Research Article

A Triparametric Family of Optimal Fourth-Order Multiple-Root Finders and Their Dynamics

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We investigate the complex dynamics of a triparametric family of optimal fourth-order multiple-root solvers by analyzing their basins of attraction along with extensive study of Möbius conjugacy maps and extraneous fixed points applied to a prototype quadratic polynomial raised to the power of the known integer multiplicity m . A 600×600 uniform grid centered at the origin covering 6×6 square region is chosen to display the initial points on each basin of attraction according to a coloring scheme based on their orbit behavior. With illustrative basins of attractions applied to various test polynomials and the corresponding statistical data for convergence as well as a number of comparisons made among the listed methods, we confirm our investigation and analysis developed in this paper.

1. Introduction

Many researchers [1–7] have shown their interest in the dynamics of iterative methods locating the multiple roots [8, 9] of a nonlinear equation. To ensure the convergence of an iterative method in a root-finding problem [10], it is very important to take a good initial value [11–17] close to the desired zero of the given nonlinear equation under consideration. In connection with such a choice of a good initial value, we pay a special attention to the complex dynamics for a number of optimal fourth-order multiple-root finders by investigating their basins of attraction.

Definition 1. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence converging to α and let $e_n = x_n - \alpha$ be the n th iterate error. If there exist real numbers p and a nonzero constant b such that the following error equation holds

$$e_{n+1} = be_n^p + O(e_n^{p+1}), \quad (1)$$

then b or $|b|$ is called the asymptotic error constant and p is called the order of convergence [18].

Definition 2. Let d be the number of distinct functional or derivative evaluations per iteration. The efficiency index [19] is defined by $EI = p^{1/d}$, where p is the order of convergence.

In our study, all the listed methods have the same $EI = 4^{1/3} \approx 1.587$ agreeing with Kung-Traub optimality [19] conjecture. We investigate the basins of attraction of a number of iterative methods with various test polynomials. Typical fourth-order multiple-root finders developed by Kanwar et al. [20], Soleymani and Babajee [21], and Shengguo et al. [6] are conveniently denoted by Kan, Sol, and Li for later use. Besides, by extending the work of Geum and Kim [22, 23], we propose a triparametric family of optimal fourth-order methods Yk 's whose developments will be described in Section 2. They are listed below in their respective order.

Kan:

$$y_n = x_n - \frac{2m}{m+2} \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{mf(x_n) \{m^2(3m-2)f'(y_n)^2 - C_1 f'(x_n) f'(y_n) + C_2 f'(y_n)^2\}}{2f'(x_n)((m-1)d^m f'(x_n) - mf'(x_n))(mf'(y_n))(mf'(y_n) - d^m(m+8)f'(x_n))},$$

$$d = \left(\frac{m}{m+2}\right)^m, \text{ where } C_1 = md(6m^2 + 17m - 14) \text{ with } C_2 = d^2(3m^3 + 19m^2 + 16m + 16). \quad (2)$$

Sol:

$$y_n = x_n - \frac{2m}{m+2} \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n + \frac{4md}{d(m^2 + 2m - 4) - m^2 v} \left[1 - \frac{m^3(m-2)}{16d^2} \left(v - \frac{m+2}{m} d \right)^2 \right] \cdot \frac{f(x_n)}{f'(x_n)}, \quad (3)$$

$$v = \frac{f'(y_n)}{f'(x_n)}, \quad d = \left(\frac{m}{m+2}\right)^m.$$

Li:

$$y_n = x_n - \frac{2m}{m+2} \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{(1/2)m(m-2)d^{-1}f'(y_n) - (m^2/2)f'(x_n)}{f'(x_n) - d^{-1}f'(y_n)} \cdot \frac{f(x_n)}{f'(x_n)}, \quad d = \left(\frac{m}{m+2}\right)^m. \quad (4)$$

Yk:

$$y_n = x_n - \frac{2m}{m+2} \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \left(\frac{a}{f'(x_n)} + \frac{b}{f'(y_n)} \right) f(x_n) - \left(\frac{c}{f'(x_n)} + \frac{d}{f'(y_n)} \right) F(y_n), \quad (5)$$

$$\text{where } F(y_n) = f(x_n) + (y_n - x_n) \frac{\lambda f'(x_n) f'(y_n)}{f'(x_n) + \rho f'(y_n)},$$

where a, b, c, d, λ , and ρ are parameters to be chosen for fourth order of optimal convergence [19, 24]. Typical cases of methods Yk's are presented in Table 1 for $1 \leq k \leq 6$ with selected parameters λ, ρ , and d .

2. Convergence Analysis

We describe the main theorem regarding the convergence behavior of proposed family of methods (5) and select

parameters a, b , and c for the quartic convergence with the aid of Taylor expansion and symbolic computation of Mathematica [25].

Theorem 3. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ have a zero α with integer multiplicity $m \geq 1$ and be analytic in a small neighborhood of α . Let x_0 be an initial guess chosen in a sufficiently small neighborhood of α . Let $\kappa = (m/(m+2))^m, \gamma = 2m/(m+2)$, and $\lambda, \rho, d \in \mathbb{R}$ be free constant parameters. Let $a = (1/8(m+2)\gamma\kappa\lambda)\tau - m(m+2)^4\kappa^3(\gamma\lambda - \rho)^2 + \kappa[8d(m+2)\gamma\lambda(\gamma\lambda - \rho) + m^3(-2(m^2 + 5m + 8)\gamma\lambda + 3(m+2)^2\rho)]$, $b = m(m+2)^3\kappa(1 + \kappa\rho)/8 - d$, $c = -(m^4 + \kappa(3m^3(m+2) - 8d\gamma\lambda)\rho + 3m^2(m+2)^2\kappa^2\rho^2 + m(m+2)^3\kappa^3\rho^3)/8\gamma\kappa\lambda$, and $\tau = m^4(m+2) - 3m^2(m+2)^3\kappa^2(\gamma\lambda - \rho)\rho$. Then iterative method (5) is of order four and defines a triparametric family of iterative methods with the following error equation:

$$e_{n+1} = \psi_4 e_n^4 + O(e_n^5), \quad n = 0, 1, 2, \dots, \quad (6)$$

where $\psi_4 = ((m(8+2m+6m^2+4m^3+m^4)+(-8+12m+14m^2+14m^3+6m^4+m^5)\kappa\rho)/3m^4(m+1)^3(m+2)[m+(m+2)\kappa\rho])\theta_1^3 - (1/m(m+1)^2(m+2))\theta_1\theta_2 + (m/(m+2)^3(m+1)(m+3))\theta_3$, $\theta_j = f^{(m+j)}(\alpha)/f^{(m)}(\alpha)$ for $j \in \mathbb{N}$, and $e_n = x_n - \alpha$.

Proof. Using Taylor's series about α , we have

$$f(x_n) = e^m \beta (1 + A_1 e + A_2 e^2 + A_3 e^3 + A_4 e^4 + O(e_n^5)), \quad (7)$$

$$f'(x_n) = e^{m-1} \beta m (1 + B_1 e + B_2 e^2 + B_3 e^3 + B_4 e^4 + O(e_n^5)), \quad (8)$$

where $\beta = f^{(m)}(\alpha)/m!$, $A_k = (m!/(m+k!))\theta_k$, $B_k = ((m-1)!/(m+k-1!))\theta_k$, and $\theta_k = f^{(m+k)}(\alpha)/f^{(m)}(\alpha)$ for $k \in \mathbb{N}$.

Dividing (7) by (8) gives us

$$\frac{f(x_n)}{f'(x_n)} = \frac{1}{m} [e_n - K_1 e_n^2 - K_2 e_n^3 + K_3 e_n^4 + O(e_n^5)], \quad (9)$$

where $K_1 = -A_1 + B_1$, $K_2 = -A_2 + A_1 B_1 - B_1^2 + B_2$, and $K_3 = -A_3 + A_2 B_1 - A_1 B_1^2 + B_1^3 + A_1 B_2 - 2B_1 B_2 + B_3$.

TABLE 1: Typical methods with selected parameters (λ, ρ) and constants (a, b, c, d) .

MT*	(λ, ρ)	(a, b, c, d)
Y1	$(-1/\kappa, 5)$	$a = \frac{m^3[m(m+2) + 5(m^2 + 2m - 4)\kappa]}{40\delta}, c = 0, d = -\frac{1}{80}(m+2)(m+5\delta)^3,$ $b = \frac{1}{80}(m+2)[m^3 + 5m(4+5m)\delta + 125m\delta^2 + 125\delta^3].$
Y2	$(1, 5)$	$a = \frac{m^3[m(m+2) + 5(m^2 + 2m - 4)\kappa]}{40\delta}, c = 0, d = \frac{m(m+5\delta)^3}{80\kappa},$ $b = \frac{(m+2)[-m^3 - 15m^2\delta - 65m\delta^2 - 25\delta^2(10+3m)\kappa]}{80\kappa}.$
Y3	$(-1, 1)$	$a = -\frac{m^3(m+2)^2 + (2+3m)^2\delta^3 + m^2(24+68m+38m^2+7m^3)\kappa + 5m(m+2)(2+3m)\delta^2}{16\delta},$ $b = 0, c = \frac{(m+2)[m(m^2+3m\delta+5\delta^2) + (2+3m)\delta^2\kappa]}{16\kappa}, d = \frac{1}{8}m(m+2)^2\delta(1+\kappa).$
Y4	$(-6, -2)$	$a = -\frac{m^3(m+2)^2 + 12m(m+2)(-2+5m)\delta^2 - 8(2-5m)^2\delta^3 + 6m^2(-8+20m+14m^2+3m^3)\kappa}{96\delta},$ $b = 0, c = \frac{(m+2)[m^3 - 6m^2\delta - 12m\delta^2 + 8\delta^2(-2+5m)\kappa]}{96\kappa}, d = \frac{1}{8}m(m+2)^2\delta(1+\kappa\delta).$
Y5	$(-1, 0)$	$a = \frac{m^3[-(m+2)^2 - 4(8+5m+m^2)\kappa]}{16\delta}, b = \frac{1}{8}m(m+2)^2\delta, c = \frac{m^3(m+2)}{16\kappa}, d = 0.$
Y6	$(1/\kappa, 1)$	$a = \frac{-m^3(28+16m+3m^2) - 3m^2(m+2)^2\delta + m(m+2)^2\delta^2 + (m+2)^2\delta^3}{16(m+2)},$ $b = \frac{1}{8}m(m+2)^2\delta(1+\kappa), c = -\frac{1}{16}(m+2)(m+\delta)^3, d = 0.$

