \in \mathbb{R}$ for each $\omega \in \Omega$,

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t \eta(\theta_s \omega) ds = 0; \quad (9)$$

and $\eta(\omega)$ is tempered.

(A2) $g = (g_i)_{i \in \mathbb{Z}} \in L_{\text{loc}}^2(\mathbb{R}, l_\rho^2)$.

(A3) Let $\lambda = \min\{\alpha/2 - \sigma, \sigma/2\}$, and

$$\epsilon = \lambda - \frac{(\lambda + 1)(\sigma(\alpha - \sigma) + \beta)}{2\lambda} > 0. \quad (10)$$

There exists a positive constant $a \in (0, \epsilon)$ such that

$$\int_{-\infty}^0 e^{as} \|g(s + \tau)\|_\rho^2 ds < \infty, \quad \forall \tau \in \mathbb{R}. \quad (11)$$

We call $\phi: [\tau, \tau + T) \rightarrow l_\rho^2 \times l_\rho^2$ a mild solution of the following random lattice differential equations:

$$\begin{aligned} \frac{d\phi}{dt} &= G(\phi, t, \theta_t \omega), \\ \phi &= (\phi_i)_{i \in \mathbb{Z}}, \\ G &= (G_i)_{i \in \mathbb{Z}}, \\ t &\in [\tau, \tau + T), \quad \tau \in \mathbb{R}, \end{aligned} \quad (12)$$

where $\omega \in \Omega$, if $\phi \in C([\tau, \tau + T), l_\rho^2 \times l_\rho^2)$ and

$$\begin{aligned} \phi_i(t) &= \phi_i(\tau) + \int_\tau^t G_i(\phi(s), s, \theta_s \omega) ds, \\ i &\in \mathbb{Z}, \quad t \in [\tau, \tau + T), \quad \tau \in \mathbb{R}. \end{aligned} \quad (13)$$

Theorem 1. Let $T > 0$ and (A1)-(A2) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and any initial data $(u_\tau, v_\tau) \in l_\rho^2 \times l_\rho^2$, problem (8) admits a unique mild solution $(u(\cdot, \tau, \omega, u_\tau, v_\tau), v(\cdot, \tau, \omega, u_\tau, v_\tau)) \in C([\tau, \tau + T), l_\rho^2 \times l_\rho^2)$ with $(u(\tau, \tau, \omega, u_\tau, v_\tau), v(\tau, \tau, \omega, u_\tau, v_\tau)) = (u_\tau, v_\tau)$, $(u(t, \tau, \omega, u_\tau, v_\tau), v(t, \tau, \omega, u_\tau, v_\tau))$ being continuous in $(u_\tau, v_\tau) \in l_\rho^2 \times l_\rho^2$; $(u(t, \tau, \omega, u_\tau, v_\tau), v(t, \tau, \omega, u_\tau, v_\tau)) \in l^2 \times l^2$ if $(u_\tau, v_\tau) \in l^2 \times l^2$. Moreover, (8) generates a continuous cocycle Ψ_c over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$ with state space $l_\rho^2 \times l_\rho^2$; for $(u_\tau, v_\tau) \in l_\rho^2 \times l_\rho^2$, $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$,

$$\begin{aligned} \Psi_c(t, \tau, \omega, u_\tau, v_\tau) &:= (u(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau, v_\tau), \\ &\quad v(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau, v_\tau)). \end{aligned} \quad (14)$$

We can prove Theorem 1 by Theorem 6.1.7 in [16] and Definition 2.1 in [2]. We omit it here.

3. Existence of Random Attractors

We first provide some sufficient conditions for the existence of random attractors for nonautonomous RDSs in weighted spaces of infinite sequences in [2].

In the following, let $(X, \|\cdot\|_X)$ be a separable Banach space and let $\mathcal{D}(X)$ be the collection of all tempered families of nonempty bounded subsets of X .

Definition 2. Let Φ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$.

- (1) A family $K = \{K(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(X)$ is called a random absorbing set for Φ if, for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and for every $D \in \mathcal{D}(X)$, there exists $T = T(D, \tau, \omega) > 0$ such that

$$\Phi(t, \tau - t, \theta_{-\tau} \omega, D(\tau - t, \theta_{-\tau} \omega)) \subseteq K(\tau, \omega) \quad \forall t \geq T. \quad (15)$$

- (2) A family $\mathcal{A} = \{\mathcal{A}(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(X)$ is called a random attractor for Φ if for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$, (i) $\mathcal{A}(\tau, \omega)$ is compact in X and is measurable in ω with respect to $\mathcal{F}$; (ii) $\mathcal{A}$ is invariant,

that is, $\Phi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \theta_t \omega)$; and (iii) for every $D = \{D(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(X)$,

$$\begin{aligned} \lim_{t \rightarrow \infty} d_H(\Phi(t, \tau - t, \theta_{-\tau} \omega, D(\tau - t, \theta_{-\tau} \omega)), \mathcal{A}(\tau, \omega)) \\ = 0, \end{aligned} \quad (16)$$

where d_H is the Hausdorff semidistance given by $d_H(F, G) = \sup_{u \in F} \inf_{v \in G} \|u - v\|_X$, for any $F, G \subset X$.

Theorem 3. Let Φ be a continuous cocycle on $l_\rho^2 \times l_\rho^2$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$. Suppose the following.

- (a) There exists a bounded closed random absorbing set $B_0 = \{B_0(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l_\rho^2 \times l_\rho^2)$ such that, for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $B = \{B(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l_\rho^2 \times l_\rho^2)$, there exists $T_1 = T_1(\tau, \omega, B) > 0$ yielding $\Phi(t, \tau - t, \theta_{-\tau} \omega, B(\tau - t, \theta_{-\tau} \omega)) \subset B_0(\tau, \omega)$, $\forall t \geq T_1$.
- (b) For each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and for any $\varepsilon > 0$, there exist $T_2 = T_2(\tau, \varepsilon, \omega, B_0) > 0$ and $I_0 = I_0(\tau, \varepsilon, \omega, B_0) \in \mathbb{N}$ such that

$$\sum_{|i| > I_0} \rho_i |\Phi_i(t, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})|^2 \leq \varepsilon, \quad (17)$$

$$\forall t \geq T_2, \quad u_{\tau-t} \in B_0(\tau - t, \theta_{-\tau} \omega).$$

Then Φ possesses a unique random attractor $\mathcal{A}$ in $\mathcal{D}(l_\rho^2 \times l_\rho^2)$ given by, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\mathcal{A}(\tau, \omega) = \bigcap_{\tau \geq T_1} \overline{\bigcup_{t \geq \tau} \Phi(t, \tau - t, \theta_{-\tau} \omega, B_0(\tau - t, \theta_{-\tau} \omega))}. \quad (18)$$

Next, we will use Theorem 3 to prove the existence of a random attractor for the continuous cocycle Ψ_c in $l_\rho^2 \times l_\rho^2$ under conditions (A1)-(A3).

Theorem 4. If (A1)-(A3) hold, then, for every $c > 0$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$ and for any $B = \{B(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l_\rho^2 \times l_\rho^2)$, there exists $T = T(\tau, \omega, B, c) > 0$ such that, for all $t \geq T$ and $(u_{\tau-t}, v_{\tau-t}) \in B(\tau - t, \theta_{-\tau} \omega)$, the solution (u, v) of (8) satisfies

$$\begin{aligned} &\|u(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau-t}, v_{\tau-t})\|_\rho^2 \\ &\quad + \|v(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau-t}, v_{\tau-t})\|_\rho^2 \\ &\quad + \frac{\sigma}{2} \int_{-\tau}^0 e^{\int_s^0 \mathcal{R}(r, \omega) dr} \|u(s, \tau - t, \theta_{-\tau} \omega, u_{\tau-t}, v_{\tau-t})\|_\rho^2 ds \\ &\leq \mathcal{F}(c, \tau, \omega), \end{aligned} \quad (19)$$

where

$$\begin{aligned} \mathcal{F}(c, \tau, \omega) &= \frac{(1+\lambda)^2}{\lambda} e^{\int_{-\infty}^0 \mathcal{H}(r, \omega) dr} \left(\|u_{\tau-t}\|_\rho^2 + \|v_{\tau-t}\|_\rho^2 \right) \\ &\quad + \frac{1+\lambda}{\alpha\lambda} \int_{-\infty}^0 e^{\int_s^0 \mathcal{H}(r, \omega) dr} \|g(s+\tau)\|_\rho^2 ds, \end{aligned} \quad (20)$$

$$\begin{aligned} \mathcal{H}(t, \omega) &= -2\epsilon + \frac{2c(\lambda + \sigma(\lambda + 1))}{\lambda} |z(\theta_t \omega)| \\ &\quad + \frac{c^2(\lambda + 1)}{\lambda} z^2(\theta_t \omega) \\ &\quad + \frac{2(\lambda + 1)(1 + q + \bar{q})}{\lambda} \eta(\theta_t \omega). \end{aligned} \quad (21)$$

