Research Article

A Fractional Trust Region Method for Linear Equality Constrained Optimization

Honglan Zhu,^{1,2} Qin Ni,¹ Liwei Zhang,³ and Weiwei Yang¹

¹College of Science, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China

²Business School, Huaiyin Institute of Technology, Huai'an 223003, China

³Jiangsu Cable, Nanjing 210000, China

Correspondence should be addressed to Honglan Zhu; zhuhonglan1981@163.com

Received 21 April 2016; Accepted 29 May 2016

Academic Editor: Josef Diblík

Copyright © 2016 Honglan Zhu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A quasi-Newton trust region method with a new fractional model for linearly constrained optimization problems is proposed. We delete linear equality constraints by using null space technique. The fractional trust region subproblem is solved by a simple dogleg method. The global convergence of the proposed algorithm is established and proved. Numerical results for test problems show the efficiency of the trust region method with new fractional model. These results give the base of further research on nonlinear optimization.

1. Introduction

In this paper, we consider the linear equality constrained optimization problem:

$$\min_{x \in R^n} f(x), \quad (1)$$

$$\text{s.t. } A^T x = b, \quad (2)$$

where $f(x)$ are continuously differentiable, $A \in R^{n \times m}$, $b \in R^m$, and $\text{rank}(A) = m$.

Trust region methods have the advantage in the theoretical analysis of convergence properties and the practice. Besides, Davidon proposed a conic model which makes use of more incorporate information at each iteration (see [1]). It is believed that combining trust region techniques with a conic model would be appealing. And it has attracted more and more attention of the researchers (see [2–8]).

For unconstrained optimization problems, we proposed a new fractional model (see [9]):

$$\begin{aligned} \psi_k(s) = & \frac{(1 + c_k^T s)}{(1 - a_k^T s)(1 - b_k^T s)} g_k^T s \\ & + \frac{1}{2} \frac{(1 + c_k^T s)^2}{(1 - a_k^T s)^2 (1 - b_k^T s)^2} s^T B_k s, \end{aligned} \quad (3)$$

where $a_k, b_k, c_k \in R^n$ are horizontal vectors, $g_k = \nabla f(x_k)$, $s \in R^n$, $B_k \in R^{n \times n}$ is symmetric and an approximate Hessian of $f(x)$ at x_k . Then, the trust region subproblem of the unconstrained optimization problems is

$$\begin{aligned} \min \quad & \psi_k(s), \\ \text{s.t.} \quad & \|s\| \leq \Delta_k, \end{aligned} \quad (4)$$

$$\left| (1 - a_k^T s)(1 - b_k^T s) \right| \geq \epsilon_0,$$

where ϵ_0 ($0 < \epsilon_0 < 1$) is a sufficiently small positive number, $\|\cdot\|$ refers to the Euclidean norm, and $\Delta_k > 0$ is a trust region radius. If $b_k = c_k = 0$, then $\psi_k(s)$ is reduced to the conic model. If $a_k = b_k = c_k = 0$, then $\psi_k(s)$ is the quadratic model. In order to ensure that the fractional model function $\psi_k(s)$ is bounded over the trust region $\{s \mid \|s\| \leq \Delta_k\}$, we assume $\exists \epsilon_1 \in (0, 1/3)$:

$$\begin{aligned} \|a_k\| \Delta_k & \leq \epsilon_1, \\ \|b_k\| \Delta_k & \leq \epsilon_1, \\ \|c_k\| \Delta_k & \leq \epsilon_1. \end{aligned} \quad (5)$$

We denote

$$\tilde{\Delta}_k = \min \left\{ \Delta_k, \frac{\epsilon_1}{\|a_k\|}, \frac{\epsilon_1}{\|b_k\|}, \frac{\epsilon_1}{\|c_k\|} \right\}; \quad (6)$$

then, (4) reduces to a simplified fractional trust region subproblem:

$$\begin{aligned} \min \quad & \psi(s), \\ \text{s.t.} \quad & \|s\| \leq \tilde{\Delta}_k. \end{aligned} \quad (7)$$

However, the subproblem (7) is solved only in quasi-Newton direction in [9]. In this paper, we made a further research, where the subproblem (7) is solved by a generalized dogleg algorithm.