* MT: method, $\delta = (m+2)\kappa, \kappa = (m/(m+2))^m$.Letting $t = 1 - \gamma/m$ with the above relation (9), we have

$$\begin{aligned}
y_n &= x_n - \gamma \frac{f(x_n)}{f'(x_n)} = \alpha + te + K_1(1-t)e_n^2 + K_2(1-t)e_n^3 + K_3(1-t)e_n^4 + O(e_n^5), \\
f'(y_n) &= mt^{m-1} + mt^m \left(B_1 + \frac{K_1(m-1+t-mt)}{t^2} \right) \\
&\cdot e + \frac{1}{2}mt^{m-3} \left(K_1^2(m-2)(m-1)(t-1)^2 \right. \\
&- 2B_1K_1m(t-1)t^2 + 2t(B_2t^3 + K_2(m-1+t-mt)) \Big) e_n^2 + \frac{1}{6}mt^{m-4} \left(-K_1^3(m-3)(m-2)(m-1)(t-1)^3 \right. \\
&+ 3B_1K_1^2(m-1)m(t-1)^2t^2 - 6K_1(t-1)t(-tK_2(m-2)(m-1)(t-1) \\
&+ B_2(m+1)t^3) + 6t^2(K_3(m-1+t-mt) \\
&+ t(-B_1K_2m(t-1) + B_3t^3)) \Big) e_n^3 + \frac{1}{24} \\
&\cdot mt^{m-5} \left(K_1^4(m-4)(m-3)(m-2)(m-1)(t-1)^4 \right. \\
&- 4B_1K_1^3(m-2)(m-1)m(t-1)^3t^2 \\
&+ 12K_1^2(t-1)^2t(-K_2(m-3)(m-2)(m-1)(t-1) \\
&- 1) + B_2m(m+1)t^3) - 24K_1(t-1)t^2(-K_3(m-2)(m-1)(t-1) + t(B_3(m+2)t^3 \\
&+ B_1K_2m(m-1+t-mt))) \Big) + 12t^2(K_2^2(m-2) \\
&\cdot (m-1)(t-1)^2 - 2B_2K_2(m+1)(t-1)t^3 \\
&+ 2t(K_4(m-1)(t-1) + t(-B_1K_3m(t-1) + B_4t^4))) \Big) e_n^4 + O(e_n^5).
\end{aligned}$$

$$\begin{aligned}
&- 1) + B_2m(m+1)t^3) - 24K_1(t-1)t^2(-K_3(m-2)(m-1)(t-1) + t(B_3(m+2)t^3 \\
&+ B_1K_2m(m-1+t-mt))) \Big) + 12t^2(K_2^2(m-2) \\
&\cdot (m-1)(t-1)^2 - 2B_2K_2(m+1)(t-1)t^3 \\
&+ 2t(K_4(m-1)(t-1) + t(-B_1K_3m(t-1) + B_4t^4))) \Big) e_n^4 + O(e_n^5).
\end{aligned}$$

Substituting (7)–(10) into (5), we get the error equation:

$$e_{n+1} = \psi_1 e_n + \psi_2 e_n^2 + \psi_3 e_n^3 + \psi_4 e_n^4 + O(e_n^5), \quad (11)$$

where $\psi_1 = t - (a(t + t^m \rho) + t^{-m}(bt(t + t^m \rho) + (dt + ct^m)(t + t^m(-\gamma\lambda + \rho))))/m(t + t^m \rho)$ and coefficients ψ_i ($i = 2, 3, 4$) depend on the parameters t, a, b, c, d, λ , and ρ and the function $f(x)$.

Solving $\psi_1 = 0$ and $\psi_2 = 0$ for a and b , we have

$$\begin{aligned}
a &= t(m - (b + d)t^{-m}) \\
&+ \frac{dt\gamma\lambda - c(t + t^m(-\gamma\lambda + \rho))}{t + t^m \rho},
\end{aligned} \quad (12)$$

$$b = -\frac{mt^m}{(t-1)(1+m(t-1)+t)} - \frac{dt^2 + 2dt^{1+m}\rho + t^{2m}(c\gamma\lambda + d\rho(-\gamma\lambda + \rho))}{(t+t^m\rho)^2}. \quad (13)$$

After substituting a and b into $\psi_3 = \psi_{31}\theta_1^2 + \psi_{32}\theta_2$, we solve $\psi_3 = 0$. Due to the fact that ψ_3 is independent of θ_1 and θ_2 , solving $\psi_{31} = \psi_{32} = 0$ for c and t , we have

$$t = \frac{m}{m+2},$$

$$c = -\frac{1}{2}t^{-m}(t+t^m\rho)^3\Omega_1 \left[2mt(2+t) + \frac{\Omega_2}{t} + m^2 \left(\frac{1}{t} - 3 + 4t^2 \right) - \frac{2dt^m\rho\Omega_1}{(t+t^m\rho)^3} \right], \quad (14)$$

where $\Omega_1 = (t-1)^2(1+m(t-1)+t)^3\gamma\lambda$ and $\Omega_2 = m^3(t-1)(1+2(t-1)t)$.

Applying $t = m/(m+2)$ into (12)–(14) with $\gamma = 2m/(m+2)$ and $\kappa = (m/(m+2))^m$, we have the following relations:

$$\begin{aligned} a &= \frac{1}{8(m+2)\gamma\kappa\lambda} m^4(m+2) - 3m^2(m+2)^3\kappa^2(\gamma\lambda - \rho)\rho - m(m+2)^4\kappa^3(\gamma\lambda - \rho)\rho^2 \\ &\quad + \kappa(8d(m+2)\gamma\lambda(\gamma\lambda - \rho) + m^3(-2(8+m(5+m))\gamma\lambda + 3(m+2)^2\rho)), \\ b &= \frac{1}{8}(-8d + m(m+2)^3\kappa(1+\kappa\rho)), \\ c &= -\frac{m^4 + \kappa(3m^3(m+2) - 8d\gamma\lambda)\rho + 3m^2(m+2)^2\kappa^2\rho^2 + m(m+2)^3\kappa^3\rho^3}{8\gamma\kappa\lambda}. \end{aligned} \quad (15)$$

Thanks to symbolic computation of Mathematica [25], we reach the error equation below:

$$e_{n+1} = \psi_4 e_n^4 + O(e_n^5), \quad (16)$$

where $\psi_4 = \psi_{41}\theta_1^3 + \psi_{42}\theta_1\theta_2 + \psi_{43}\theta_3$ with

$$\begin{aligned} \psi_{41} &= \frac{m(8+2m+6m^2+4m^3+m^4) + (-8+12m+14m^2+14m^3+6m^4+m^5)\kappa\rho}{3m^4(m+1)^3(m+2)[m+(m+2)\kappa\rho]}, \\ \psi_{42} &= -\frac{1}{m(m+1)^2(m+2)}, \\ \psi_{43} &= \frac{m}{(m+2)^3(m+1)(m+3)}, \end{aligned} \quad (17)$$

completing the proof. $\square$

Remark 4. If $\lambda = -1/\kappa\gamma$, $\rho = -1/\kappa$, and $d = 0$ are selected, then we find relations:

$$\begin{aligned} a &= -\frac{m^3}{2(m+2)}, \\ b &= 0, \\ c &= -m. \end{aligned} \quad (18)$$

In this case, proposed method (5) reduces to method Li given by (4).

3. Conjugacy Maps and Dynamics

Multipoint iterative methods [19] solving a nonlinear equation of the form $f(x) = 0$ can be generally written as a discrete dynamical system

$$x_{n+1} = R_f(x_n), \quad (19)$$

where R_f is the iteration function. We begin by writing (5) in the form of a complex discrete dynamical system:

$$z_{n+1} = R_f(z_n) = z_n - \frac{f(z_n)}{f'(z_n)} H_f(z_n), \quad (20)$$

where $H_f(z_n) = \gamma + (a+c\sigma) + (b+d\sigma)(v/s)$, $\gamma = 2m/(m+2)$, $\sigma = 1 - \gamma\lambda s/(v+\rho s)$, $s = f(z_n)/f'(z_n)$, and $v = f(z_n)/f'(y_n)$; $a, b,$

and c are given, respectively, by (12), (13), and (14); $\lambda, \rho, d \in \mathbb{C}$ are free parameters.

Definition 5. Let $F : X \rightarrow X$ and $G : Y \rightarrow Y$ be two functions (dynamical systems). One says that F and G are conjugate if there is a function $h : X \rightarrow Y$ such that $h \circ F = G \circ h$. Then the map h is called a conjugacy [26].

Remark 6. Note that a conjugacy indeed preserves the dynamical behavior between the two dynamical systems; for example, if F is conjugate to G via h and ξ is a fixed point of F , then $h(\xi)$ is a fixed point of G .

Furthermore, if h is a homeomorphism, that is, if F is topologically conjugate to G via h , and ζ is a fixed point of G , then $h^{-1}(\zeta)$ is a fixed point of F . Also, we find $G = h \circ F \circ h^{-1}$ and $G^n = (h \circ F \circ h^{-1}) \circ (h \circ F \circ h^{-1}) \circ \dots \circ (h \circ F \circ h^{-1}) = h \circ F^n \circ h^{-1}$. If F and G are invertible, then the topological conjugacy h maps an orbit

$$\dots, F^{-2}(x), F^{-1}(x), x, F(x), F^2(x), \dots, \quad (21)$$

of F , onto an orbit

$$\dots, G^{-2}(y), G^{-1}(y), y, G(y), G^2(y), \dots, \quad (22)$$

of G , where $y = h(x)$ and the order of points is preserved. Hence, the orbits of the two maps behave similarly under homeomorphism h or h^{-1} .

Via Möbius conjugacy map $M(z) = (z-A)/(z-B)$, $z, A \neq B, A, B \in \mathbb{C} \cup \{\infty\}$, considered by Blanchard [27], R_f in (20) is conjugated to J satisfying

$$J(z; A, B, \lambda, \rho, d) = \frac{H(z; A, B, \lambda, \rho, d)}{D(z; A, B, \lambda, \rho, d)}, \quad (23)$$

when applied to a quadratic polynomial $f(z) = [(z-A)(z-B)]^m$ raised to the power of m , where H and D are polynomials with no common factors whose coefficients are generally dependent upon parameters A, B, λ, ρ , and d . The following theorem favorably indicates that J is dependent only on ρ but independent of parameters A, B, λ , and d .

Theorem 7. Let $f(z) = [(z-A)(z-B)]^m$ with $m \in \mathbb{N}$ and $M(z) = (z-A)/(z-B)$, $A \neq B, A, B \in \mathbb{C} \cup \{\infty\}$. Then $R_f(z; \lambda, \rho, d)$ is conjugate to $J(z; \rho)$ satisfying

$$J(z; \rho) = z \cdot \frac{r_1\mu + (1 + \kappa\rho)\tau - r_2\omega}{r_3\mu + z(1 + \kappa\rho)\tau - r_4\omega}, \quad (24)$$

where $r_1 = m^3 + (m^3 - 4m - 8z)\kappa\rho$, $r_2 = 4(z - \kappa\rho) + m[2 + m(m+3)(1 + \kappa\rho)]$, $r_3 = -8\kappa\rho + z[m^3 + (m-2)m(m+2)\kappa\rho]$, $r_4 = 4 + (m+2)z[m(m+1) + (m-1)(m+2)\kappa\rho]$, $\mu = \beta_1^{2m-2}\beta_2^{2m-2}\beta^2$, $\tau = m^{4m-2}(m+2)^3(z+1)^{4m}\kappa^2$, $\omega = 2m^{2m-1}(z+1)^{2m}\beta_1^{m-1}\beta_2^{m-1}\beta\kappa$, $\beta_1 = mz + m - \gamma$, $\beta_2 = m + z(m - \gamma)$, $\beta = m(z+1)^2 - 2z\gamma$, $\kappa = (m/(m+2))^m$, and $\gamma = 2m/(m+2)$.

Proof. Since the inverse of $M(z)$ is easily found to be $M^{-1}(z) = (Bz-A)/(z-1)$, we find after a lengthy computation with the aid of Mathematica [25] symbolic capability:

$$J(z; \rho) = M \circ R_f \circ M^{-1}(z) = z \cdot \frac{\mathcal{H}(z; \rho)}{\mathcal{D}(z; \rho)}, \quad (25)$$

where $\mathcal{H}(z; \rho) = r_1\mu + (1 + \kappa\rho)\tau - r_2\omega$ and $\mathcal{D}(z; \rho) = r_3\mu + z(1 + \kappa\rho)\tau - r_4\omega$ are polynomials of degree at most $4m+1$ in z with a single free parameter $\rho \in \mathbb{C}$. This gives the desired result, completing the proof. $\square$

The result of Theorem 7 enables us to discover that $z = 0$ (corresponding to fixed point A of R_f or root A of $f(z) = [(z-A)(z-B)]^m$) and $z = \infty$ (corresponding to fixed point B of R_f or root B of $f(z)$) are clearly two of their fixed points of the conjugate map $J(z; \rho)$, regardless of ρ -values. Besides, by direct computation, we find that $z = 1$ is a strange fixed point [28–30] of J (that is not a root of $f(z) = [(z-A)(z-B)]^m$) due to the fact that $J(1; \rho) = 1$, regardless of ρ -values.