Proof. For each $\omega \in \Omega$, there exists a sequence $\eta_{i,j}^{(m)}(t, \omega)$ of continuous functions in $t \in \mathbb{R}$ (see [17]) such that

$$\lim_{m \rightarrow \infty} \int_\tau^t |\eta_{i,j}^{(m)}(s, \omega) - \eta_{i,j}(\theta_s \omega)| ds = 0, \quad (22)$$

$$\forall t > 0, \tau \in \mathbb{R}, j \in \{-q, \dots, 0, \dots, q\}, q \in \mathbb{N}, i \in \mathbb{Z},$$

and $|\eta_{i,j}^{(m)}(t, \omega)| \leq |\eta_{i,j}(\theta_t \omega)| \leq \eta(\theta_t \omega)$ for $t \in \mathbb{R}$. Consider the following random differential equations:

$$\begin{aligned} \frac{du^{(m)}}{dt} &= -\sigma u^{(m)} + v^{(m)} + cu^{(m)}z(\theta_t \omega), \\ \frac{dv^{(m)}}{dt} &= (\sigma(\alpha - \sigma) - \lambda)u^{(m)} + (\sigma - \alpha)v^{(m)} \\ &\quad + \mathbb{B}_m(t, \omega)u^{(m)} \\ &\quad + (2c\sigma z(\theta_t \omega) - c^2 z^2(\theta_t \omega))u^{(m)} \\ &\quad - cz(\theta_t \omega)v^{(m)} - \beta \sin u^{(m)} + g(t), \end{aligned} \quad (23)$$

$$u^{(m)}(\tau) = u_\tau,$$

$$v^{(m)}(\tau) = v_\tau = u_{1\tau} + \sigma u_\tau - cu_\tau z(\omega),$$

where $(\mathbb{B}_m(t, \omega)u^{(m)})_i = \sum_{j=-q}^q \eta_{i,j}^{(m)}(t, \omega)u_{i+j}^{(m)}$. It is easy to see that (23) has a unique mild solution $(u^{(m)}(\cdot, \tau, \omega, u_\tau, v_\tau), v^{(m)}(\cdot, \tau, \omega, u_\tau, v_\tau)) \in C([\tau, +\infty), l^2 \times l^2) \cap C^1((\tau, +\infty), l^2 \times l^2)$ satisfying (23). Taking the inner product of the first equation in (23) with $u^{(m)}(t)$ in $l_{\lambda, \rho}^2$ and the second equation with $v^{(m)}(t)$ in l_ρ^2 , then we have that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u^{(m)}\|_{\lambda, \rho}^2 &= -\sigma(u^{(m)}, u^{(m)})_{\lambda, \rho} + (v^{(m)}, u^{(m)})_{\lambda, \rho} \\ &\quad + cz(\theta_t \omega)(u^{(m)}, u^{(m)})_{\lambda, \rho}, \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v^{(m)}\|_\rho^2 &= (\sigma(\alpha - \sigma) - \lambda)(u^{(m)}, v^{(m)})_\rho \\ &\quad + (\sigma - \alpha)(v^{(m)}, v^{(m)})_\rho + (\mathbb{B}_m(t, \omega)u^{(m)}, v^{(m)})_\rho \\ &\quad + (2c\sigma z(\theta_t \omega) - c^2 z^2(\theta_t \omega))(u^{(m)}, v^{(m)})_\rho \\ &\quad - cz(\theta_t \omega)(v^{(m)}, v^{(m)})_\rho - \beta(\sin u^{(m)}, v^{(m)})_\rho \\ &\quad + (g(t), v^{(m)})_\rho. \end{aligned} \quad (24)$$

Summing the two equations of (24), we find that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|u^{(m)}\|_{\lambda, \rho}^2 + \|v^{(m)}\|_\rho^2) &= (-\sigma + cz(\theta_t \omega))\|u^{(m)}\|_{\lambda, \rho}^2 \\ &\quad + \sigma(\alpha - \sigma)(u^{(m)}, v^{(m)})_\rho \\ &\quad + (\sigma - \alpha - cz(\theta_t \omega))\|v^{(m)}\|_\rho^2 \\ &\quad + (\mathbb{B}_m(t, \omega)u^{(m)}, v^{(m)})_\rho \\ &\quad + (2c\sigma z(\theta_t \omega) - c^2 z^2(\theta_t \omega))(u^{(m)}, v^{(m)})_\rho \\ &\quad - \beta(\sin u^{(m)}, v^{(m)})_\rho + (g(t), v^{(m)})_\rho. \end{aligned} \quad (25)$$

Note that

$$\begin{aligned} \sigma(\alpha - \sigma)(u^{(m)}, v^{(m)})_\rho &\leq \frac{\sigma(\alpha - \sigma)}{2} \left(\frac{1}{\lambda} \|v^{(m)}\|_\rho^2 + \lambda \|u^{(m)}\|_\rho^2 \right) \\ &\leq \frac{\sigma(\alpha - \sigma)}{2} \left(\frac{1}{\lambda} \|v^{(m)}\|_\rho^2 + \|u^{(m)}\|_{\lambda, \rho}^2 \right) \\ &\leq \frac{\sigma(\alpha - \sigma)}{2} \left(\frac{1}{\lambda} + 1 \right) (\|v^{(m)}\|_\rho^2 + \|u^{(m)}\|_{\lambda, \rho}^2), \end{aligned} \quad (26)$$

$$\begin{aligned} (2c\sigma z(\theta_t \omega) - c^2 z^2(\theta_t \omega))(u^{(m)}, v^{(m)})_\rho &\leq \left(c\sigma |z(\theta_t \omega)| + \frac{c^2 z^2(\theta_t \omega)}{2} \right) \\ &\quad \cdot \left(\frac{1}{\lambda} \|v^{(m)}\|_\rho^2 + \lambda \|u^{(m)}\|_\rho^2 \right) \\ &\leq \left(c\sigma |z(\theta_t \omega)| + \frac{c^2 z^2(\theta_t \omega)}{2} \right) \\ &\quad \cdot \left(\frac{1}{\lambda} \|v^{(m)}\|_\rho^2 + \|u^{(m)}\|_{\lambda, \rho}^2 \right) \\ &\leq \left(c\sigma |z(\theta_t \omega)| + \frac{c^2 z^2(\theta_t \omega)}{2} \right) \left(\frac{1}{\lambda} + 1 \right) \\ &\quad \cdot (\|v^{(m)}\|_\rho^2 + \|u^{(m)}\|_{\lambda, \rho}^2), \end{aligned} \quad (27)$$

$$\begin{aligned}
-\beta \left(\sin u^{(m)}, v^{(m)} \right)_\rho &= -\beta \sum_{i \in \mathbb{Z}} \rho_i v_i^{(m)} \sin u_i^{(m)} \\
&\leq \beta \sum_{i \in \mathbb{Z}} \rho_i |v_i^{(m)}| \cdot |u_i^{(m)}| \\
&\leq \frac{\beta}{2} \left(\frac{1}{\lambda} \|v^{(m)}\|_\rho^2 + \lambda \|u^{(m)}\|_\rho^2 \right) \leq \frac{\beta}{2} \left(\frac{1}{\lambda} + 1 \right) \\
&\quad \cdot \left(\|v^{(m)}\|_\rho^2 + \|u^{(m)}\|_{\lambda, \rho}^2 \right),
\end{aligned} \tag{28}$$

$$\left(g(t), v^{(m)} \right)_\rho \leq \frac{1}{2\alpha} \|g(t)\|_\rho^2 + \frac{\alpha}{2} \|v^{(m)}\|_\rho^2. \tag{29}$$

By (3), we get

$$\begin{aligned}
&\left(\mathbb{B}_m(t, \omega) u^{(m)}, v^{(m)} \right)_\rho \\
&= \sum_{i \in \mathbb{Z}} \left(\rho_i v_i^{(m)} \cdot \sum_{j=-q}^q \eta_{i,j}^{(m)}(t, \omega) u_{i+j}^{(m)} \right) \leq \eta(\theta_t \omega) \\
&\quad \cdot \sum_{i \in \mathbb{Z}} \left(\rho_i |v_i^{(m)}| \cdot \sum_{j=-q}^q |u_{i+j}^{(m)}| \right) \leq \frac{1}{2} \eta(\theta_t \omega) \\
&\quad \cdot \left((1+2q) \|v^{(m)}\|_\rho^2 + (1+2\tilde{q}) \|u^{(m)}\|_\rho^2 \right) \leq \eta(\theta_t \omega) \\
&\quad \cdot (1+q+\tilde{q}) \left(\frac{1}{\lambda} + 1 \right) \left(\|v^{(m)}\|_\rho^2 + \|u^{(m)}\|_{\lambda, \rho}^2 \right),
\end{aligned} \tag{30}$$

where $\tilde{q} = \sum_{k=1}^q c_0^k$.