For linear equality constraints problem, we focus on the problems which are solved by a trust region method with new fractional model. If the constraints are linear inequality constraints or some constraint functions are nonlinear, then a few difficulties may arise. However, for these cases, the linear equality constraints problem may be as the subproblems of them. For example, the inequality constraints can be removed by an active set technique or a barrier transformation, and then the nonlinear constraints are linearized.

In [10], Sun et al. established the algorithm for the problem (1)-(2) and prove the convergence. However, they do not consider the model computation. In [11], Lu and Ni proposed a trust region method with new conic model for solving (1)-(2) and carried out the numerical experiments.

In this paper, we use a simple dogleg method to solve fractional model subproblems and present a quasi-Newton trust region algorithm for solving linear equality constrained optimization problems. This is a continuing work of the fractional model (see [9]), where the linear equality constraints (2) are deleted by using null space techniques.

This paper is organized as follows. In Section 2, we give a description of the fractional trust region subproblem. In Section 3, we give a generalized dogleg algorithm for solving the fractional subproblem. In Section 4, we propose a new quasi-Newton method based on the fractional model for solving linearly constrained optimization problems and prove the global convergence of the proposed method under the reasonable assumptions. The numerical results are presented in Section 5.

2. The Fractional Trust Region Subproblem

In order to solve the problem (1)-(2), we assume that A is column full rank and constraints are consistent. That is, the current point x_k always satisfies $A^T x_k = \mathbf{b}$. Obviously, the constrained condition is equivalent to $A^T s = 0$ if $x = x_k + s$. Therefore, combining with (1)-(4), we can obtain that the trial step s_k is computed by the following subproblem:

$$\min \quad \psi_k(s), \quad (8)$$

$$\text{s.t.} \quad A^T s = 0, \quad (9)$$

$$\left| (1 - a_k^T s) (1 - b_k^T s) \right| \geq \epsilon_0, \quad (10)$$

$$\|s\| \leq \Delta_k. \quad (11)$$

It can be found that our trust region subproblem (8)-(11) is the minimization of a fractional function subject to the trust region constraint and the linear constraints.

In order to solve (8)-(11), firstly we consider removing the constraint (10) by the same assumption as in [9]. That is, we assume the parameters a_k , b_k , and c_k satisfied (5); then, subproblem (8)-(11) can be rewritten as the following reduced subproblem:

$$\min \quad \psi_k(s) \quad (12)$$

$$\text{s.t.} \quad A^T s = 0, \quad (13)$$

$$\|s\| \leq \tilde{\Delta}_k, \quad (14)$$

where $\tilde{\Delta}_k$ is defined as (6).

The null space technology (see [4, 12, 13]) is an important technique for solving equality constraints programming problems. In the following, we use this technology to eliminate constraint (13). Since A has full column rank, then there exist an orthogonal matrix $Q \in R^{n \times n}$ and a nonsingular upper triangular matrix $R \in R^{m \times m}$ such that

$$A = QR = [Q_1 \ Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q_1 R_1, \quad (15)$$

where $Q_1 \in R^{n \times m}$, $Q_2 \in R^{n \times (n-m)}$, and $R_1 \in R^{m \times m}$. Then, (13) can be rewritten as

$$R_1^T Q_1^T s = 0. \quad (16)$$

Therefore, the feasible point for (13) can be presented by

$$s = Q_2 u, \quad (17)$$

for any $u \in R^{n-m}$, where $Q_2 u$ lies in the null space of A . Then, the subproblem (12)-(14) becomes

$$\begin{aligned} \min \quad & \tilde{\psi}_k(u) \\ & = \frac{(1 + \tilde{c}_k^T u) \tilde{g}_k^T u}{(1 - \tilde{a}_k^T u) (1 - \tilde{b}_k^T u)} \end{aligned} \quad (18)$$

$$+ \frac{(1 + \tilde{c}_k^T u)^2 u^T \tilde{B}_k u}{2(1 - \tilde{a}_k^T u)^2 (1 - \tilde{b}_k^T u)^2},$$

$$\text{s.t.} \quad \|u\| \leq \tilde{\Delta}_k, \quad (19)$$

where $\tilde{a}_k$, $\tilde{b}_k$, and $\tilde{c}_k$ are reduced horizontal vectors, $\tilde{g}_k$ is the reduced gradient, $\tilde{B}_k$ is Hessian approximation, and

$$\begin{aligned} \tilde{g}_k &= Q_2^T g_k, \\ \tilde{B}_k &= Q_2^T B_k Q_2, \\ \tilde{a}_k &= Q_2^T a_k, \\ \tilde{b}_k &= Q_2^T b_k, \\ \tilde{c}_k &= Q_2^T c_k. \end{aligned} \quad (20)$$