We now seek further strange fixed points including $z = 1$ (corresponding to the original convergence to infinity in view of the fact that $M^{-1}(1) = \infty$ or $M(\infty) = 1$). To do so, we first investigate some properties of $J(z; \rho) = z \cdot (\mathcal{H}(z; \rho)/\mathcal{D}(z; \rho))$ stated in the following theorem.

Theorem 8. Let $\mathcal{H}(z; \rho)$ and $\mathcal{D}(z; \rho)$ be given by (25). Then the following hold:

- The leading highest-order term of $\mathcal{H}(z; \rho)$ is given by $-(8m^{7m-4}/(m+2)^{3m-2})[((m+2)/m)^{m-1} + \rho]z^{4m+1}$, provided that $\rho \neq -((m+2)/m)^{m-1}$.
- $\mathcal{H}(z; \rho)$ has a factor z^3 , provided that $\rho \neq -(3m+4)(5m^2+4)\gamma/2(15m^3+20m^2+12m-8)\kappa$.
- $\mathcal{H}(1; \rho) = \mathcal{D}(1; \rho)$, and $\mathcal{H}(1; \rho)/\mathcal{D}(1; \rho) = 1$, provided that $\rho \neq -(m^3(2m-\gamma)^{4m-2} + \tau_1 - (m^3 + 3m^2 + 2m + 4)\tau_2)/\kappa[(m^3 - 4m - 8)(2m - \gamma)^{4m-2} + \tau_1 - (m-1)(m+2)^2\tau_2]$, with $\tau_1 = 4^{2m-1}m^{4m-2}(m+2)^3\kappa^2$ and $\tau_2 = 4^m m^{2m-1}(2m - \gamma)^{2m-1}\kappa$.
- $J(z; \rho)$ approaches ∞ as z tends to ∞ , provided that $\rho \neq -((m+2)/m)^{m-1}$.

Proof. After a lengthy computation and careful algebraic treatments with the aid of Mathematica, (a) and (c) follow without difficulty. For the proof of (b), we directly compute the values of $\mathcal{H}(0; \rho) = \mathcal{H}'(0; \rho) = \mathcal{H}''(0; \rho) = 0$ and $\mathcal{H}'''(0; \rho) \neq 0$. The proof of (d) follows from the fact that $J(\infty; \rho) = \infty$, by using (a) along with a highest-order term of $\mathcal{D}(z; \rho)$ having degree at most $4m+1$. $\square$

We now will begin with locating the fixed points of the iteration function $J(z; \rho)$. Let $\phi(z; \rho) = z - J(z; \rho)$, whose zeros are the desired fixed points of J . The result of Theorem 8 shows that $z = 0$ and $z = 1$ are the roots of ϕ . Hence the expression of $\phi(z; \rho)$ will take the following form:

$$\phi(z; \rho) = z(z-1) \cdot \frac{\Psi(z; \rho)}{T(z; \rho)}, \quad (26)$$

where $\Psi(z; \rho) = m^7 q(z+1)^4 + m^2 r_1 \beta_1^{2m} \beta_2^{2m} \beta^2 - 2m^6 q(z+1)^4 (\gamma - 3) + 8qz^2 \gamma^4 - 2m^{2m+1} (z+1)^{2m} \beta_1^{m+1} \beta_2^{m+1} \beta \kappa \nu_0 - 4mqz\gamma^3 \nu_1 + 2m^2 q\gamma^2 \nu_2 + m^5 q(z+1)^2 \nu_3 + 2m^4 q(z+1)^2 \nu_4 + m^3 q\gamma \nu_5$ and $T(z; \rho) = m^7 qz(z+1)^4 - 2m^6 qz(z+1)^4 (\gamma - 3) + 8qz^3 \gamma^4 - 2m^{2m+1} r_4 (z+1)^{2m} \beta_1^{m+1} \beta_2^{m+1} \beta \kappa - 4mqz^2 \gamma^3 \nu_1 + m^2 (r_3 \beta_1^{2m} \beta_2^{2m} \beta^2 + 2qz\gamma^2 \nu_2) + m^5 qz(z+1)^2 \nu_3 + 2m^4 qz(z+1)^2 \nu_4 + m^3 qz\gamma \nu_5$ are polynomials in z with $q = (z+1)^{4m} m^{4m} \kappa^2 (1 + \kappa \rho)$, $\nu_0 = -4(1 + \kappa \rho) + m(2 + m(m+3)(1 + \kappa \rho))$, $\nu_1 = 4(z+1)^2 - 3z\gamma$, $\nu_2 = 4(z+1)^2 (1 + 4z + z^2) - 12z(z+1)^2 \gamma + 3z^2 \gamma^2$, $\nu_3 = 12(z+1)^2 - 12(z+1)^2 \gamma + (1 + 4z + z^2) \gamma^2$, $\nu_4 = 4(z+1)^2 - 12(z+1)^2 \gamma + 3(1 + 4z + z^2) \gamma^2 - z\gamma^3$, and $\nu_5 = -16(z+1)^4 + 12(z+1)^2 (1 + 4z + z^2) \gamma - 12z(z+1)^2 \gamma^2 + z^2 \gamma^3$ and with $\beta_1, \beta_2, \beta, r_1, r_3$, and r_4 given in Theorem 7.

As a result, $z = 0$, $z = 1$, and $z = \infty$ are clearly the fixed points of J . Among these fixed points, $z = 1$ is a strange fixed point that is not the root A or B . Further strange fixed points are possible from the roots of $\Psi(z; \rho)$. The following theorem describes some properties of $\phi(z; \rho)$.

Theorem 9. Let $\phi(z; \rho)$ be given by (26). Then the following hold:

- (a) $\Psi(1/z; \rho) = (1/z^{4(m+1)})\Psi(z; \rho)$ for $z \neq 0$, regardless of ρ -values.
- (b) $\Psi(z; \rho)$ has double roots at $z = -(m+2)/m$ and $z = -m/(m+2)$; that is, it has a factor $(z + (m+2)/m)^2 (z + m/(m+2))^2$, provided that $\rho \neq -1/\kappa$.
- (c) $T(1/z; \rho) = (1/z^{4m+5})\tilde{T}(z; \rho)$ for $z \neq 0$, regardless of ρ -values, where $\tilde{T}(z; \rho) = (T(z; \rho) - \delta_0)/z$, $\delta_0 = 8m^2(z^2 - 1)\beta_1^{m+1}\beta_2^{m+1}\beta\kappa(m^{2m-1}(z+1)^{2m} + \beta_1^{m-1}\beta_2^{m-1}\beta\rho)$.
- (d) $T(z; \rho)$ has also double roots at $z = -(m+2)/m$ and $z = -m/(m+2)$; that is, it has a common factor $(z + (m+2)/m)^2 (z + m/(m+2))^2$ as shown in $\Psi(z; \rho)$, provided that $\rho = -1/\kappa$.
- (e) $\phi(1/z; \rho) = -(z-1)/z$, and $(\Psi(z; \rho)/\tilde{T}(z; \rho))$, for $z \neq 0$, provided that $\rho = -1/\kappa$.

Proof. Via careful algebraic treatments and symbolic computation with the aid of Mathematica, (a), (c), and (e) follow without difficulty. For the proof of (b), we directly compute the values of $\Psi(-(m+2)/m; \rho) = \Psi'(-(m+2)/m; \rho) = 0$ and $\Psi''(-(m+2)/m; \rho) = 2^{4(m+1)} m^3 \gamma \kappa^2 (m+1)^2 (1 + \kappa \rho) \neq 0$, for $\rho \neq -1/\kappa$. In view of the relations, $\Psi(z) = z^{4m+4} \Psi(1/z)$, $\Psi'(z) = z^{4m+2} [4(m+1)z\Psi(1/z) - \Psi'(1/z)]$, and $\Psi''(z) = z^{4m} [4(4m^2 + 7m + 3)z^2\Psi(1/z) - 2(4m+3)z\Psi'(1/z) + \Psi''(1/z)]$, we also find $\Psi(-m/(m+2); \rho) = \Psi'(-m/(m+2); \rho) = 0$ and $\Psi''(-m/(m+2); \rho) = 2^{4(m+1)} m^3 \gamma \kappa^2 (m+1)^2 (1 + \kappa \rho) \neq 0$. The proof of (d) follows from the fact that $T(-(m+2)/m; \rho) = T'(-(m+2)/m; \rho) = 0$ and $T''(-(m+2)/m; \rho) = \infty$, for any ρ . We also find $T(-m/(m+2); \rho) = T'(-m/(m+2); \rho) = 0$ and $T''(-m/(m+2); \rho) = -2^{4m+5} m^3 (m+1)^2 \kappa^2 (1 + \kappa \rho) \neq 0$, for $\rho \neq -1/\kappa$. We also find $T(-m/(m+2); \rho) = T'(-m/(m+2); \rho) = 0$ and $T''(-m/(m+2); \rho) = -2^{4m+3} m^3 (m+1)^2 \gamma^2 \kappa^6 (1 + \kappa \rho) \neq 0$, for $\rho \neq -1/\kappa$. $\square$

With the use of properties of $\phi(z; \rho)$, we now consider some strange fixed points along with their stability for selected values of $m = 1$ and $m = 2$.

To continue our investigation of dynamics behind iterative map (20) applied to a quadratic polynomial raised to the power of m , $f(z) = p(z) = [(z-A)(z-B)]^m$, we will describe the fixed points of J in (25) and their stability. In view of the fact that $M(z)$ is a fixed point of J for a fixed point z of R_p with $M^{-1}(z) = (zB - A)/(z - 1)$, we are interested in the explicit form of $\phi(z; \rho) = z - J(z; \rho)$ for $m \in \{1, 2\}$ below:

$$\phi(z; \rho) = \begin{cases} -\frac{z(z-1)\Psi_1(z)}{q_1(z)}, & \text{if } m = 1, \\ -\frac{z(z-1)\Psi_2(z)}{q_2(z)}, & \text{if } m = 2, \end{cases} \quad (27)$$

where we conveniently denote

$$\begin{aligned} \Psi_1(z) &= 9(1 + \rho) + 9z^4(1 + \rho) + 3z(11 + 7\rho) \\ &\quad + 3z^3(11 + 7\rho) + z^2(54 + 34\rho), \\ q_1(z) &= 9(1 + \rho) + 12z(2 + \rho) + z^2(21 + 13\rho), \\ \Psi_2(z) &= (4(2 + \rho) + 4z^8(2 + \rho) + 4z(17 + 8\rho) \\ &\quad + 4z^7(17 + 8\rho) + 3z^4(236 + 97\rho) \\ &\quad + z^2(256 + 113\rho) + z^6(256 + 113\rho) \\ &\quad + z^3(548 + 230\rho) + z^5(548 + 230\rho)), \\ q_2(z) &= 4(2 + \rho) + 10z^5(12 + 5\rho) + 4z(15 + 7\rho) \\ &\quad + z^6(20 + 9\rho) + 6z^3(52 + 21\rho) + z^2(188 + 81\rho) \\ &\quad + z^4(280 + 111\rho). \end{aligned} \quad (28)$$

This enables us to discover that $z = 0$ (corresponding to fixed point a of R_p or root A of $p(z)$) and $z = \infty$ (corresponding to fixed point B of R_p or root B of $p(z)$) are clearly two of their fixed points regardless of m . To find further strange fixed points, we solve equations $\phi(z; \rho) = 0$ in (27) for z with typical values of $m \in \{1, 2\}$.