From (25)–(28), we obtain that, for $t > 0$,

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(\|u^{(m)}\|_{\lambda, \rho}^2 + \|v^{(m)}\|_\rho^2 \right) + \frac{\sigma}{2} \|u^{(m)}\|_{\lambda, \rho}^2 \leq \left(\frac{1}{\lambda} + 1 \right) \\
&\quad \cdot \left(c\sigma |z(\theta_t \omega)| + \frac{c^2 z^2(\theta_t \omega)}{2} + \frac{\sigma(\alpha - \sigma)}{2} + \frac{\beta}{2} \right. \\
&\quad \left. + \eta(\theta_t \omega) (1+q+\tilde{q}) \right) \left(\|v^{(m)}\|_\rho^2 + \|u^{(m)}\|_{\lambda, \rho}^2 \right) \\
&\quad + \left(-\frac{\sigma}{2} + cz(\theta_t \omega) \right) \|u^{(m)}\|_{\lambda, \rho}^2 + (\sigma - \alpha - cz(\theta_t \omega)) \\
&\quad \cdot \|v^{(m)}\|_\rho^2 + \frac{1}{2\alpha} \|g(t)\|_\rho^2 + \frac{\alpha}{2} \|v^{(m)}\|_\rho^2.
\end{aligned} \tag{31}$$

Recalling that $\lambda = \min\{\alpha/2 - \sigma, \sigma/2\}$ and ϵ in (10), then we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(\|u^{(m)}\|_{\lambda, \rho}^2 + \|v^{(m)}\|_\rho^2 \right) + \frac{\sigma}{2} \|u^{(m)}\|_{\lambda, \rho}^2 \leq \left(\frac{1}{\lambda} + 1 \right) \\
&\quad \cdot \left(c\sigma |z(\theta_t \omega)| + \frac{1}{2} c^2 z^2(\theta_t \omega) + \frac{\sigma(\alpha - \sigma)}{2} + \frac{\beta}{2} \right. \\
&\quad \left. + \eta(\theta_t \omega) (1+q+\tilde{q}) \right) \left(\|v^{(m)}\|_\rho^2 + \|u^{(m)}\|_{\lambda, \rho}^2 \right)
\end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{2\alpha} \|g(t)\|_\rho^2 + \left(-\frac{\sigma}{2} + cz(\theta_t \omega) \right) \|u^{(m)}\|_{\lambda, \rho}^2 + \left(\sigma \right. \\
&\quad \left. - \frac{\alpha}{2} - cz(\theta_t \omega) \right) \|v^{(m)}\|_\rho^2 \leq \left(-\lambda + c |z(\theta_t \omega)| \right. \\
&\quad \left. + \frac{\lambda+1}{\lambda} \left(c\sigma |z(\theta_t \omega)| + \frac{c^2 z^2(\theta_t \omega)}{2} + \frac{\sigma(\alpha - \sigma)}{2} \right. \right. \\
&\quad \left. \left. + \frac{\beta}{2} + \eta(\theta_t \omega) (1+q+\tilde{q}) \right) \right) \cdot \left(\|v^{(m)}\|_\rho^2 \right. \\
&\quad \left. + \|u^{(m)}\|_{\lambda, \rho}^2 \right) + \frac{1}{2\alpha} \|g(t)\|_\rho^2 = \left(-\epsilon \right. \\
&\quad \left. + \frac{c(\lambda + \sigma(\lambda + 1))}{\lambda} |z(\theta_t \omega)| + \frac{c^2(\lambda + 1)}{2\lambda} z^2(\theta_t \omega) \right. \\
&\quad \left. + \frac{(\lambda + 1)(1+q+\tilde{q})}{\lambda} \eta(\theta_t \omega) \right) \cdot \left(\|v^{(m)}\|_\rho^2 \right. \\
&\quad \left. + \|u^{(m)}\|_{\lambda, \rho}^2 \right) + \frac{1}{2\alpha} \|g(t)\|_\rho^2 = \frac{\mathcal{H}(t, \omega)}{2} \left(\|v^{(m)}\|_\rho^2 \right. \\
&\quad \left. + \|u^{(m)}\|_{\lambda, \rho}^2 \right) + \frac{1}{2\alpha} \|g(t)\|_\rho^2,
\end{aligned} \tag{32}$$

where $\mathcal{H}(t, \omega)$ is as in (21). Then we obtain from (32) that, for $t > 0$,

$$\begin{aligned}
&\|u^{(m)}(\tau, \tau - t, \omega, u_{\tau-t}, v_{\tau-t})\|_{\lambda, \rho}^2 \\
&+ \|v^{(m)}(\tau, \tau - t, \omega, u_{\tau-t}, v_{\tau-t})\|_\rho^2 \\
&+ \sigma \int_{\tau-t}^\tau e^{\int_s^\tau \mathcal{H}(r, \omega) dr} \|u^{(m)}(s, \tau - t, \omega, u_{\tau-t}, v_{\tau-t})\|_{\lambda, \rho}^2 ds \\
&\leq e^{\int_{\tau-t}^\tau \mathcal{H}(r, \omega) dr} \left(\|u_{\tau-t}\|_{\lambda, \rho}^2 + \|v_{\tau-t}\|_\rho^2 \right) \\
&\quad + \frac{1}{\alpha} \int_{\tau-t}^\tau e^{\int_s^\tau \mathcal{H}(r, \omega) dr} \|g(s)\|_\rho^2 ds.
\end{aligned} \tag{33}$$

From (33) and by replacing ω by $\theta_{-\tau} \omega$, we have

$$\begin{aligned}
&\|u^{(m)}(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau-t}, v_{\tau-t})\|_{\lambda, \rho}^2 + \|v^{(m)}(\tau, \tau - t, \\
&\quad \theta_{-\tau} \omega, u_{\tau-t}, v_{\tau-t})\|_\rho^2 + \sigma \int_{-\tau}^0 e^{\int_s^0 \mathcal{H}(r, \omega) dr} \|u^{(m)}(s, \tau \\
&\quad - t, \theta_{-\tau} \omega, u_{\tau-t}, v_{\tau-t})\|_{\lambda, \rho}^2 ds \\
&\leq e^{\int_{-\tau}^\tau \mathcal{H}(r-\tau, \omega) dr} \left(\|u_{\tau-t}\|_{\lambda, \rho}^2 + \|v_{\tau-t}\|_\rho^2 \right) + \frac{1}{\alpha} \\
&\quad \cdot \int_{-\tau}^\tau e^{\int_s^\tau \mathcal{H}(r-\tau, \omega) dr} \|g(s)\|_\rho^2 ds \\
&\leq e^{\int_{-\tau}^0 \mathcal{H}(r, \omega) dr} \left(\|u_{\tau-t}\|_{\lambda, \rho}^2 + \|v_{\tau-t}\|_\rho^2 \right) + \frac{1}{\alpha} \\
&\quad \cdot \int_{-\tau}^0 e^{\int_s^0 \mathcal{H}(r, \omega) dr} \|g(s+\tau)\|_\rho^2 ds.
\end{aligned} \tag{34}$$

Note that (34) holds with $u^{(m)}(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}, v_{\tau-t})$ and $v^{(m)}(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}, v_{\tau-t})$ being replaced by $u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}, v_{\tau-t})$ and $v(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}, v_{\tau-t})$; then we have

$$\begin{aligned} & \|u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}, v_{\tau-t})\|_\rho^2 \\ & + \|v(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}, v_{\tau-t})\|_\rho^2 \\ & + \sigma \int_{-\tau}^0 e^{\int_s^0 \mathcal{H}(r, \omega) dr} \|u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}, v_{\tau-t})\|_\rho^2 ds \\ & \leq \frac{(1 + \lambda)^2}{\lambda} e^{\int_{-\tau}^0 \mathcal{H}(r, \omega) dr} (\|u_{\tau-t}\|_\rho^2 + \|v_{\tau-t}\|_\rho^2) \\ & + \frac{1 + \lambda}{\alpha \lambda} \int_{-\tau}^0 e^{\int_s^0 \mathcal{H}(r, \omega) dr} \|g(s + \tau)\|_\rho^2 ds. \end{aligned} \quad (35)$$