It can be seen that this subproblem has the same form as the subproblem (7) of the unconstrained optimization problems and it can be considered as the subproblem of the unconstrained minimization over u . Therefore, we can find a solution of (18)-(19) by the dogleg method. Besides, it is easy to find that $\|s\| = \|Q_2 u\| \leq \tilde{\Delta}_k$ is equivalent to $\|u\| \leq \tilde{\Delta}_k$ due to $Q_2^T Q_2 = I$.

3. The Dogleg Method of Fractional Trust Region Subproblem

Now we consider calculating the trial step u_k of the new subproblem (18)-(19) by a simple dogleg method. Firstly, we need to recall the choice of the Newton point u_N as the following subalgorithm (see [9]).

Subalgorithm 3.1. Given $B_k > 0$,

$$\epsilon_1 \in \left(0, \frac{1}{3}\right). \quad (21)$$

Step 1. Calculate $\tilde{a}_k, \tilde{b}_k, \tilde{c}_k, \tilde{g}_k$, and $\tilde{B}_k$ as defined in (20) and

$$\xi_a = \tilde{c}_k^T \tilde{B}_k^{-1} \tilde{g}_k + \tilde{a}_k^T \tilde{B}_k^{-1} \tilde{g}_k \tilde{b}_k^T \tilde{B}_k^{-1} \tilde{g}_k, \quad (22)$$

$$\xi_b = \tilde{a}_k^T \tilde{B}_k^{-1} \tilde{g}_k + \tilde{b}_k^T \tilde{B}_k^{-1} \tilde{g}_k - 1.$$

Step 2. If $\xi_a < 0$, then $u_N = -\tau_N \tilde{B}_k^{-1} \tilde{g}_k$, where

$$\tau_N = \frac{\xi_b + \sqrt{\xi_b^2 + (-4\xi_a)}}{-2\xi_a}. \quad (23)$$

Step 3. If $\xi_a \geq 0$, then set $\tilde{b}_k = \tilde{c}_k = 0$. If $|\tilde{a}_k^T \tilde{B}_k^{-1} \tilde{g}_k| \leq \epsilon_1$, then $u_N = -\tilde{B}_k^{-1} \tilde{g}_k / (1 - \tilde{a}_k^T \tilde{B}_k^{-1} \tilde{g}_k)$. If $|\tilde{a}_k^T \tilde{B}_k^{-1} \tilde{g}_k| > \epsilon_1$, then set $\tilde{a}_k = 0$ and $u_N = -\tilde{B}_k^{-1} \tilde{g}_k$.

In the following, we consider determining the steepest descent point of (18)-(19), where the steepest descent point is defined by Definition 1. Let $u = -\tau \tilde{g}_k$ ($\tilde{g}_k \neq 0$) and from (18) we have

$$\hat{\psi}(\tau) = \tilde{\psi}_k(-\tau \tilde{g}_k); \quad (24)$$

then, (18)-(19) becomes

$$\begin{aligned} \min \quad & \hat{\psi}(\tau), \\ \text{s.t.} \quad & 0 \leq \tau \leq \tau_\Delta, \end{aligned} \quad (25)$$

where

$$\begin{aligned} \hat{\psi}(\tau) &= \frac{\tau(1 - \tau \tilde{c}_k^T \tilde{g}_k)(-2v_\tau \tilde{g}_k^T \tilde{g}_k + \tau(1 - \tau \tilde{c}_k^T \tilde{g}_k) \tilde{g}_k^T \tilde{B}_k \tilde{g}_k)}{2v_\tau^2}, \end{aligned} \quad (26)$$

$$v_\tau = (1 + \tau \tilde{a}_k^T \tilde{g}_k)(1 + \tau \tilde{b}_k^T \tilde{g}_k), \quad (27)$$

$$\tau_\Delta = \frac{\tilde{\Delta}_k}{\|\tilde{g}_k\|}.$$

Definition 1. Let τ_{cp} be the solution of (25). Then, $u_{cp} = -\tau_{cp} \tilde{g}$ is called a steepest descent point of (18)-(19).