We now investigate further strange fixed points including $z = 1$ (corresponding to the original convergence to infinity in view of the fact that $M^{-1}(1) = \infty$ or $M(\infty) = 1$). By direct computation, we will describe the roots of $\phi(z; \rho) = 0$ for $m \in \{1, 2\}$. To this end, we first check the existence of ρ -values for common factors (divisors) of $\Psi_i(z)$ and $q_i(z)$. Besides, $q_i(z)$ will be checked if it has a divisor $(z - 1)$ or z . The following theorem best describes relevant properties of such existence as well as explicit strange fixed points.

Theorem 10. Let $m = 1$ in (27). Then the following hold:

- (a) If $\rho = -3$, then $\phi(z; \rho) = -(z-1)z(z^2 + z + 1)$ and the strange fixed points z are given by $z = 1$ and $z = (-1 \pm \sqrt{3})/2$.

- (b) If $\rho = -1$, then $\phi(z; \rho) = -(z-1)z(3z^2+5z+3)/(2z+3)$ and the strange fixed points are given by $z = 1$ and $z = (-5 \pm i\sqrt{11})/6$.
- (c) If $\rho = -3/5$, then $\phi(z; \rho) = -(z-1)z(z+1)(3z^2+11z+3)/(11z+3)$ and the strange fixed points are given by $z = \pm 1$, $z = (-11 \pm \sqrt{85})/6$.
- (d) If $\rho = -27/17$, then $\phi(z; \rho) = -z(15z^4 + z^3 + z + 15)/(z+15)$ and the strange fixed points are given by $z = (1/60)(-1 - \sqrt{1801} \pm i\sqrt{1798 - 2\sqrt{1801}})$ and $z = (1/60)(-1 + \sqrt{1801} \pm i\sqrt{1798 + 2\sqrt{1801}})$.
- (e) Let $\rho \notin \{-3, -1, -3/5, -27/17\}$. Then $\Psi_1(1/z) = z^{-4}\Psi_1(z)$ holds for $z \neq 0$. Hence, if $z \neq 0$ is a root of $\Psi_1(z; \rho)$, then so is $1/z$.

Proof. (a)–(c) Suppose that $\Psi_1(z) = 0$ and $q_1(z) = 0$ for some values of z . Observe that parameter ρ exists in a linear fashion in all coefficients of both polynomials. By eliminating ρ from the two polynomials, we obtain the relation $G(z) = z(1+z)(3+2z+3z^2) = 0$. Hence, any root of G is a candidate for a common divisor of $\Psi_1(z)$ and $q_1(z)$. Substituting all the roots of G into $\Psi_1(z) = 0$ and $q_1(z) = 0$, we find required relations for ρ and solving them for ρ , we find $\rho = -3, -1$. The remaining part of the proof is straightforward. (d) If z is a divisor of $q_1(z)$, then $q_1(0) = 9(1+\rho) = 0$ yielding $\rho = -1$, which is already handled in (b). If $(z-1)$ is a divisor of $q_1(z)$, then $q_1(1) = 2(27+17\rho) = 0$, yielding $\rho = -27/17$. Then remaining proof is trivial. (e) By direct substitution, we find $\Psi_1(1/z) = z^{-4}\Psi_1(z)$ without difficulty. Hence $\Psi_1(1/z) = 0$ if and only if $\Psi_1(z) = 0$ for $z \neq 0$. This completes the proof. $\square$

Theorem 11. Let $m = 2$ in (27). Then the following hold:

- (a) If $\rho = -2$, then $\phi(z; \rho) = -(z-1)z(2z^4+9z^3+15z^2+9z+2)/(z^3+7z^2+7z+2)$ and the strange fixed points are given by $z = 1$, $z = -1.80198 \pm 0.880308i$, and $z = -0.448023 \pm 0.21887i$.
- (b) If $\rho = -4$, then $\phi(z; \rho) = -(z-1)z(1+z)^3/(2z+1)$ and the strange fixed points are given by $z = 1$ and $z = -1$ (triple).
- (c) If $\rho = -988/409$, then $\phi(z; \rho) = -z(170z^8 + 951z^7 + 1735z^6 + 777z^5 - 516z^4 + 777z^3 + 1735z^2 + 951z + 170)/(178z^5 + 258z^4 - 955z^3 - 1735z^2 - 951z - 170)$ and the strange fixed points are given by $z = -2.35462, -0.424696, -1.68698 \pm 0.790067i, -0.486146 \pm 0.227678i, 0.765729 \pm 0.643163i$.
- (d) Let $\rho \notin \{-2, -4, -988/409\}$. Then $\Psi_2(1/z) = z^{-8}\Psi_2(z)$ holds for $z \neq 0$. Hence, if $z \neq 0$ is a root of $\Psi_2(z; \rho)$, then so is $1/z$.

Proof. The proofs immediately follow from the same argument as used in the proofs of Theorem 10. $\square$

As a result of Theorem 9 (a), we find the fixed points of $J(z; \rho)$, that is, the roots of $\phi(z; \rho)$ explicitly as stated in the following corollary.

Corollary 12. Let $z \notin \{0, 1\}$ be a root of $\phi(z; \rho)$, that is, a root of $\Psi_i(z; \rho)$ for $1 \leq i \leq 2$ in (27). Suppose that $\Psi(z; \rho)$ and $q_i(z)$ have no common factors for some suitable ρ -values. Then the roots of $\phi(z; \rho)$ for $1 \leq i \leq 2$ are explicitly given by the following:

- (a) The four roots of $\Psi_1(z; \rho)$ are explicitly found to be

$$\begin{aligned} z_1^{(1)} &= -\frac{11+t_1+7\rho+i\sqrt{2}\sqrt{-11t_1+t_2-7t_1\rho}}{12(1+\rho)}, \\ z_2^{(1)} &= \frac{1}{z_1^{(1)}}, \\ z_3^{(1)} &= \frac{-11+t_1-7\rho-i\sqrt{2}\sqrt{11t_1+t_2+7t_1\rho}}{12(1+\rho)}, \\ z_4^{(1)} &= \frac{1}{z_3^{(1)}}, \end{aligned} \quad (29)$$

where $t_1 = \sqrt{-23-3\rho(18+5\rho)}$ and $t_2 = 23+\rho(94+55\rho)$.

- (b) The eight roots of $\Psi_2(z; \rho)$ are explicitly found to be

$$\begin{aligned} z_1^{(2)} &= \frac{1}{2} \left(-s_1 - \sqrt{-4+s_1^2} \right), \\ z_2^{(2)} &= \frac{1}{z_1^{(2)}}, \\ z_3^{(2)} &= \frac{1}{2} \left(-s_2 - \sqrt{-4+s_2^2} \right), \\ z_4^{(2)} &= \frac{1}{z_3^{(2)}}, \\ z_5^{(2)} &= \frac{1}{2} \left(-u_1 - \sqrt{-4+u_1^2} \right), \\ z_6^{(2)} &= \frac{1}{z_5^{(2)}}, \\ z_7^{(2)} &= \frac{1}{2} \left(-u_2 - \sqrt{-4+u_2^2} \right), \\ z_8^{(2)} &= \frac{1}{z_7^{(2)}}, \end{aligned} \quad (30)$$

where $s_1 = (1/2)(c_1 - \sqrt{8+c_1^2-4c_2})$, $s_2 = (1/2)(c_1 + \sqrt{8+c_1^2-4c_2})$, $u_1 = (1/2)(d_1 - \sqrt{8+d_1^2-4d_2})$, $u_2 = (1/2)(d_1 + \sqrt{8+d_1^2-4d_2})$, $c_1 = (17+8\rho - \sqrt{-7-6\rho-C^2})/2(2+\rho)$, $c_2 = 3(9+4\rho - \sqrt{-7-6\rho-\rho^2})/2(2+\rho)$, $d_1 = (17+8\rho + \sqrt{-7-6\rho-\rho^2})/2(2+\rho)$, and $d_2 = 3(9+4\rho + \sqrt{-7-6\rho-\rho^2})/2(2+\rho)$.

Proof. Since z is a root of $\Psi_i(z; \rho)$ for $1 \leq i \leq 2$, so is $1/z$ from the result of Theorem 9 (a). For the proof of (a), $\Psi_1(z; \rho)$ can be written as a product of two factors:

$$\Psi_1(z; \rho) = (z^2 + cz + 1)(z^2 + dz + 1). \quad (31)$$

By expanding the right side of the above equation and comparing the coefficients of the first- and second-order terms, we find two relations:

$$\begin{aligned} c + d &= \frac{11 + 7\rho}{3(1 + \rho)}, \\ 2 + cd &= \frac{54 + 34\rho}{9(1 + \rho)}, \end{aligned} \quad (32)$$

which gives the desired values of c and d . Then the four roots can be found explicitly from $(z^2 + cz + 1) = 0$ or $(z^2 + dz + 1) = 0$. Similarly for the proof of (b), $\Psi_2(z; \rho)$ can be written as a product of two factors each of which is further decomposed into two factors:

$$\begin{aligned} \Psi_2(z; \rho) &= (1 + c_1z + c_2z^2 + c_1z^3 + z^4) \\ &\cdot (1 + d_1z + d_2z^2 + d_1z^3 + z^4) = (z^2 + s_1z + 1) \\ &\cdot (z^2 + s_2z + 1)(z^2 + u_1z + 1)(z^2 + u_2z + 1). \end{aligned} \quad (33)$$

By the same argument as used in the proof of (a), the desired result follows. This completes the proof. $\square$

We are now ready to determine the stability of the fixed points. To do so, it is necessary to compute the derivative of J from (24):

$$J'(z; \rho) = \begin{cases} \frac{4z^3 Q_1(z)}{w_1(z)^2}, & \text{if } m = 1, \\ \frac{2z^3 Q_2(z)}{w_2(z)^2}, & \text{if } m = 2, \end{cases} \quad (34)$$

where

$$\begin{aligned} Q_1(z) &= 36z(18 + 23\rho + 7\rho^2) \\ &+ 36z^3(18 + 23\rho + 7\rho^2) \\ &+ 9(21 + 34\rho + 13\rho^2) \\ &+ 9z^4(21 + 34\rho + 13\rho^2) \\ &+ 2z^2(459 + 546\rho + 175\rho^2), \\ w_1(z) &= 9(1 + \rho) + 12z(2 + \rho) + z^2(21 + 13\rho), \\ Q_2(z) &= 8(40 + 38\rho + 9\rho^2) + 8z^{12}(40 + 38\rho + 9\rho^2) \\ &+ z(4200 + 3850\rho + 878\rho^2) \\ &+ z^{11}(4200 + 3850\rho + 878\rho^2) \end{aligned}$$

$$\begin{aligned} &+ z^2(24880 + 22056\rho + 4861\rho^2) \\ &+ z^{10}(24880 + 22056\rho + 4861\rho^2) \\ &+ 2z^3(43848 + 37745\rho + 8088\rho^2) \\ &+ 2z^9(43848 + 37745\rho + 8088\rho^2) \\ &+ 6z^5(55820 + 46235\rho + 9581\rho^2) \\ &+ 6z^7(55820 + 46235\rho + 9581\rho^2) \\ &+ 2z^4(102448 + 86152\rho + 18081\rho^2) \\ &+ 2z^8(102448 + 86152\rho + 18081\rho^2) \\ &+ z^6(393440 + 324176\rho + 66891\rho^2), \\ w_2(z) &= 4(2 + \rho) + 10z^5(12 + 5\rho) + 4z(15 + 7\rho) \\ &+ z^6(20 + 9\rho) + 6z^3(52 + 21\rho) \\ &+ z^2(188 + 81\rho) + z^4(280 + 111\rho). \end{aligned} \quad (35)$$

We first check the existence of ρ -values for common factors (divisors) of $Q_i(z)$ and $w_i(z)$. Besides, $w_i(z)$ will be checked if it has divisors z , z^2 , z^3 . The following theorem best describes relevant properties of such existence as well as explicit strange fixed points.