Let

$$\begin{aligned} \mathcal{H}_1(t, \omega) &= \frac{c(\lambda + \sigma(\lambda + 1))}{\lambda} |z(\theta_t \omega)| \\ &+ \frac{c^2(\lambda + 1)}{2\lambda} z^2(\theta_t \omega) \\ &+ \frac{(\lambda + 1)(1 + q + \tilde{q})}{\lambda} \eta(\theta_t \omega); \end{aligned} \quad (36)$$

then $\mathcal{H}(t, \omega) = 2(-\epsilon + \mathcal{H}_1(t, \omega))$. By (9), we find that there exists $T_1 = T_1(\omega) > 0$ such that, for $t > T_1$,

$$\begin{aligned} & \int_{-\tau}^0 z(\theta_s \omega) ds \leq \frac{\epsilon \lambda}{4c(\lambda + \sigma(\lambda + 1))} t, \\ & \int_{-\tau}^0 |z(\theta_s \omega)|^2 ds \leq \frac{\epsilon \lambda}{2c^2(\lambda + 1)} t, \\ & \int_{-\tau}^0 \eta(\theta_s \omega) ds \leq \frac{\epsilon \lambda}{4(\lambda + 1)(1 + q + \tilde{q})} t, \end{aligned} \quad (37)$$

from which, along with $(u_{\tau-t}, v_{\tau-t}) \in B(\tau - t, \theta_{-\tau}\omega) \in \mathcal{D}(l_\rho^2 \times l_\rho^2)$, we have

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \frac{(1 + \lambda)^2}{\lambda} e^{\int_{-\tau}^0 \mathcal{H}(r, \omega) dr} (\|u_{\tau-t}\|_\rho^2 + \|v_{\tau-t}\|_\rho^2) \\ & \leq \limsup_{t \rightarrow +\infty} \frac{(1 + \lambda)^2}{\lambda} e^{-\epsilon t/4} \|B(\tau - t, \theta_{-\tau}\omega)\|_\rho^2 = 0. \end{aligned} \quad (38)$$

Therefore, there exists $T_2 = T_2(\tau, \omega, B, c) > 0$ such that, for all $t \geq T_2$,

$$\frac{(1 + \lambda)^2}{\lambda} e^{\int_{-\tau}^0 \mathcal{H}(r, \omega) dr} (\|u_{\tau-t}\|_\rho^2 + \|v_{\tau-t}\|_\rho^2) \leq 1. \quad (39)$$

Since $z(\theta_t \omega)$ and $\eta(\theta_t \omega)$ are tempered, then $\mathcal{H}(t, \omega)$ is tempered. Then by (A3), we can verify that the following integrals are convergent:

$$\frac{1 + \lambda}{\alpha \lambda} \int_{-\tau}^0 e^{\int_s^0 \mathcal{H}(r, \omega) dr} \|g(s + \tau)\|_\rho^2 ds < \infty. \quad (40)$$

Thus the theorem follows from (35), (39), and (40). $\square$

Theorem 5. Assume that (A1)–(A3) hold. Then the continuous cocycle Ψ_c associated with (8) has a unique random attractor $\mathcal{A}_c = \{\mathcal{A}_c(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l_\rho^2 \times l_\rho^2)$.

Proof. By Theorem 3, it suffices to prove that, for every $\varepsilon > 0$, $c > 0$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$ and for any $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l_\rho^2 \times l_\rho^2)$, there exist $T = T(\tau, \omega, B, c, \varepsilon) > 0$ and $R = R(\tau, \omega, c, \varepsilon) > 1$ such that, for all $t \geq T$ and $(u_{\tau-t}, v_{\tau-t}) \in B(\tau - t, \theta_{-\tau}\omega)$, the solution (u, v) of (8) satisfies

$$\begin{aligned} & \sum_{|i| > R} \rho_i (|u_i(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}, v_{\tau-t})|^2 \\ & + |v_i(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}, v_{\tau-t})|^2) \leq \varepsilon. \end{aligned} \quad (41)$$

Choose a smooth increasing function $\chi : \mathbb{R}^+ \rightarrow [0, 1]$ such that

$$\chi(s) = \begin{cases} 0, & 0 \leq s \leq 1, \\ 1, & s \geq 2, \end{cases} \quad (42)$$

and there exists a positive constant c_χ such that $|\chi'(s)| \leq c_\chi$ for $s \in \mathbb{R}^+$.

Let $(u(\tau, \tau - t, \omega, u_{\tau-t}, v_{\tau-t}), v(\tau, \tau - t, \omega, u_{\tau-t}, v_{\tau-t}))$ be a mild solution of (8) with $(u_{\tau-t}, v_{\tau-t}) \in l_\rho^2 \times l_\rho^2$. For any given $N > 0$ define $\mathcal{Q}_N : l_\rho^2 \times l_\rho^2 \rightarrow l^2 \times l^2$, $(u, v) = (u_i, v_i)_{i \in \mathbb{Z}} \mapsto \mathcal{Q}_N(u, v) = ((\mathcal{Q}_N u)_i, (\mathcal{Q}_N v)_i)_{i \in \mathbb{Z}}$ by $((\mathcal{Q}_N u)_j, (\mathcal{Q}_N v)_j) = (u_j, v_j)$ if $|j| \leq N$ and $((\mathcal{Q}_N u)_j, (\mathcal{Q}_N v)_j) = (0, 0)$ otherwise.

For any $n \geq 1$, let $(u^{(m)}, v^{(m)}) = (u^{(m)}(\tau, \tau - t, \omega, \mathcal{Q}_n u_{\tau-t}, \mathcal{Q}_n v_{\tau-t}), v^{(m)}(\tau, \tau - t, \omega, \mathcal{Q}_n u_{\tau-t}, \mathcal{Q}_n v_{\tau-t})) = (u_i^{(m)}, v_i^{(m)})_{i \in \mathbb{Z}}$ be the solution of (23).

Then taking the inner product of the first equation in (23) with $\hat{\chi}(r)u^{(m)} = (\chi(|i|/r)u_i^{(m)})_{i \in \mathbb{Z}}$ in $l_{\lambda, \rho}^2$ and the second equation with $\hat{\chi}(r)v^{(m)} = (\chi(|i|/r)v_i^{(m)})_{i \in \mathbb{Z}}$ in $l_{\lambda, \rho}^2$, we have that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) (\lambda |u_i^{(m)}|^2 + |v_i^{(m)}|^2) \\ & = -\sigma (u^{(m)}, \hat{\chi}(r) u^{(m)})_{\lambda, \rho} + (v^{(m)}, \hat{\chi}(r) u^{(m)})_{\lambda, \rho} \\ & + cz(\theta_t \omega) (u^{(m)}, \hat{\chi}(r) u^{(m)})_{\lambda, \rho} \\ & + (\sigma(\alpha - \sigma) - \lambda) (u^{(m)}, \hat{\chi}(r) v^{(m)})_{\lambda, \rho} \\ & + (\sigma - \alpha) (v^{(m)}, \hat{\chi}(r) v^{(m)})_{\lambda, \rho} \\ & + (\mathbb{B}_m(t, \omega) u^{(m)}, \hat{\chi}(r) v^{(m)})_{\lambda, \rho} \end{aligned}$$

$$\begin{aligned}
& + \left(2c\sigma z (\theta_t \omega) - c^2 z^2 (\theta_t \omega) \right) \left(u^{(m)}, \widehat{\chi}(r) v^{(m)} \right)_\rho \\
& - cz (\theta_t \omega) \left(v^{(m)}, \widehat{\chi}(r) v^{(m)} \right)_\rho \\
& - \beta \left(\sin u^{(m)}, \widehat{\chi}(r) v^{(m)} \right)_\rho + \left(g(t), \widehat{\chi}(r) v^{(m)} \right)_\rho \\
& = \lambda (-\sigma + cz (\theta_t \omega)) \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) |u_i^{(m)}|^2 \\
& + \sigma (\alpha - \sigma) \left(u^{(m)}, \widehat{\chi}(r) v^{(m)} \right)_\rho \\
& + (\sigma - \alpha - cz (\theta_t \omega)) \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) |v_i^{(m)}|^2 \\
& + \left(\mathbb{B}_m(t, \omega) u^{(m)}, \widehat{\chi}(r) v^{(m)} \right)_\rho \\
& + \left(2c\sigma z (\theta_t \omega) - c^2 z^2 (\theta_t \omega) \right) \left(u^{(m)}, \widehat{\chi}(r) v^{(m)} \right)_\rho \\
& - \beta \left(\sin u^{(m)}, \widehat{\chi}(r) v^{(m)} \right)_\rho + \left(g(t), \widehat{\chi}(r) v^{(m)} \right)_\rho.
\end{aligned} \tag{43}$$