In order to discuss the stationary points of $\hat{\psi}(\tau)$, by computation we have that the derivative of $\hat{\psi}(\tau)$ is

$$\hat{\psi}'(\tau) = \frac{\varphi_\tau \tilde{\varphi}_\tau}{v_\tau^3}, \quad (28)$$

where

$$\varphi_\tau = a_\tau \tau^2 + b_\tau \tau + \tilde{g}_k^T \tilde{g}_k, \quad (29)$$

$$\tilde{\varphi}_\tau = \tilde{a}_\tau \tau^2 + 2\tilde{c}_k^T \tilde{g}_k \tau - 1, \quad (30)$$

$$a_\tau = \tilde{c}_k^T \tilde{g}_k \tilde{g}_k^T \tilde{B}_k \tilde{g}_k + \tilde{a}_k^T \tilde{g}_k \tilde{b}_k^T \tilde{g}_k \tilde{g}_k^T \tilde{g}_k, \quad (31)$$

$$b_\tau = \tilde{a}_k^T \tilde{g}_k \tilde{g}_k^T \tilde{g}_k + \tilde{b}_k^T \tilde{g}_k \tilde{g}_k^T \tilde{g}_k - \tilde{g}_k^T \tilde{B}_k \tilde{g}_k, \quad (32)$$

$$\tilde{a}_\tau = \tilde{c}_k^T \tilde{g}_k \tilde{a}_k^T \tilde{g}_k + \tilde{c}_k^T \tilde{g}_k \tilde{b}_k^T \tilde{g}_k + \tilde{a}_k^T \tilde{g}_k \tilde{b}_k^T \tilde{g}_k. \quad (33)$$

By direct calculating, we have

$$a_\tau = \tilde{a}_\tau \tilde{g}_k^T \tilde{g}_k - b_\tau \tilde{c}_k^T \tilde{g}_k. \quad (34)$$

During the analysis, we find that if $b_\tau \geq 0$ and $\tilde{c}_k^T \tilde{g}_k \leq 0$, then (29) and (30) may have no positive real zero points in most of the cases, respectively. Thus, in order to simplify the discussion, we assume

$$\begin{aligned} b_\tau &< 0, \\ \tilde{c}_k^T \tilde{g}_k &> 0. \end{aligned} \quad (35)$$

In order to discuss the feasible stationary points of φ_τ and $\tilde{\varphi}_\tau$, we first define the sets T_φ and $T_{\tilde{\varphi}}$ which contain all the positive extremum points of φ_τ and $\tilde{\varphi}_\tau$, respectively, and these extremum points should be inside the feasible region $[0, \tau_\Delta]$. It is easy to obtain that

$$\begin{aligned} T_\varphi &= \begin{cases} \emptyset, & \text{if } a_\tau > 0, \\ \{\tau_1\}, & \text{if } a_\tau = 0, \\ \{\tau_2\}, & \text{otherwise,} \end{cases} \\ T_{\tilde{\varphi}} &= \begin{cases} \{\tau_3\}, & \text{if } \tilde{a}_\tau > 0, \\ \{\tau_3\}, & \text{if } \tilde{a}_\tau = 0, \\ \{\tau_3, \tau_4\}, & \text{if } -(\tilde{c}_k^T \tilde{g}_k)^2 < \tilde{a}_\tau < 0, \\ \emptyset, & \text{otherwise,} \end{cases} \end{aligned} \quad (36)$$

where

$$\begin{aligned}\hat{\tau}_1 &= \frac{\tilde{g}_k^T \tilde{g}_k}{-b_\tau}, \\ \tau_1 &= \frac{-b_\tau + \sqrt{\Delta_\tau}}{2a_\tau},\end{aligned}\quad (37)$$