Theorem 13. *Let $m = 1$ in (34). Then the following hold:*

- (a) *If $\rho = -3$, then $J'(z; \rho) = 4z^3$.*
- (b) *If $\rho = -1$, then $J'(z; \rho) = 2z^2(9z^2 + 22z + 9)/(2z + 3)^2$.*
- (c) *If $\rho = -3/5$, then $J'(z; \rho) = 12z^3(11z^2 + 34z + 11)/(11z + 3)^2$.*
- (d) *If $\rho = -1/3$, then $J'(z; \rho) = 4z^3(3z + 5)(15z^2 + 26z + 15)/(5z + 3)^3$.*
- (e) *If $\rho = -5/3$, then $J'(z; \rho) = 4z^3(3z - 1)(3z^2 - 14z + 3)/(z - 3)^3$.*
- (f) *If $\rho = -27/17$, then $J'(z; \rho) = -12z^3(5 + 94z + 5z^2)/(z + 15)^2$.*
- (g) *Let $\rho \notin \{-3, -1, -3/5, -1/3, -5/3, -27/17\}$. Let z be a fixed point of $J(z; \rho)$ satisfying $\phi(z; \rho) = 0$. Then $J'(z; \rho) = J'(1/z; \rho)$ holds for $z \neq 0$.*

Proof. The proofs of (a)–(f) immediately follow from the same argument as used in the proofs of Theorem 10. Eliminating ρ from the two polynomials $Q_i(z)$ and $w_i(z)$ plays a key role in obtaining the relation $G(z) = (-3 + z)(-1 + z)z(1 + z)(3 + 5z)(3 + 2z + 3z^2) = 0$, whose roots enable us to deduce some desired ρ -values. Additional requirement that z, z^2, z^3 are candidates for common divisors of $Q_i(z)$ and $w_i(z)$ gives only $\rho = -1$. For the proof of (g), via direct computation with the aid of Mathematica symbolic capability, we find $J'(z; \rho) - J'(1/z; \rho) = -(4Q_1(z)(z + 1)\Delta_1/q_1(z)q_1(1/z)^2z^8)\phi(z; \rho) = 0$,

TABLE 2: Stability check from $|J'(\zeta; \rho)|$ of strange fixed points ζ for special ρ -values with $1 \leq m \leq 2$.

m	ρ	ζ $ J'(\zeta; \rho) : \mathbf{t}^*$			Number of ζ
1	$-3/5$	-1	1	$(-11 \pm \sqrt{85})/6$	4
		$9/4: \mathbf{r}$	$24/7: \mathbf{r}$	$76/9: \mathbf{r}$	
	-3	1	$\pm(1 + \sqrt{3})/2$		3
		$4: \mathbf{r}$	$4: \mathbf{r}$		
	-1	1	$(-5 \pm i\sqrt{11})/6$		3
		$16/5: \mathbf{r}$	$14/3: \mathbf{r}$		
	$-1/3$	1	$-1.70469 \pm 0.885458i$	$-0.461974 \pm 0.239960i$	5
		$7/2: \mathbf{r}$	$4.76095: \mathbf{r}$	$4.76095: \mathbf{r}$	
	$-5/3$	1	$-0.712486 \pm 0.701686i$	$0.545819 \pm 0.837902i$	5
		$8: \mathbf{r}$	$5.12085: \mathbf{r}$	$6.37915: \mathbf{r}$	
2	$-27/17$	$-0.723969 \pm 0.689831i$	$0.690636 \pm 0.723202i$		4
		$5.09653: \mathbf{r}$	$4.90792: \mathbf{r}$		
	-2	1	$-1.80197 \pm 0.880307i$	$-0.448023 \pm 0.218869i$	5
		$54/17: \mathbf{r}$	$6.49038: \mathbf{r}$	$6.49038: \mathbf{r}$	
	-4	-1 (triple)	1		4
		$1: \mathbf{p}$	$11/3: \mathbf{r}$		
	$-988/409$	-2.35462	$-1.68698 \pm 0.790067i$	$0.765729 \pm 0.643163i$	8
		$834.411: \mathbf{r}$	$6.04594: \mathbf{r}$	$4.98848: \mathbf{r}$	
				$-0.486146 \pm 0.227678i$	
				-0.424696	

* $|J'(\zeta; \rho)|: \mathbf{t}$ implies that ζ is attractive, parabolic, and repulsive, if $\mathbf{t} = \mathbf{a}(|J'| < 1)$, $\mathbf{t} = \mathbf{p}(|J'| = 1)$, and $\mathbf{t} = \mathbf{r}(|J'| > 1)$, respectively.

where $\Delta_1 = 9(1 + \rho) + 3z(5 + \rho) + 2z^2(3 + 5\rho) + 3z^3(5 + \rho) + 9z^4(1 + \rho)$. This completes the proof. $\square$

Theorem 14. Let $m = 2$ in (34). Then the following hold:

- If $\rho = -2$, then $J'(z; \rho) = 2z^2(3 + 35z + 112z^2 + 159z^3 + 112z^4 + 35z^5 + 3z^6)/(2 + 7z + 7z^2 + z^3)^2$.
- If $\rho = -4$, then $J'(z; \rho) = z^3(8 + 17z + 8z^2)/(1 + 2z)^2$.
- If $\rho = -988/409$, then $J'(z; \rho) = z^3\sigma_1/(-170 - 951z - 1735z^2 - 955z^3 + 258z^4 + 178z^5)^2$, where $\sigma_1 = 121040 + 727134z + 624992z^2 - 5092795z^3 - 17328190z^4 - 24412987z^5 - 17328190z^6 - 5092795z^7 + 624992z^8 + 727134z^9 + 121040z^{10}$.
- Let $\rho \notin \{-2, -4, -988/409\}$. Let z be a fixed point of $J(z; \rho)$ satisfying $\phi(z; \rho) = 0$. Then $J'(z; \rho) = J'(1/z; \rho)$ holds for $z \neq 0$.

Proof. The proofs of (a)–(c) immediately follow from the same argument as used in the proofs of Theorem 13. For the proof of (d), via direct computation with the aid of Mathematica symbolic capability, we find $J'(z; \rho) - J'(1/z; \rho) = -(2Q_2(z)(z+1)\Delta_2/q_2(z)q_2(1/z)^2z^{16})\phi(z; \rho) = 0$, where $\Delta_2 = 4(2+\rho)+4z^8(2+\rho)+4z(13+6\rho)+4z^7(13+6\rho)+2z^3(98+39\rho)+$

$2z^5(98+39\rho)+z^2(136+57\rho)+z^6(136+57\rho)+z^4(204+83\rho)$. This completes the proof. $\square$

Table 2 summarizes the stability results for the strange fixed points ζ of J for special ρ -values with $m \in \{1, 2\}$.

We are ready to discuss the stability of the fixed points described in Theorems 10 and 11 in terms of parameter ρ .

Theorem 15. Let $m = 1$ and $\rho \notin \{-3, -1, -3/5, -1/3, -5/3, -27/17\}$. Then the following hold:

- The strange fixed point $z = 1$ becomes an attractor, a parabolic (indifferent, neutral) point, and a repulser, respectively, when $|(3 + 2\rho)/(27 + 17\rho)| < 1/32$, $|(3 + 2\rho)/(27 + 17\rho)| = 1/32$, and $|(3 + 2\rho)/(27 + 17\rho)| > 1/32$.
- The strange fixed point $z = 1$ is a superattractor if $\rho = -3/2$.

Proof. (a) From the case of $m = 1$ in (34), we find $J'(1; \rho) = 32((3 + 2\rho)/(27 + 17\rho))$. Solving $|J'(1; \rho)| = 32|(3 + 2\rho)/(27 + 17\rho)| = 1$ for ρ , we obtain an ellipse $(x + 5685/3807)^2/(32/1269)^2 + y^2/(32/(3\sqrt{206095}))^2 = 1$ in the cross-sectional ρ -parameter plane for $z = 1$ to be a parabolic point, where $x = \text{Re}(\rho)$, $y = \text{Im}(\rho)$. (b) Solving $|J'(1; \rho)| = 0$ easily yields $\rho = -3/2$. $\square$

FIGURE 1: Stability surfaces of the strange fixed points $J(z; \rho)$ for $m = 1$.

Theorem 16. Let $m = 2$ and $\rho \notin \{-2, -4, -988/409\}$. Then the following hold:

- (a) The strange fixed point $z = 1$ is a parabolic (neutral, indifferent) point, respectively, when $|(64 + 27\rho)/(988 + 409\rho)| < 1/54$, $|(64 + 27\rho)/(988 + 409\rho)| = 1/54$, and $|(64 + 27\rho)/(988 + 409\rho)| > 1/54$.
- (b) The strange fixed point $z = 1$ is a superattractor if $\rho = -64/27$.

Proof. From the case of $m = 2$ in (34), we find $J'(1; \rho) = 54((64 + 27\rho)/(988 + 409\rho))$. Solving $|J'(1; \rho)| = 54|(64 + 27\rho)/(988 + 409\rho)| = 1$ for ρ , we obtain an ellipse $(x + 4634756/1958483)^2/(27000/1958483)^2 + y^2/(5400\sqrt{5/898177930147})^2 = 1$ in the cross-sectional ρ -parameter plane for $z = 1$ to be a parabolic point, where $x = \operatorname{Re}(\rho)$, $y = \operatorname{Im}(\rho)$. (b) Solving $|J'(1; \rho)| = 0$ easily yields $\rho = -64/27$. $\square$

We now proceed to discuss the stability of the strange fixed points ζ for conjugate map $J(z; \rho)$ with $m \in \{1, 2\}$

using $J'(\zeta; \rho)$. As a consequence of Theorems 13 (g) and 14 (d) together with Corollary 12, the stability can be stated at most five strange fixed points including $z = 1$. Then the stability of these fixed points can be best described by illustrative conical surfaces shown in Figures 1-2. The top row of each figure refers to a stability surface for strange fixed point $z = 1$. The stability surfaces for the remaining fixed points z_i ($1 \leq i \leq 8$) are displayed in order from top to bottom and from left to right in each case of $m = 1$ and $m = 2$. The underlying theory is clearly verified via cross-sectional views of the stability surfaces with ρ -parameter domains.

4. Extraneous Fixed Points

In this section, we will consider different complex dynamics behind the extraneous fixed points to be defined now. The fixed points of R_f are zeros of $f(x)$ under consideration. The iteration function R_f , however, might possess other fixed points that are not zeros of f . Such fixed points different from zeros of f are called the *extraneous fixed points* [31, 32] of the iteration function R_f . Extraneous fixed points may

FIGURE 2: Stability surfaces of the strange fixed points $J(z; \rho)$ for $m = 2$.

form attractive, indifferent, repulsive cycles or periodic orbits to display chaotic dynamics behind the basin of attraction under investigation. The existence of such extraneous fixed points would affect the global iteration dynamics, which was demonstrated via König functions by Vrscay and Gilbert [32]. Particularly the presence of attractive cycles induced by the extraneous fixed points of R_f may alter the basin of attractions due to the trapped sequence $\{x_n\}$. Even in the case of repulsive or indifferent fixed points, an initial value x_0 chosen near a desired root may converge to another unwanted remote root. Indeed, these aspects of the Schröder functions [32, 33] were observed in an application to the

family of functions $\{f_k(x) = x^k - 1, k \geq 2\}$ for simple-root finders. By taking a particular member of the family into account for multiple-root finders, we are further interested in the dynamics applied to prototype quadratic polynomial $(z^2 - 1)$ raised to the power m , being the multiplicity of zero α under consideration. Such dynamical aspects motivate our investigation of the extraneous fixed points that may affect the basins of attraction for the proposed methods (5).

The structure of $H_f(x_n)$ in (20) clearly characterizes a variety of iterative methods. The zero α of $f(x)$ is obviously a fixed point of R_f . The points $\xi \neq \alpha$ for which $H_f(\xi) = 0$ are extraneous fixed points of R_f .