For each term of (43), it has been checked that

$$\begin{aligned}
\left(u^{(m)}, \widehat{\chi}(r) v^{(m)} \right)_\rho &= \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) u_i^{(m)} \cdot v_i^{(m)} \leq \frac{1 + \lambda}{2\lambda} \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) (\lambda |u_i^{(m)}|^2 + |v_i^{(m)}|^2), \\
\left(\mathbb{B}_m(t, \omega) u^{(m)}, \widehat{\chi}(r) v^{(m)} \right)_\rho &= \sum_{i \in \mathbb{Z}} \left(\rho_i \chi \left(\frac{|i|}{r} \right) v_i^{(m)} \sum_{j=-q}^q \eta_{i,j}^{(m)}(t, \omega) u_{i+j}^{(m)} \right) \\
&\leq \frac{\eta(\theta_t \omega)}{2} \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) \left((1 + 2q) |v_i^{(m)}|^2 + \sum_{j=-q}^q |u_{i+j}^{(m)}|^2 \right) \\
&\leq \frac{\eta(\theta_t \omega)}{2} \sum_{i \in \mathbb{Z}} \left[\rho_i \sum_{j=-q}^q \left(\left(\chi \left(\frac{|i|}{r} \right) - \chi \left(\frac{|i+j|}{r} \right) \right) |u_{i+j}^{(m)}|^2 + \chi \left(\frac{|i+j|}{r} \right) |u_{i+j}^{(m)}|^2 \right) \right] \\
&\quad + \frac{(1 + 2q) \eta(\theta_t \omega)}{2} \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) |v_i^{(m)}|^2 \\
&\leq \frac{c_\chi q (1 + 2\bar{q}) \eta(\theta_t \omega)}{2r} \|u^{(m)}\|_\rho^2 + \frac{(1 + 2\bar{q}) \eta(\theta_t \omega)}{2} \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) |u_i^{(m)}|^2 \\
&\quad + \frac{(1 + 2q) \eta(\theta_t \omega)}{2} \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) |v_i^{(m)}|^2 \\
&\leq \frac{c_\chi q (1 + 2\bar{q}) \eta(\theta_t \omega)}{2r} \|u^{(m)}\|_\rho^2 \\
&\quad + \frac{(1 + \lambda) (1 + q + \bar{q}) \eta(\theta_t \omega)}{\lambda} \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) (\lambda |u_i^{(m)}|^2 + |v_i^{(m)}|^2), \\
-\beta \left(\sin u^{(m)}, \widehat{\chi}(r) v^{(m)} \right)_\rho &= -\beta \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) \sin u_i^{(m)} \cdot v_i^{(m)} \leq \beta \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) |u_i^{(m)}| \cdot |v_i^{(m)}| \\
&\leq \frac{\beta (1 + \lambda)}{2\lambda} \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) (\lambda |u_i^{(m)}|^2 + |v_i^{(m)}|^2), \\
\left(g(t), \widehat{\chi}(r) v^{(m)} \right)_\rho &= \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) g_i(t) \cdot v_i^{(m)} \leq \frac{1}{2\alpha} \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) |g_i(t)|^2 + \frac{\alpha}{2} \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) |v_i^{(m)}|^2.
\end{aligned} \tag{44}$$

Putting above inequalities into (43), we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) \left(\lambda |u_i^{(m)}|^2 + |v_i^{(m)}|^2 \right) \\
& \leq \frac{1+\lambda}{\lambda} \left(c\sigma |z(\theta_t \omega)| + \frac{1}{2} c^2 z^2(\theta_t \omega) + \frac{\sigma(\alpha - \sigma)}{2} \right. \\
& \quad \left. + \frac{\beta}{2} + \eta(\theta_t \omega)(1 + q + \bar{q}) \right) \cdot \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) \\
& \quad \cdot \left(\lambda |u_i^{(m)}|^2 + |v_i^{(m)}|^2 \right) + \lambda(-\sigma + cz(\theta_t \omega)) \\
& \quad \cdot \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) |u_i^{(m)}|^2 + \left(\sigma - \frac{\alpha}{2} - cz(\theta_t \omega) \right) \\
& \quad \cdot \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) |v_i^{(m)}|^2 + \frac{1}{2\alpha} \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) |g_i(t)|^2 \\
& \quad + \frac{c_\chi q(1 + 2\bar{q}) \eta(\theta_t \omega)}{2r} \|u^{(m)}\|_\rho^2 \leq \frac{\mathcal{H}(t, \omega)}{2} \\
& \quad \cdot \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) \left(\lambda |u_i^{(m)}|^2 + |v_i^{(m)}|^2 \right) + \frac{1}{2\alpha} \\
& \quad \cdot \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) |g_i(t)|^2 \\
& \quad + \frac{c_\chi q(1 + 2\bar{q}) \eta(\theta_t \omega)}{2r} \|u^{(m)}\|_\rho^2,
\end{aligned} \tag{45}$$

where $\mathcal{H}(t, \omega)$ is as in (21). Multiplying (45) by $e^{\int_0^t \mathcal{H}(r, \omega) dr}$ and then integrating over $[\tau - t, \tau]$ with $t > 0$, we obtain

$$\begin{aligned}
& \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) \left(\lambda |u_i^{(m)}(\tau, \tau - t, \omega, \mathcal{Q}_n u_{\tau-t}, \mathcal{Q}_n v_{\tau-t})|^2 \right. \\
& \quad \left. + |v_i^{(m)}(\tau, \tau - t, \omega, \mathcal{Q}_n u_{\tau-t}, \mathcal{Q}_n v_{\tau-t})|^2 \right) \\
& \leq e^{\int_{\tau-t}^\tau \mathcal{H}(r, \omega) dr} \left(\lambda \|\mathcal{Q}_n u_{\tau-t}\|_\rho^2 + \|\mathcal{Q}_n v_{\tau-t}\|_\rho^2 \right) + \frac{1}{\alpha} \\
& \quad \cdot \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) \int_{\tau-t}^\tau e^{\int_s^\tau \mathcal{H}(r, \omega) dr} |g_i(s)|^2 ds \\
& \quad + \frac{c_\chi q(1 + 2\bar{q})}{r} \int_{\tau-t}^\tau e^{\int_s^\tau \mathcal{H}(r, \omega) dr} \eta(\theta_s \omega) \\
& \quad \cdot \|u(s, \tau - t, \omega, \mathcal{Q}_n u_{\tau-t}, \mathcal{Q}_n v_{\tau-t})\|_\rho^2 ds.
\end{aligned} \tag{46}$$

Replacing ω and m in (46) by $\theta_{-\tau} \omega$ and m_k , respectively, and letting $k \rightarrow \infty$, then we have that

$$\begin{aligned}
& \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) \left(\lambda |u_i(\tau, \tau - t, \theta_{-\tau} \omega, \mathcal{Q}_n u_{\tau-t}, \mathcal{Q}_n v_{\tau-t})|^2 \right. \\
& \quad \left. + |v_i(\tau, \tau - t, \theta_{-\tau} \omega, \mathcal{Q}_n u_{\tau-t}, \mathcal{Q}_n v_{\tau-t})|^2 \right) \\
& \leq e^{\int_{-\tau}^0 \mathcal{H}(r, \omega) dr} \left(\lambda \|\mathcal{Q}_n u_{\tau-t}\|_\rho^2 + \|\mathcal{Q}_n v_{\tau-t}\|_\rho^2 \right) + \frac{1}{\alpha} \\
& \quad \cdot \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) \int_{-\tau}^0 e^{\int_s^0 \mathcal{H}(r, \omega) dr} |g_i(s + \tau)|^2 ds \\
& \quad + \frac{c_\chi q(1 + 2\bar{q})}{r} \int_{-\tau}^0 e^{\int_s^0 \mathcal{H}(r, \omega) dr} \eta(\theta_s \omega) \\
& \quad \cdot \|u(s + \tau, \tau - t, \theta_{-\tau} \omega, \mathcal{Q}_n u_{\tau-t}, \mathcal{Q}_n v_{\tau-t})\|_\rho^2 ds.
\end{aligned} \tag{47}$$

We now estimate each term on the right-hand side of (47). For the first term on the right-hand side of (47), since $(\mathcal{Q}_n u_{\tau-t}, \mathcal{Q}_n v_{\tau-t}) \in B(\tau - t, \theta_{-\tau} \omega)$ and B is tempered, then there exists $T_1 = T_1(\tau, \varepsilon, \omega, B) > 0$ such that if $t > T_1$, then

$$e^{\int_{-\tau}^0 \mathcal{H}(r, \omega) dr} \left(\lambda \|\mathcal{Q}_n u_{\tau-t}\|_\rho^2 + \|\mathcal{Q}_n v_{\tau-t}\|_\rho^2 \right) \leq \varepsilon. \tag{48}$$

For the second term on the right-hand side of (47), by (A3), there exist $R_1 = R_1(\varepsilon, \omega) > 0$ and $T_2 = T_2(\varepsilon, \omega) > 0$, such that if $r > R_1$ and $t > T_2$, then

$$\frac{1}{\alpha} \sum_{i \in \mathbb{Z}} \rho_i \chi \left(\frac{|i|}{r} \right) \int_{-\tau}^0 e^{\int_s^0 \mathcal{H}(r, \omega) dr} |g_i(s + \tau)|^2 ds \leq \varepsilon. \tag{49}$$

For the third term on the right-hand side of (47), by (A1) and Theorem 4, there exist $T_3 = T_3(\tau, \varepsilon, \omega, B) > 0$ and $R_2 = R_2(\varepsilon, \omega) > 0$ such that, for all $t > T_3$ and $r > R_2$,

$$\begin{aligned}
& \frac{c_\chi q(1 + 2\bar{q})}{r} \int_{-\tau}^0 e^{\int_s^0 \mathcal{H}(r, \omega) dr} \eta(\theta_s \omega) \\
& \quad \cdot \|u(s + \tau, \tau - t, \theta_{-\tau} \omega, \mathcal{Q}_n u_{\tau-t}, \mathcal{Q}_n v_{\tau-t})\|_\rho^2 ds \leq \varepsilon.
\end{aligned} \tag{50}$$

Let $T = \max\{T_1, T_2, T_3\}$ and $R = 2 \max\{R_1, R_2\}$. Then it follows from (48)–(50) that, for all $t > T$ and $r > R$, we obtain

$$\begin{aligned}
& \sum_{|i| > R} \rho_i \left(\lambda |u_i(\tau, \tau - t, \theta_{-\tau} \omega, \mathcal{Q}_n u_{\tau-t}, \mathcal{Q}_n v_{\tau-t})|^2 \right. \\
& \quad \left. + |v_i(\tau, \tau - t, \theta_{-\tau} \omega, \mathcal{Q}_n u_{\tau-t}, \mathcal{Q}_n v_{\tau-t})|^2 \right) \leq 3\varepsilon,
\end{aligned} \tag{51}$$

for any $n \geq 1$. Let $n \rightarrow \infty$; we have that (41) holds. This completes the proof. $\square$

4. Upper Semicontinuity of Random Attractors

First, we present a criteria concerning upper semicontinuity of nonautonomous random attractors with respect to a parameter in [18].

Theorem 6. Let Φ_c be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$. Suppose the following.

(i) Φ_c has a closed measurable random absorbing set $K_c = \{K_c(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $\mathcal{D}(X)$ and a unique random attractor $\mathcal{A}_c = \{\mathcal{A}_c(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $\mathcal{D}(X)$.

(ii) For each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $K_0(\tau, \omega) = \{u \in X : \|u\|_X \leq r_0(\tau, \omega)\}$ and

$$\limsup_{c \rightarrow 0} \|K_c(\tau, \omega)\|_X = \limsup_{c \rightarrow 0} \sup_{x \in K_c(\tau, \omega)} \|x\|_X \leq r_0(\tau, \omega), \quad (52)$$

where $r_0(\tau, \omega)$ is a positive-valued tempered random variable.

(iii) There exists $\varepsilon > 0$ such that, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\bigcup_{|c| \leq \varepsilon} \mathcal{A}_c(\tau, \omega)$ is precompact in X .

(iv) For $t > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $c_n \rightarrow 0$ when $n \rightarrow \infty$, and $x_n, x_0 \in X$ with $x_n \rightarrow x_0$ when $n \rightarrow \infty$, it holds:

$$\lim_{n \rightarrow \infty} \Phi_{c_n}(t, \tau, \omega, x_n) = \Phi_0(t, \tau, \omega, x_0). \quad (53)$$

Then for $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$d_H(\mathcal{A}_c(\tau, \omega), \mathcal{A}_0(\tau, \omega)) = \sup_{u \in \mathcal{A}_c(\tau, \omega)} \inf_{v \in \mathcal{A}_0(\tau, \omega)} \|u - v\|_X \rightarrow 0 \quad \text{as } c \rightarrow 0. \quad (54)$$

Next, we will use Theorem 6 to consider an upper semicontinuity of random attractors $\mathcal{A}_c(\tau, \omega)$ to $\mathcal{A}_0(\tau, \omega)$ as $c \rightarrow 0$.

Theorem 7. Assume that (A1)–(A3) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, one has

$$\begin{aligned} d_H(\mathcal{A}_c(\tau, \omega), \mathcal{A}_0(\tau, \omega)) &= \sup_{(u^c, v^c) \in \mathcal{A}_c(\tau, \omega)} \inf_{(u^0, v^0) \in \mathcal{A}_0(\tau, \omega)} \left(\|u^c - u^0\|_\rho^2 + \|v^c - v^0\|_\rho^2 \right)^{1/2} \rightarrow 0 \quad \text{as } c \rightarrow 0. \end{aligned} \quad (55)$$

Proof. Let $\mathcal{J}(c, \tau, \omega)$ be as in Theorem 4.

(i) By Theorems 4 and 5, Φ_c has a closed measurable random absorbing set $B_c = \{B_c(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l_\rho^2 \times l_\rho^2)$, where $B_c(\tau, \omega) = \{(u^c, v^c) \in l_\rho^2 \times l_\rho^2 : \|u^c\|_\rho^2 + \|v^c\|_\rho^2 \leq \mathcal{J}(c, \tau, \omega)\}$, and a unique random attractor $\mathcal{A}_c = \{\mathcal{A}_c(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $\mathcal{D}(l_\rho^2 \times l_\rho^2)$, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\mathcal{A}_c(\tau, \omega) \subseteq B_c(\tau, \omega)$.

(ii) Given $|c| < 1$. By (20), we have

$$\begin{aligned} \mathcal{J}(c, \tau, \omega) &\leq \mathcal{J}(1, \tau, \omega) < \infty, \\ \limsup_{c \rightarrow 0} \mathcal{J}(c, \tau, \omega) &\leq \mathcal{J}(1, \tau, \omega). \end{aligned} \quad (56)$$

So, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\begin{aligned} \limsup_{c \rightarrow 0} \|B_c(\tau, \omega)\|_\rho &= \limsup_{c \rightarrow 0} \sup_{x \in B_c(\tau, \omega)} \|x\|_{l_\rho^2 \times l_\rho^2} \\ &\leq \mathcal{J}^{1/2}(1, \tau, \omega). \end{aligned} \quad (57)$$

Moreover, $B_0(\tau, \omega) = \{(u^0, v^0) \in l_\rho^2 \times l_\rho^2 : \|u^0\|_\rho^2 + \|v^0\|_\rho^2 \leq \mathcal{J}(1, \tau, \omega)\}$ is a closed tempered random absorbing set for the continuous cocycle Φ_0 associated with the limiting system:

$$\begin{aligned} \frac{du^0}{dt} &= -\sigma u^0 + v^0, \\ \frac{dv^0}{dt} &= (\sigma(\alpha - \sigma) - \lambda) u^0 + (\sigma - \alpha) v^0 \\ &\quad + \mathbb{B}(\theta_t \omega) u^0 - \beta \sin u^0 + g(t), \end{aligned} \quad (58)$$

$$u^0(\tau) = u_\tau^0,$$

$$v^0(\tau) = v_\tau^0 = u_{1\tau}^0 + \sigma u_\tau^0,$$

$$\bigcup_{|c| \leq 1} \mathcal{A}_c(\tau, \omega) \subseteq \bigcup_{|c| \leq 1} B_c(\tau, \omega) \subseteq B_0(\tau, \omega). \quad (59)$$