Let $H(z)$ represent $H_f(z)$ when $f(z)$ is a finite-order rational function of z . Then it would be of great interest for us to investigate the complex dynamics of the rational iterative map R_p of the form [26]

$$z_{n+1} = R_p(z_n) = z_n - \frac{p(z_n)}{p'(z_n)} H_p(z_n), \quad (36)$$

in connection with the basins of attraction for a variety of polynomials $p(z_n)$. Clearly, $R_p(z)$ represents classical Newton's method with weight function $H_p(z)$ and may possess its fixed points as zeros of $p(z)$ or extraneous fixed points associated with $H(z)$.

We now turn to complex dynamics [28–30] behind the basins of attraction of iterative map (36) applied to a prototype quadratic polynomial raised to the power of m . We are interested also in the investigation of unified dynamics associated with these extraneous fixed points. To this end, we apply a simple quadratic polynomial raised to the power of multiplicity m , that is, $f(z) = (z^2 - 1)^m$ to $H_f(z)$, simple-root cases of which were introduced by Cayley [34] and Vrscay and Gilbert [32] in dynamical studies of the Schröder and König functions for a family of functions $f_k(z) = z^k - 1$, $k \in \mathbb{N}$, to minimize perturbations of the Julia set boundaries.

Hence in this section we will discuss the complex dynamics of (36) associated with its extraneous fixed points. To this end, we first write $H(z)$ associated with H_f applied to $f(z) = (z^2 - 1)^m$ in the form of

$$H(z) = \omega + \frac{F(\zeta)}{D(\zeta)}, \quad (37)$$

where $\zeta = z^2$ and $\omega = a + c + \gamma(1 - d\lambda)$ with $\gamma = 2m/(m + 2)$ is a constant independent of z ;

$$F(\zeta) = (b + d)(2m)^{4m-2} \zeta^{2m} + \rho(b + d)(2m)^{2m-1} \delta_1^{m-1} \delta_2 \zeta^m$$

$$H(z) = \begin{cases} \frac{m \left[-(3m-2) \delta_1^{2m-2} \delta_2^2 + 2^{2m-1} m^{2m-2} (2m+7) (3m-2) \delta_1^{m-1} \delta_2 \zeta^m \kappa - 2^{4m-2} m^{4m-4} \eta_1 \zeta^{2m} \kappa^2 \right]}{2 \left[\delta_1^{2m-2} \delta_2^2 - 2^{2m-1} m^{2m-2} (2m+7) \delta_1^{m-1} \delta_2 \zeta^m \kappa + 2^{4m-2} (m-1) (m+8) m^{4m-4} \zeta^{2m} \kappa^2 \right]} & \text{for Kan,} \\ \frac{-2^{-2m} (m-2) m^{2-2m} \delta_1^{2m-2} \delta_2^2 + (m-2) (m+2) \delta_1^{m-1} \delta_2 \zeta^m \kappa - 2^{2m-2} m^{2m-3} \eta_2 \zeta^{2m} \kappa^2}{2 \kappa \zeta^m \left[\delta_1^{m-1} \delta_2 - 2^{2m-1} m^{2m-3} (m^2 + 2m - 4) \zeta^m \kappa \right]} & \text{for Sol,} \\ \frac{m \left[-(m-2) \delta_1^{m-1} \delta_2 + 2^{2m-1} m^{2m} \zeta^m \kappa \right]}{2 \left[\delta_1^{m-1} \delta_2 - (2m)^{2m-1} \zeta^m \kappa \right]} & \text{for Li,} \end{cases} \quad (41)$$

where $\eta_1 = 3m^3 + 19m^2 + 16m + 16$ and $\eta_2 = m^4 + 2m^3 - 4m^2 - 8m - 16$.

Since $H(z)$ in (40) or (41) defines a high-order rational function as the multiplicity m increases, it is convenient to study the typical cases of $m \in \{2, 3, 4, 5\}$ for locating the corresponding extraneous fixed points ξ by solving $H(\xi) = 0$ for $\xi = \zeta^{1/2}$. In fact, Table 3 lists the extraneous fixed

$$- \lambda \gamma (c - d\rho) \delta_1^{2m-2} \delta_2^2, \\ D(\zeta) = \delta_1^{m-1} \delta_2 \left[(2m)^{2m-1} \zeta^m + \delta_1^{m-1} \delta_2 \rho \right] \quad (38)$$

are polynomials having no common factors with

$$\delta_1 = (2m - \gamma)^2 \zeta - \gamma^2, \\ \delta_2 = (2m - \gamma) \zeta + \gamma. \quad (39)$$

Hence, the roots of $F(\zeta) + \omega \cdot D(\zeta) = 0$ may indeed express the desired extraneous fixed points of H , provided that $D(\zeta) \neq 0$. In this paper, we limit ourselves to considering a simple form of $H(z)$ by selecting $\lambda = -m(m+2)/\tau\gamma\kappa$, $\rho = -m(m+2)/\tau\kappa$, and $d = -m^2(m+4)^3\kappa/(m+2)^2\tau$ with $\tau = m^2 + 2m - 4$ and $\kappa = (m/(m+2))^m$ so that we have $a = c = 0$ and $b = -m(m^3 + 6m^2 + 8m - 8)\kappa/2(m+2)^2$. Consequently, $F(\zeta)$ and $D(\zeta)$ reduce to the following:

$$F(\zeta) = \frac{m^4}{\tau} \left[-\frac{(m+4)^3 \delta_1^{2m-2} \delta_2^2}{\kappa \tau^2} + \frac{2^{2m-2} m^{2m-3} (m+2)^4 \delta_1^{m-1} \delta_2 \zeta^m}{\tau} - 2^{4m-3} m^{4m-5} (m+2)^3 \zeta^{2m} \kappa \right], \quad (40)$$

$$D(\zeta) = \delta_1^{m-1} \delta_2 \left[(2m)^{2m-1} \zeta^m - \frac{m(m+2)}{\tau\kappa} \delta_1^{m-1} \delta_2 \right].$$

In order to compare the dynamics behavior of (36) behind the extraneous fixed points, let us now investigate the corresponding $H(z)$ of the existing three optimal methods, Kan, Sol, and Li, introduced in Section 1. By similarly following the development procedure of $H(z)$ as shown in (37), we find with $\zeta = z^2$

points ξ and their stability from the value of $|R'_p(\xi)|$ applied to a prototype polynomial $p(z) = (z^2 - 1)^m$, respectively, for values of $2 \leq m \leq 5$. As can be seen in the table, all extraneous fixed points of the listed methods are found to be repulsive. Observe that methods Sol and Li do not possess the extraneous fixed points when $m = 2$. In addition, critical points of the proposed methods applied to a polynomial

TABLE 3: Extraneous fixed points $\xi = \zeta^{1/2}$ with stability check for selected cases with $2 \leq m \leq 5$.

Method	m	ξ	Number of ξ points	$ R'_p(\xi) $		
				<1	1	>1
Yk	2	$\pm 0.243414 \pm 0.415875i, \pm 0.326691 \pm 0.0719699i$	8	0	0	8
	3	$\pm 0.248244 \pm 0.321071i, \pm 0.201519 \pm 0.00811083i, \pm 0.287319 \pm 0.157274i$	12	0	0	12
	4	$\pm 0.180609 \pm 0.209976i, \pm 0.154424 \pm 0.0318211i, \pm 0.165971 \pm 0.0281012i, \pm 0.341376 \pm 0.250384i$	16	0	0	16
	5	$(\pm 0.134189 \pm 0.189108i, \pm 0.129334 \pm 0.0459184i, \pm 0.12138, \pm 0.132268, \pm 0.144672 \pm 0.0387865i, \pm 0.380633 \pm 0.254445i)$	20	0	0	20
Kan	2	$\pm 0.372622 \pm 0.342462i, \pm 0.364216 \pm 0.112918i$	8	0	0	8
	3	$\pm 0.370756 \pm 0.347075i, \pm 0.206707 \pm 0.00369212i, \pm 0.361833 \pm 0.157914i$	12	0	0	12
	4	$\pm 0.36607 \pm 0.331711i, \pm 0.166419 \pm 0.0262382i, \pm 0.167623 \pm 0.029141i, \pm 0.362159 \pm 0.184595i$	16	0	0	16
	5	$(\pm 0.132208 \pm 0.000478097i, \pm 0.144187 \pm 0.038215i, \pm 0.145007 \pm 0.039469i, \pm 0.362075 \pm 0.308059i, \pm 0.361766 \pm 0.208745i)$	20	0	0	20
Sol	2	—	—	—	—	—
	3	$\pm 0.679565i, \pm 0.205392, \pm 0.218593, \pm 0.37247 \pm 0.238124i, \pm 0.329022$	12	0	0	12
	4	$\pm 0.778239i, \pm 0.167222 \pm 0.0275903i, \pm 0.171541 \pm 0.0256375i, \pm 0.402522 \pm 0.246961i, \pm 0.351918$	16	0	0	16
	5	$(\pm 0.831769i, \pm 0.132545, \pm 0.134193, \pm 0.145034 \pm 0.0385469i, \pm 0.147144 \pm 0.0370658i, \pm 0.419276 \pm 0.24999i, \pm 0.367148)$	20	0	0	20
Li	2	—	—	—	—	—
	3	$\pm 0.202285, \pm 0.316011 \pm 0.237459i$	6	0	0	6
	4	$\pm 0.166502 \pm 0.0278872i, \pm 0.363341 \pm 0.25042i$	8	0	0	8
	5	$\pm 0.132338, \pm 0.144764 \pm 0.0387259i, \pm 0.389278 \pm 0.253836i$	10	0	0	10

TABLE 4: Critical points w of R_f applied to a polynomial $f(z) = (z^2 - 1)^m$ for selected values of $2 \leq m \leq 5$.

m	w	Number of w points
2	$\pm 0.61585i, \pm 0.300682i, \pm 0.22275, \pm 0.323127 \pm 0.165034i, \pm 0.340261, \pm 1.0, \pm 1.0, \pm 1.0$	18
3	$(\pm 0.555234i, \pm 0.250157i, \pm 0.158737 \pm 0.0554075i, \pm 0.274347 \pm 0.210868i, \pm 0.198344, \pm 0.227118, \pm 0.291901, \pm 1.0, \pm 1.0, \pm 1.0)$	24
4	$(\pm 0.509498i, \pm 0.214305i, \pm 0.245897 \pm 0.225504i, \pm 0.125963 \pm 0.0723105i, \pm 0.16229 \pm 0.0295239i, \pm 0.178323 \pm 0.0217879i, \pm 0.272811, \pm 1.0, \pm 1.0, \pm 1.0)$	30
5	$(\pm 0.47428i, \pm 0.188956i, \pm 0.22628 \pm 0.228735i, \pm 0.105228 \pm 0.0783695i, \pm 0.10314 \pm 0.0206427i, \pm 0.129732, \pm 0.141316 \pm 0.0409263i, \pm 0.138474, \pm 0.152133 \pm 0.0327313i, \pm 0.261805, \pm 1.0, \pm 1.0, \pm 1.0)$	36

$f(z) = (z^2 - 1)^m$ are found and displayed in Table 4 for values of $2 \leq m \leq 5$.

In the latter part of Section 6, complex dynamics behind the extraneous fixed points will be discussed along with chaotic behavior of rational iterative maps (36) when applied to various polynomials $p(z)$, based on visual description of their basins of attraction along with comparison of their dynamic properties and characteristics.

5. Numerical Experiment

We have conducted numerical experiments with a number of test functions using Mathematica Version 7 to confirm

the optimal fourth-order convergence. We have assigned 200 significant digits to the minimum number of precision digits and prescribed error bound of $\varepsilon = 10^{-150}$ throughout the current experiment. The initial values x_0 are selected close to the sought zero α for guaranteed convergence to the desired root. All computations have been performed by Mathematica Version 7 with AMD Kaveri 7850 CPU having 3700 Mhz of clock core speed under Windows 7 operating system.