(iii) Given $|c| < 1$. Let us prove the precompactness of $\bigcup_{|c| \leq 1} \mathcal{A}_c(\tau, \omega)$ for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$. For one thing, by Theorem 5, for every $\varepsilon > 0$, $c > 0$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$, there exist $T = T(\tau, \omega, B, c, \varepsilon) > 0$ and $R = R(\tau, \omega, c, \varepsilon) > 1$ such that, for all $t \geq T$ and $(u_{\tau-t}^c, v_{\tau-t}^c) \in B_0(\tau - t, \theta_{-t}\omega)$, the solution (u^c, v^c) of (8) satisfies

$$\begin{aligned} \sum_{|i| > R} \rho_i \left(|u_i^c(\tau, \tau - t, \theta_{-t}\omega, u_{\tau-t}^c, v_{\tau-t}^c)|^2 + |v_i^c(\tau, \tau - t, \theta_{-t}\omega, u_{\tau-t}^c, v_{\tau-t}^c)|^2 \right) &\leq \varepsilon, \end{aligned} \quad (60)$$

from which, along with (59) and the invariance of $\mathcal{A}_c(\tau, \omega)$, we have, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $t \geq T$,

$$\sup_{(u^c, v^c) \in \bigcup_{|c| \leq 1} \mathcal{A}_c(\tau, \omega)} \sum_{|i| > R} \rho_i \left(|u_i^c|^2 + |v_i^c|^2 \right) \leq \varepsilon. \quad (61)$$

For another, by (59) we find that the set $\{(u_i^c, v_i^c)_{|i| \leq R} : (u^c, v^c) \in \bigcup_{|c| \leq 1} \mathcal{A}_c(\tau, \omega)\}$ is bounded in a finite-dimensional space and hence $\bigcup_{|c| \leq 1} \mathcal{A}_c(\tau, \omega)$ is precompact in $l_\rho^2 \times l_\rho^2$.

(iv) Let $\varphi^c = (u^c, v^c)$ and $\varphi^0 = (u^0, v^0)$ be mild solutions of (8) and (58) with initial data (u_τ^c, v_τ^c) and (u_τ^0, v_τ^0) , respectively, and set $U = u^c - u^0$, $V = v^c - v^0$, $(U, V) = \varphi^c - \varphi^0$.

Let $(u^{c,m}(t, \tau, \omega, u_\tau^c, v_\tau^c), v^{c,m}(t, \tau, \omega, u_\tau^c, v_\tau^c))$ and $(u^{0,m}(t, \tau, \omega, u_\tau^0, v_\tau^0), v^{0,m}(t, \tau, \omega, u_\tau^0, v_\tau^0))$ be the solutions of the following random differential equations with initial data:

$$\begin{aligned} \frac{du^{c,m}}{dt} &= -\sigma u^{c,m} + v^{c,m} + cu^{c,m}z(\theta_t \omega), \\ \frac{dv^{c,m}}{dt} &= (\sigma(\alpha - \sigma) - \lambda) u^{c,m} + (\sigma - \alpha) v^{c,m} \\ &\quad + \mathbb{B}_m(t, \omega) u^{c,m} \\ &\quad + (2c\sigma z(\theta_t \omega) - c^2 z^2(\theta_t \omega)) u^{c,m} \\ &\quad - cz(\theta_t \omega) v^{c,m} - \beta \sin u^{c,m} + g(t), \end{aligned} \quad (62)$$

$$u^{c,m}(\tau) = u_\tau^c,$$

$$v^{c,m}(\tau) = v_\tau^c = u_{1\tau}^c + \sigma u_\tau^c - cu_\tau^c z(\omega),$$

$$\begin{aligned}
\frac{du^{0,m}}{dt} &= -\sigma u^{0,m} + v^{0,m}, \\
\frac{dv^{0,m}}{dt} &= (\sigma(\alpha - \sigma) - \lambda) u^{0,m} + (\sigma - \alpha) v^{0,m} \\
&\quad + \mathbb{B}_m(t, \omega) u^{0,m} - \beta \sin u^{0,m} + g(t), \\
u^{0,m}(\tau) &= u_\tau^0, \\
v^{0,m}(\tau) &= v_\tau^0 = u_{1\tau}^0 + \sigma u_\tau^0,
\end{aligned} \tag{63}$$

respectively. Then, $\varphi^{c,m}(\cdot, \tau, \omega, u_\tau^c, v_\tau^c)$, $\varphi^{0,m}(\cdot, \tau, \omega, u_\tau^0, v_\tau^0) \in C([\tau, +\infty), l_\rho^2 \times l_\rho^2)$ and satisfy the differential equations (62) and (63), respectively. Moreover, $\varphi^c(t, \tau, \omega, u_\tau^c, v_\tau^c)$ and $\varphi^0(t, \tau, \omega, u_\tau^0, v_\tau^0)$ are limit functions of subsequences of $\{\varphi^{c,m}(t, \tau, \omega, u_\tau^c, v_\tau^c)\}$ and $\{\varphi^{0,m}(t, \tau, \omega, u_\tau^0, v_\tau^0)\} \in l_\rho^2 \times l_\rho^2$. So $\varphi^c(t, \tau, \omega, u_\tau^c, v_\tau^c) - \varphi^0(t, \tau, \omega, u_\tau^0, v_\tau^0)$ is a limit function of a subsequence of $\{\varphi^{c,m}(t, \tau, \omega, u_\tau^c, v_\tau^c) - \varphi^{0,m}(t, \tau, \omega, u_\tau^0, v_\tau^0)\}$ in $l_\rho^2 \times l_\rho^2$ and $(U^m(t, \tau, \omega, u_\tau^c, v_\tau^c, u_\tau^0, v_\tau^0), V^m(t, \tau, \omega, u_\tau^c, v_\tau^c, u_\tau^0, v_\tau^0))$ satisfies

$$\begin{aligned}
\frac{dU^m}{dt} &= -\sigma U^m + V^m + cz(\theta_t \omega) u^{c,m}, \\
\frac{dV^m}{dt} &= (\sigma(\alpha - \sigma) - \lambda) U^m + (\sigma - \alpha) V^m \\
&\quad + \mathbb{B}_m(t, \omega) U^m \\
&\quad + (2c\sigma z(\theta_t \omega) - c^2 z^2(\theta_t \omega)) u^{c,m} \\
&\quad - cz(\theta_t \omega) v^{c,m} - \beta \sin u^{c,m} + \beta \sin u^{0,m}, \\
U^m(\tau) &= u_\tau^c - u_\tau^0, \\
V^m(\tau) &= v_\tau^c - v_\tau^0.
\end{aligned} \tag{64}$$

Now let us estimate $\|U^m\|_\rho^2 + \|V^m\|_\rho^2 = \|u^{c,m} - u^{0,m}\|_\rho^2 + \|v^{c,m} - v^{0,m}\|_\rho^2$. Taking the inner product of (64) with $U^m(t)$ and $V^m(t)$ in $l_{\lambda,\rho}^2$ and l_ρ^2 , respectively, then we get

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} (\|U^m\|_{\lambda,\rho}^2 + \|V^m\|_\rho^2) \\
&= -\sigma \|U^m\|_{\lambda,\rho}^2 + (cu^{c,m} z(\theta_t \omega), U^m)_{\lambda,\rho} \\
&\quad + \sigma(\alpha - \sigma) (U^m, V^m)_\rho + (\sigma - \alpha) \|V^m\|_\rho^2 \\
&\quad + (\mathbb{B}_m(t, \omega) U^m, V^m)_\rho \\
&\quad + ((2c\sigma z(\theta_t \omega) - c^2 z^2(\theta_t \omega)) u^{c,m}, V^m)_\rho \\
&\quad - (cz(\theta_t \omega) v^{c,m}, V^m)_\rho \\
&\quad + \beta (\sin u^{0,m} - \sin u^{c,m}, V^m)_\rho.
\end{aligned} \tag{65}$$

We now estimate the terms in (65):

$$\begin{aligned}
&(cz(\theta_t \omega) u^{c,m}, U^m)_{\lambda,\rho} \\
&\leq \frac{c^2 z^2(\theta_t \omega)}{2\sigma} \|u^{c,m}\|_{\lambda,\rho}^2 + \frac{\sigma}{2} \|U^m\|_{\lambda,\rho}^2, \\
&\sigma(\alpha - \sigma) (U^m, V^m)_\rho \\
&\leq \frac{\sigma(\alpha - \sigma)(\lambda + 1)}{2\lambda} (\|U^m\|_{\lambda,\rho}^2 + \|V^m\|_\rho^2), \\
&(\mathbb{B}_m(t, \omega) U^m, V^m)_\rho \\
&\leq \frac{\eta(\theta_t \omega)(1 + q + \tilde{q})(\lambda + 1)}{\lambda} (\|U^m\|_{\lambda,\rho}^2 + \|V^m\|_\rho^2), \\
&((2c\sigma z(\theta_t \omega) - c^2 z^2(\theta_t \omega)) u^{c,m}, V^m)_\rho \\
&\leq \frac{\alpha}{4} \|V^m\|_\rho^2 \\
&\quad + \frac{(2c\sigma z(\theta_t \omega) - c^2 z^2(\theta_t \omega))^2}{\alpha} \|u^{c,m}\|_\rho^2, \\
&(cz(\theta_t \omega) v^{c,m}, V^m)_\rho \\
&\leq \frac{\alpha}{4} \|V^m\|_\rho^2 + \frac{c^2 z^2(\theta_t \omega)}{\alpha} \|v^{c,m}\|_\rho^2, \\
&\beta (\sin u^{0,m} - \sin u^{c,m}, V^m)_\rho \\
&\leq \frac{\beta(\lambda + 1)}{2\lambda} (\|V^m\|_\rho^2 + \|U^m\|_{\lambda,\rho}^2).
\end{aligned} \tag{66}$$

Then it follows from (66) that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} (\|U^m\|_{\lambda,\rho}^2 + \|V^m\|_\rho^2) \leq \frac{\lambda + 1}{\lambda} \left(\frac{\sigma(\alpha - \sigma)}{2} \right. \\
&\quad + \eta(\theta_t \omega)(1 + q + \tilde{q}) + \frac{\beta}{2} \Big) (\|U^m\|_{\lambda,\rho}^2 + \|V^m\|_\rho^2) \\
&\quad + \frac{c^2 z^2(\theta_t \omega)}{2\sigma} \|u^{c,m}\|_{\lambda,\rho}^2 - \frac{\sigma}{2} \|U^m\|_{\lambda,\rho}^2 + \left(\sigma - \frac{\alpha}{2} \right) \\
&\quad \cdot \|V^m\|_\rho^2 + \frac{(2c\sigma z(\theta_t \omega) - c^2 z^2(\theta_t \omega))^2}{\alpha} \|u^{c,m}\|_\rho^2 \\
&\quad + \frac{c^2 z^2(\theta_t \omega)}{\alpha} \|v^{c,m}\|_\rho^2 \leq \left(-\lambda \right. \\
&\quad + \frac{\lambda + 1}{\lambda} \left(\frac{\sigma(\alpha - \sigma)}{2} + \eta(\theta_t \omega)(1 + q + \tilde{q}) + \frac{\beta}{2} \right) \Big) \\
&\quad \cdot (\|U^m\|_{\lambda,\rho}^2 + \|V^m\|_\rho^2) + \left(\frac{\lambda c^2 z^2(\theta_t \omega)}{2\sigma} \right. \\
&\quad \left. + \frac{(2c\sigma z(\theta_t \omega) - c^2 z^2(\theta_t \omega))^2}{\alpha} \right) \|u^{c,m}\|_\rho^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{c^2 z^2 (\theta_t \omega)}{\alpha} \|v^{c,m}\|_\rho^2 = \left(-\epsilon \right. \\
& + \frac{(\lambda + 1) \eta (\theta_t \omega) (1 + q + \tilde{q})}{\lambda} \left. \right) (\|U^m\|_{\lambda,\rho}^2 + \|V^m\|_\rho^2) \\
& + \mathcal{H}_2 (c, t, \omega) (\|u^{c,m}\|_\rho^2 + \|v^{c,m}\|_\rho^2),
\end{aligned} \tag{67}$$

where $\lambda = \min\{\alpha/2 - \sigma, \sigma/2\}$, ϵ is as in (10) and $\mathcal{H}_2(c, t, \omega) = \lambda c^2 z^2 (\theta_t \omega) / 2\sigma + (2c\sigma z (\theta_t \omega) - c^2 z^2 (\theta_t \omega))^2 / \alpha + c^2 z^2 (\theta_t \omega) / \alpha$. Applying Gronwall's inequality to (67) from τ to $t + \tau$, we have

$$\begin{aligned}
& \|U^m(t + \tau, \tau, \omega, u_\tau^c, v_\tau^c, u_\tau^0, v_\tau^0)\|_{\lambda,\rho}^2 + \|V^m(t \\
& + \tau, \tau, \omega, u_\tau^c, v_\tau^c, u_\tau^0, v_\tau^0)\|_\rho^2 \\
& \leq e^{2 \int_\tau^{t+\tau} (-\epsilon + (\lambda+1)\eta(\theta_r \omega)(1+q+\tilde{q})/\lambda) dr} \left(\|u_\tau^c - u_\tau^0\|_{\lambda,\rho}^2 \right. \\
& + \|v_\tau^c - v_\tau^0\|_\rho^2 \left. \right) \\
& + 2 \int_\tau^{t+\tau} e^{2 \int_s^{t+\tau} (-\epsilon + (\lambda+1)\eta(\theta_r \omega)(1+q+\tilde{q})/\lambda) dr} \mathcal{H}_2(c, s, \omega) \\
& \cdot (\|u^{c,m}(s, \tau, \omega, u_\tau^c, v_\tau^c)\|_\rho^2 \\
& + \|v^{c,m}(s, \tau, \omega, u_\tau^c, v_\tau^c)\|_\rho^2) ds.
\end{aligned} \tag{68}$$

We now replace ω in the above by $\theta_{-\tau}\omega$ to yield

$$\begin{aligned}
& \|U^m(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau^c, v_\tau^c, u_\tau^0, v_\tau^0)\|_{\lambda,\rho}^2 + \|V^m(t \\
& + \tau, \tau, \theta_{-\tau}\omega, u_\tau^c, v_\tau^c, u_\tau^0, v_\tau^0)\|_\rho^2 \\
& \leq e^{2 \int_0^t (-\epsilon + (\lambda+1)\eta(\theta_r \omega)(1+q+\tilde{q})/\lambda) dr} \left(\|u_\tau^c - u_\tau^0\|_{\lambda,\rho}^2 + \|v_\tau^c \right. \\
& - v_\tau^0\|_\rho^2 \left. \right) \\
& + 2 \int_0^t e^{2 \int_s^t (-\epsilon + (\lambda+1)\eta(\theta_r \omega)(1+q+\tilde{q})/\lambda) dr} \mathcal{H}_2(c, s, \omega) \\
& \cdot (\|u^{c,m}(s, \tau, \theta_{-\tau}\omega, u_\tau^c, v_\tau^c)\|_\rho^2 \\
& + \|v^{c,m}(s, \tau, \theta_{-\tau}\omega, u_\tau^c, v_\tau^c)\|_\rho^2) ds.
\end{aligned} \tag{69}$$

From (69), we see that, for $\tau \in \mathbb{R}$, $t \in \mathbb{R}^+$, $\omega \in \Omega$, $c \rightarrow 0$, and $(u_\tau^c, v_\tau^c), (u_\tau^0, v_\tau^0) \in l_\rho^2 \times l_\rho^2$ with $(u_\tau^c, v_\tau^c) \rightarrow (u_\tau^0, v_\tau^0)$,

$$\begin{aligned}
& \lim_{c \rightarrow 0} \varphi^{c,m}(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau^c, v_\tau^c) \\
& = \varphi^{0,m}(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau^0, v_\tau^0) \quad \text{in } l_\rho^2 \times l_\rho^2.
\end{aligned} \tag{70}$$

Let $\{c_n\} \subset [-1, 1]$ be a sequence of numbers with $c_n \rightarrow 0$ when $n \rightarrow +\infty$. Then $\varphi^c(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau^c, v_\tau^c)$ and $\varphi^0(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau^0, v_\tau^0)$ being limit functions of subsequences of

$\{\varphi^{c_n,m}(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau^{c_n}, v_\tau^{c_n})\}$ and $\{\varphi^{0,m}(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau^0, v_\tau^0)\}$ in $l_\rho^2 \times l_\rho^2$ imply that, for $\tau \in \mathbb{R}$, $t \in \mathbb{R}^+$, $\omega \in \Omega$, $c_n \rightarrow 0$, and $(u_\tau^{c_n}, v_\tau^{c_n}), (u_\tau, v_\tau) \in l_\rho^2 \times l_\rho^2$ with $(u_\tau^{c_n}, v_\tau^{c_n}) \rightarrow (u_\tau^0, v_\tau^0)$; the following holds:

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \varphi^{c_n}(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau^{c_n}, v_\tau^{c_n}) \\
& = \varphi^0(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau^0, v_\tau^0) \quad \text{in } l_\rho^2 \times l_\rho^2.
\end{aligned} \tag{71}$$

The proof is completed. $\square$

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

The authors would like to express their sincere thanks to the anonymous referees for their time and comments. This work is supported by the National Natural Science Foundation of China under Grant nos. 11326114, 11401244, and 11471290; Natural Science Research Project of Ordinary Universities in Jiangsu Province under Grant no. 14KJB110003; Zhejiang Normal University Foundation under Grant no. ZC304011068; and Zhejiang Natural Science Foundation under Grant no. LY14A010012.

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