Definition 17 (computational convergence order). Assume that theoretical asymptotic error constant $\eta = \lim_{n \rightarrow \infty} (|e_n|/|e_{n-1}|^p)$ and convergence order $p \geq 1$ are known. Define $p_n = \log|e_n/\eta|/\log|e_{n-1}|$ as the computational convergence order. Note that $\lim_{n \rightarrow \infty} p_n = p$.

TABLE 5: Convergence for test functions $F_1(x)$ – $F_6(x)$ with selected methods Y1–Y6.

MT	F	n	x_n	$ F(x_n) $	$ x_n - \alpha $	$ e_n/e_{n-1}^4 $	η	p_n
Y1	F_1	0	0.4900000000000000 – 2.5700000000000000i	0.000678292	0.0298039			
		1	0.500000115882539 – 2.59807622867990i	1.67×10^{-25}	1.17×10^{-7}	0.1484996662	0.1356337586	3.97420
		2	0.5000000000000000 – 2.59807621135332i	3.78×10^{-112}	2.55×10^{-29}	0.1356337326		4.00000
		3	0.5000000000000000 – 2.59807621135332i	9.98×10^{-459}	5.79×10^{-116}	0.1356337586		4.00000
		4	0.5000000000000000 – 2.59807621135332i	0.0×10^{-795}	0.0×10^{-119}			
Y2	F_2	0	–0.8900000000000000	3.66×10^{-7}	0.0154886			
		1	–0.905488640138161	1.61×10^{-47}	2.90×10^{-9}	0.05047432426	0.04879773270	3.99189
		2	–0.905488637233314	4.73×10^{-200}	3.47×10^{-36}	0.04879773239		4.00000
		3	–0.905488637233314	3.48×10^{-855}	7.11×10^{-144}	0.04879773270		4.00000
		4	–0.905488637233314	0.0×10^{-1196}	0.0×10^{-200}			
Y3	F_3	0	1.1900000000000000	0.102779	0.190000			
		1	0.999178734348023	2.17×10^{13}	0.0008×10^{-13}	0.6301867327	0.2584060085	3.46320
		2	0.9999999999999883	1.27×10^{-62}	1.17×10^{-53}	0.2572719538		4.00062
		3	1.0000000000000000	1.55×10^{-259}	4.84×10^{-53}	0.2584060085		4.00000
		4	1.0000000000000000	0.0×10^{-1000}	0.0×10^{-200}			
Y4	F_4	0	2.6500000000000000	0.486088	0.00424869			
		1	2.64575131054487	8.73×10^{-22}	5.19×10^{-10}	1.594967181	1.992474266	4.04075
		2	2.64575131106459	1.91×10^{-104}	1.45×10^{-37}	1.992474316		4.00000
		3	2.64575131106459	4.38×10^{-435}	8.89×10^{-148}	1.992474266		4.00000
		4	2.64575131106459	0.0×10^{-593}	0.0×10^{-199}			
Y5	F_5	0	1.1900000000000000	1.41×10^{12}	0.0215466			
		1	1.06845276583309	2.37×10^{-53}	6.27×10^{-7}	2.911081724	2.437575882	3.95374
		2	1.06845339326982	2.47×10^{-217}	3.77×10^{-25}	2.4375758821		4.00000
		3	1.06845339326982	2.89×10^{-873}	4.96×10^{-98}	2.437575882		4.00000
		4	1.06845339326982	0.0×10^{-1799}	0.0×10^{-199}			
Y6	F_6	0	1.8900000000000000	0.0000253422	0.0229312			
		1	1.91292827592121	1.07×10^{-36}	2.90×10^{-6}	10.51276543	14.87454509	4.09193
		2	1.91293118277239	3.41×10^{-160}	1.06×10^{-21}	14.87411967		4.00000
		3	1.91293118277239	3.46×10^{-654}	1.89×10^{-83}	14.87454509		4.00000
		4	1.91293118277239	0.0×10^{-1597}	0.0×10^{-199}			

Typical methods Y1, Y2, ..., Y6 have been applied to the test functions F_1 – F_6 below:

$$F_1(x) = (1 + e^{-2x})(x^2 - x + 7)^4,$$

$$m = 4, \alpha = \frac{1 - 3\sqrt{3}i}{2}, i = \sqrt{-1},$$

$$F_2(x) = (4 + 3 \sin x - 2x^2)^6, \quad m = 6, \alpha \approx -0.9,$$

$$F_3(x) = \left(\cos\left(\frac{\pi}{2}x\right) + e^{1-x^2} - 1\right)^5, \quad m = 5, \alpha = 1,$$

$$F_4(x) = e^{x^2+x+1}(x^2 - 7)^2 \log(x^2 - 6),$$

$$m = 3, \alpha = \sqrt{7},$$

$$F_5(x) = (\cos(x^2 + x) - x \log(x^2 - \pi + 3) + 1)^8$$

$$\cdot (x^2 + 2 - \pi), \quad m = 9, \alpha = \sqrt{\pi - 2},$$

$$F_6(x) = \log^4(x^3 - 6) \cos\left(x^3 - 7 + \frac{\pi}{2}\right) \log(x^3 - 7),$$

$$m = 8, \alpha = 7^{1/3}. \quad (42)$$

As seen in Table 5, the order of convergence is four and the values of computational asymptotic error constant well approach theoretical value η .

The following test functions $f_1, f_2, \dots, f_7$ listed below further confirm the convergence behavior of our proposed methods (5). Consider

$$f_1(x) = x^9 + 4x^7 - 5,$$

$$\alpha = -0.92 + 0.39i, \quad m = 1, \quad x_0 = -0.91 + 0.41i,$$

$$f_2(x) = (e^{x^6 - x^3 - 7} - 1)(x^6 - x^3 - 7),$$

$$\alpha = \left(\frac{1 + \sqrt{29}}{2}\right)^{1/3} = 1.47, \quad m = 2, \quad x_0 = 1.46,$$

TABLE 6: Comparison of $|x_n - \alpha|$ for high-order iterative methods.

$f(x)$	x_0	$ x_n - \alpha $	Kan	Sol	Li	Y1	Y2	Y3	Y4	Y5	Y6
f_1	$-0.91 + 0.41i$	$ x_1 - \alpha $	$3.02e - 5^*$	$4.05e - 5$	$7.1e - 6$	$1.67e - 5$	$1.67e - 5$	$2.17e - 5$	$2.52e - 7$	$2.92e - 5$	$2.17e - 5$
		$ x_2 - \alpha $	$5.49e - 17$	$2.13e - 16$	$3.69e - 20$	$2.66e - 18$	$2.66e - 18$	$9.63e - 18$	$1.19e - 27$	$4.2e - 17$	$9.63e - 18$
		$ x_3 - \alpha $	$6.00e - 64$	$1.64e - 61$	$2.71e - 77$	$1.71e - 69$	$1.71e - 69$	$3.73e - 67$	$6.11e - 109$	$1.8e - 64$	$3.73e - 67$
f_2	1.46	$ x_1 - \alpha $	$3.3e - 5$	$1.99e - 5$	$1.99e - 5$	$2.88e - 5$	$2.88e - 5$	$3.17e - 5$	$8.65e - 5$	$7.36e - 5$	$3.43e - 5$
		$ x_2 - \alpha $	$1.55e - 15$	$1.21e - 16$	$1.21e - 16$	$7.92e - 16$	$7.92e - 16$	$1.27e - 15$	$3.67e - 11$	$9.57e - 15$	$1.90e - 15$
		$ x_3 - \alpha $	$7.65e - 57$	$1.67e - 61$	$1.67e - 61$	$4.52e - 58$	$4.52e - 58$	$3.38e - 57$	$2.8e - 30$	$2.73e - 54$	$1.79e - 56$
f_3	-2.61	$ x_1 - \alpha $	$1.09e - 5$	$1.12e - 5$	$1.11e - 5$	$1.09e - 5$	$1.09e - 5$	$1.09e - 5$	$1.07e - 5$	$1.09e - 5$	$1.09e - 5$
		$ x_2 - \alpha $	$6.94e - 20$	$8.08e - 20$	$7.46e - 20$	$6.94e - 20$	$6.94e - 20$	$6.83e - 20$	$6.08e - 20$	$6.76e - 20$	$6.83e - 20$
		$ x_3 - \alpha $	$1.10e - 76$	$2.15e - 76$	$1.51e - 76$	$1.10e - 76$	$1.10e - 76$	$1.03e - 76$	$6.17e - 77$	$9.84e - 77$	$1.03e - 76$
f_4	$0.49 - 2.25i$	$ x_1 - \alpha $	$1.09e - 5$	$3.78e - 6$	$4.11e - 6$	$4.29e - 6$	$4.29e - 6$	$4.32e - 6$	$4.44e - 6$	$4.33e - 6$	$4.32e - 6$
		$ x_2 - \alpha $	$6.94e - 20$	$6.67e - 24$	$1.01e - 23$	$1.26e - 23$	$1.26e - 23$	$1.30e - 23$	$1.50e - 23$	$1.32e - 23$	$1.30e - 23$
		$ x_3 - \alpha $	$1.10e - 76$	$6.46e - 95$	$3.82e - 94$	$9.42e - 94$	$9.42e - 94$	$1.07e - 93$	$1.99e - 93$	$1.17e - 93$	$1.07e - 93$
f_5	9.03	$ x_1 - \alpha $	$1.09e - 5$	$4.50e - 5$	$4.47e - 5$	$4.45e - 5$	$4.45e - 5$	$4.45e - 5$	$4.45e - 5$	$4.45e - 5$	$4.45e - 5$
		$ x_2 - \alpha $	$6.94e - 20$	$3.27e - 16$	$3.17e - 16$	$3.13e - 16$	$3.13e - 16$	$3.13e - 16$	$3.11e - 16$	$3.13e - 16$	$3.13e - 16$
		$ x_3 - \alpha $	$1.10e - 76$	$9.08e - 61$	$8.06e - 61$	$7.70e - 61$	$7.70e - 61$	$7.65e - 61$	$7.50e - 61$	$7.63e - 61$	$7.65e - 61$
f_6	$-2.19i$	$ x_1 - \alpha $	$1.09e - 5$	$1.72e - 4$	$1.53e - 4$	$1.56e - 4$	$1.56e - 4$	$1.56e - 4$	$1.58e - 4$	$1.57e - 4$	$1.56e - 4$
		$ x_2 - \alpha $	$6.94e - 20$	$6.35e - 15$	$5.99e - 15$	$7.02e - 15$	$7.02e - 15$	$7.18e - 15$	$7.73e - 15$	$7.27e - 15$	$7.18e - 15$
		$ x_3 - \alpha $	$1.10e - 76$	$1.18e - 56$	$1.37e - 56$	$2.88e - 56$	$2.88e - 56$	$3.17e - 56$	$4.39e - 56$	$3.35e - 56$	$3.17e - 56$

* $3.02e - 5 = 3.02 \times 10^{-5}$.

$$f_3(x) = e^{x^2+1} (x^2 - 7)^2 \sec x \log(x^2 - 6),$$

$$\alpha = -\sqrt{7} = -2.64575, \quad m = 3, \quad x_0 = -2.59,$$

$$f_4(x) = (5 - x + x^2)^4 \cot(x^3 + 7),$$

$$\alpha = \frac{1 - \sqrt{19}i}{2}, \quad m = 4, \quad x_0 = 0.49 - 2.25i,$$

$$f_5(x) = (x^5 - 9x^4 + 3x - 27) e^{x^2+x} \sin\left(\frac{\pi x}{9}\right) \cdot (\log(x^2 - 6x - 26))^3,$$

$$\alpha = 9, \quad m = 5, \quad x_0 = 9.12,$$

$$f_6(x) = (x^2 + 5)^4 (\log(x^3 - 3x^2 + 5x - 14))^2,$$

$$\alpha = -\sqrt{5}i, \quad m = 6, \quad x_0 = -2.18i.$$

(43)

Table 6 shows the comparison of $|x_n - \alpha|$ among listed methods Kan, Sol, and Li and Y1–Y6 described in Section 1. In Table 6, the least errors within the prescribed error bound are highlighted in bold face. Although we are limited to the selected current experiments, within three iterations, a strict comparison shows that method Y4 displays slightly better convergence for test functions f_1 and f_3 , and method Sol displays slightly better convergence for test function f_4 , while method Kan displays slightly better convergence for test functions f_5 and f_6 . In addition, both methods Sol and Li show similar convergence for test function f_2 . If we closely

view the definition of the asymptotic error constant, we will find that the local convergence is dependent on the function $f(x)$, an initial guess x_0 , the multiplicity m , and zero α . Consequently, for a given set of test functions, one method is hardly expected to always show better performance than the others.

It is important to properly select initial values guaranteeing the convergence of iterative schemes. For ensured convergence of iterative map (5), it requires good initial values close to zero α . It is not easy to determine how close the initial values are to zero α , since initial values are generally dependent upon computational precision, error bound, and the given function $f(x)$ under consideration. One efficient way of selecting stable initial guesses is to directly use visual basins of attraction [27, 35, 36]. Since the area of convergence can be seen on the basins of attraction, it would be a reasonable measure of convergence behavior. One would say that a method having a larger area of convergence implies a more stable method. Obviously a quantitative analysis becomes an essential tool for measuring the size of area of convergence. In the next section, we will illustrate the basins of attraction of the listed methods when applied to a variety of polynomials with multiple zeros and discuss underlying relevant dynamics.

6. Basins of Attraction

Throughout the current dynamics experiment, for effective constructing basins of attraction, we have employed a tolerance $\varepsilon = 10^{-4}$ for convergence within a maximum of 40 iterations. To illustrate the desired basins of attraction, we first take a 600×600 uniform grid point in a 6×6 square region

FIGURE 3: (a) Kan, (b) Sol, (c) Zhou, (d) Y1, (e) Y2, (f) Y3, (g) Y4, (h) Y5, and (i) Y6, for the roots of the polynomial $P_1(z) = (z^3 - z)^2$.

centered at the origin of the complex plane, which contains all roots of test functions selected. We then paint the initial points on the basins of attraction, with a diversity of colors ranging from bright ones to dark ones based on the iteration number for convergence. In Figures 3–8, the black points are regarded as the points for which the corresponding iteration scheme starting from an initial point does not converge to

any root of the test polynomial under consideration. We have applied all the methods mentioned in Section 1 to a variety of polynomials having multiple roots with multiplicity of $m = 2, 3, 4, 5, 6$. In Tables 7–12, abbreviations CPU, TCON, AVG, and TDIV denote CPU time measured in units of seconds for convergence, the number of total convergent points, the number of average iteration for convergence, and the number

FIGURE 4: (a) Kan, (b) Sol, (c) Zhou, (d) Y1, (e) Y2, (f) Y3, (g) Y4, (h) Y5, and (i) Y6, for the roots of the polynomial $P_2(z) = (z^3 + 5z - 7)^3$.

of divergent points, respectively. At this point, we now begin by presenting various examples to display the desired basins of attraction.

As a first example, we have taken the following polynomial:

$$P_1(z) = (z^3 - z)^2 \quad (44)$$

whose roots $z = 0, \pm 1$ are all real with multiplicity $m = 2$. Based on Table 7 and Figure 3, we find that Y1 has shown

best AVG and TDIV, followed by Kan and Y2. As can be seen in Figure 3, Sol has shown considerable amount of black points. These points causing divergence behavior were observed from the last column of Table 7. The best result for CPU is given by Zhou and the worst one is given by Y6.

Our next sample has triple roots. The polynomial

$$P_2(z) = (z^3 + 5z - 7)^3 \quad (45)$$

FIGURE 5: (a) Kan, (b) Sol, (c) Zhou, (d) Y1, (e) Y2, (f) Y3, (g) Y4, (h) Y5, and (i) Y6, for the roots of the polynomial $P_3(z) = (z^2 + 3z + 5)^4$.

has three roots $z = -0.559719 - 2.43718i, -0.559719 + 2.43718i, 1.11944$ of multiplicity 3. The statistical results are listed in Table 8 and relevant basins of attraction are illustrated in Figure 4. The method Y5 has performed best AVG and TDIV. As can be seen in Figure 4, Sol has shown considerable amount of black points, while Zhou has shown a few black ones. The best result for CPU is given by Y5 and the worst one is given by Y6.

As a third example, we have taken the following polynomial whose roots are all of multiplicity four:

$$P_3(z) = (z^2 + 3z + 5)^4 \quad (46)$$

whose roots are $z = -1.5 - 1.65831i, -1.5 + 1.65831i$. The statistical results are presented in Table 9 and relevant basins of attraction are illustrated in Figure 5. The method Y1 has

FIGURE 6: (a) Kan, (b) Sol, (c) Zhou, (d) Y1, (e) Y2, (f) Y3, (g) Y4, (h) Y5, and (i) Y6, for the roots of the polynomial $P_4(z) = (z^2 - z)^6$.

shown best AVG and TDIV. As can be seen in Figure 5, Kan and Y4 have shown considerable amount of black points, while Y5 and Y6 have shown a few black ones. The best result for CPU is given by Zhou and the worst one is given by Y2.

In the fourth example, we have taken the polynomial having two roots of unity

$$P_4(z) = (z^2 - z)^6. \quad (47)$$

The statistical results are presented in Table 10 and relevant basins of attraction are illustrated in Figure 6. The method Y1 has shown best AVG and TDIV. As can be seen in Figure 6, Y6 has shown considerable amount of black points, while Kan and Sol have shown a few black ones. We find that Zhou shows the best result for CPU, while Y5 shows the worst one.

In the fifth example, we have chosen the following polynomial:

$$P_5(z) = (z^2 - 1)^2. \quad (48)$$

FIGURE 7: (a) Kan, (b) Sol, (c) Zhou, (d) Y1, (e) Y2, (f) Y3, (g) Y4, (h) Y5, and (i) Y6, for the roots of the polynomial $P_4(z) = (z^2 - 1)^2$.

The statistical results are presented in Table 11 and relevant basins of attraction are displayed in Figure 7. The method Y3 has shown best AVG and TDIV. As can be seen in Figure 7, Sol has shown considerable amount of black points. We find that Zhou shows the best result for CPU, while Y5 shows the worst one.

In the last example, we have selected the polynomial whose roots are all of multiplicity five

$$P_6(z) = (z^2 - 1)^5. \quad (49)$$

The experimented results are presented in Table 12 and relevant basins of attraction are shown in Figure 8. The

FIGURE 8: (a) Kan, (b) Sol, (c) Zhou, (d) Y1, (e) Y2, (f) Y3, (g) Y4, (h) Y5, and (i) Y6, for the roots of the polynomial $P_4(z) = (z^2 - 1)^5$.

method Sol has shown best AVG and TDIV. We find that Zhou shows the best result for CPU, while Y6 shows the worst one.

7. Conclusion

We have developed a triparametric family of optimal fourth-order methods and investigated their complex dynamics via Möbius conjugacy map applied to a polynomial of the form

$f(z) = [(z - A)(z - B)]^m$ along with the stability analysis of strange fixed points. Besides, different complex dynamics have been also investigated from a viewpoint of extraneous fixed points of the proposed methods (5) when applied to a prototype quadratic polynomial $f(z) = (z^2 - 1)^m$ raised to the power of known multiplicity m . To extract information on better initial guesses of an iterative method, we need to carefully examine the basins of attraction. In this paper, we have illustrated a number of basins of attraction for the listed

TABLE 7: Typical Example 1 with $P_1(z) = (z^3 - z)^2, m = 2$.

Method	CPU	TCON	AVG	TDIV
Kan	205.469	360,000	6.20867	0
Sol	111.884	351,220	5.88726	8780
Zhou	76.331	360,000	7.34681	0
Y1	184.503	360,000	6.03626	0
Y2	195.422	360,000	6.22663	0
Y3	200.024	360,000	6.34617	0
Y4	118.436	360,000	6.35492	0
Y5	308.039	360,000	9.56036	0
Y6	488.376	360,000	13.786	0

TABLE 8: Typical Example 1 with $P_2(z) = (z^3 + 5z - 7)^3, m = 3$.

Method	CPU	TCON	AVG	TDIV
Kan	442.013	360,000	5.73999	0
Sol	296.418	359,898	5.57455	102
Zhou	84.599	359,996	6.72502	4
Y1	357.679	360,000	5.44998	0
Y2	397.303	360,000	5.84558	0
Y3	411.795	360,000	6.03896	0
Y4	423.153	360,000	6.01186	0
Y5	206.343	360,000	5.35253	0
Y6	973.993	360,000	13.2845	0

TABLE 9: Typical Example 1 with $P_3(z) = (z^2 + 3z + 5)^4, m = 4$.

Method	CPU	TCON	AVG	TDIV
Kan	201.023	359,594	5.54699	406
Sol	145.612	360,000	5.36095	0
Zhou	54.226	360,000	6.00988	0
Y1	176.749	360,000	5.32411	0
Y2	187.17	360,000	5.58189	0
Y3	189.401	359,998	5.66574	2
Y4	198.277	359,725	5.73911	275
Y5	189.385	359,996	7.25579	4
Y6	384.09	359,994	10.9976	6

optimal fourth-order iterative methods when applied to various complex polynomials as well as statistical analysis featuring data for CPU time and other tabulated numbers for convergence behavior. From the area of convergence on the basins of attraction, more effective information can be obtained for selecting stable initial values of the iterative methods under consideration. As a future research work, we will devote ourselves to developing a higher-order family of iterative methods along with statistical data analysis as well as an illustrative investigation of the desired basins of attraction.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

TABLE 10: Typical Example 1 with $P_4(z) = (z^2 - z)^6, m = 6$.

Method	CPU	TCON	AVG	TDIV
Kan	331.018	359,994	5.76332	6
Sol	235.64	359,998	5.56602	2
Zhou	55.615	360,000	6.45457	0
Y1	251.661	360,000	5.70224	0
Y2	252.815	360,000	5.78377	0
Y3	253.236	360,000	5.79558	0
Y4	252.316	360,000	5.78629	0
Y5	267.401	360,000	8.74116	0
Y6	416.866	359,891	8.59493	109

TABLE 11: Typical Example 1 with $P_5(z) = (z^2 - 1)^2, m = 2$.

Method	CPU	TCON	AVG	TDIV
Kan	60.231	360,000	5.28004	0
Sol	48.375	350,408	5.00397	9592
Zhou	39.952	360,000	5.92232	0
Y1	61.948	360,000	5.31739	0
Y2	46.723	360,000	5.38721	0
Y3	60.481	360,000	5.07764	0
Y4	62.557	360,000	5.31739	0
Y5	464.556	360,000	13.8001	0
Y6	158.309	360,000	13.8001	0

TABLE 12: Typical Example 1 with $P_6(z) = (z^2 - 1)^5, m = 5$.

Method	CPU	TCON	AVG	TDIV
Kan	101.743	360,000	5.45698	0
Sol	75.38	360,000	5.26937	0
Zhou	37.472	360,000	5.92232	0
Y1	119.169	360,000	5.59266	0
Y2	119.84	360,000	5.58321	0
Y3	96.877	360,000	5.33871	0
Y4	100.932	360,000	5.59266	0
Y5	150.634	360,000	8.93996	0
Y6	164.113	359,891	8.93996	0

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