$$\begin{aligned}\tau_2 &= \frac{-b_\tau - \sqrt{\Delta_\tau}}{2a_\tau}, \\ \hat{\tau}_3 &= \frac{1}{2\tilde{c}_k^T \tilde{g}_k}, \\ \tau_3 &= \frac{-\tilde{c}_k^T \tilde{g}_k + \sqrt{\tilde{\Delta}_\tau}}{\tilde{a}_\tau},\end{aligned}\quad (38)$$

$$\begin{aligned}\tau_4 &= \frac{-\tilde{c}_k^T \tilde{g}_k - \sqrt{\tilde{\Delta}_\tau}}{\tilde{a}_\tau}, \\ \Delta_\tau &= b_\tau^2 - 4a_\tau \tilde{g}_k^T \tilde{g}_k, \\ \tilde{\Delta}_\tau &= (\tilde{c}_k^T \tilde{g}_k)^2 + \tilde{a}_\tau.\end{aligned}\quad (39)$$

Now, we have the the following conclusions.

Remark 2. Suppose that (5) and (35) hold. (i) If $\tilde{a}_\tau \geq 0$, then from (34) we know that $a_\tau > 0$. (ii) If $-(\tilde{c}_k^T \tilde{g}_k)^2 \leq \tilde{a}_\tau < 0$, $a_\tau = 0$, then $\hat{\tau}_1 > \tau_\Delta$. From (37) and (33), we have

$$1 - \hat{\tau}_1 \tilde{c}_k^T \tilde{g}_k = \frac{\tilde{a}_\tau + (\tilde{c}_k^T \tilde{g}_k)^2}{\tilde{a}_\tau} \leq 0, \quad (40)$$

where $\hat{\tau}_1$ is defined in (37). However, from (37) and (35) we have $\hat{\tau}_1 > 0$. We assume $\hat{\tau}_1 \in (0, \tau_\Delta]$; then, from (5) and (27) we have

$$1 - \hat{\tau}_1 \tilde{c}_k^T \tilde{g}_k \geq 1 - \tau_\Delta \|\tilde{c}_k\| \|\tilde{g}_k\| \geq 1 - \epsilon_1 > 0, \quad (41)$$

where the last inequality is obtained by (21) and this conflicts with (40). Therefore, $\hat{\tau}_1 > \tau_\Delta$.

Similarly, we can prove that if $-(\tilde{c}_k^T \tilde{g}_k)^2 \leq \tilde{a}_\tau < 0$, $a_\tau < 0$, then $\tau_2 > \tau_\Delta$.

Then, combining with (36) and Remark 2., we define a set

$$T = \begin{cases} \{\tau_3\}, & \text{if } \tilde{a}_\tau > 0, \\ \{\hat{\tau}_3\}, & \text{if } \tilde{a}_\tau = 0, \\ \{\tau_3, \tau_4\}, & \text{if } -(\tilde{c}_k^T \tilde{g}_k)^2 < \tilde{a}_\tau < 0, \\ \{\hat{\tau}_1\}, & \text{if } \tilde{a}_\tau < -(\tilde{c}_k^T \tilde{g}_k)^2, \tilde{a}_\tau = 0, \\ \{\tau_2\}, & \text{if } \tilde{a}_\tau < -(\tilde{c}_k^T \tilde{g}_k)^2, \tilde{a}_\tau < 0, \\ \emptyset, & \text{otherwise,} \end{cases} \quad (42)$$

where $\hat{\tau}_1$, τ_2 , $\hat{\tau}_3$, τ_3 , and τ_4 are determined by (37)–(39).

Hence, it is easy to get the following theorems.

Theorem 3. Suppose that (5) and (35) hold. Then, the solution of (25) is

$$\tau_{cp} = \arg \min \{\hat{\psi}(\tau), \tau \in T^*\}, \quad (43)$$

where

$$T^* = (T \cap \{[0, \tau_\Delta]\}) \cup \{0, \tau_\Delta\} \quad (44)$$

and T is defined by (36).

Therefore, the steepest descent point $u_{cp} = -\tau_{cp}g$ is an approximate solution of the fractional trust region subproblem (18)–(19), where τ_{cp} is defined by (43).

Similarly, the fractional trust region subproblem (18)–(19) has the following property. The proof of this theorem is similar to Theorem 3.1 in [14], so we omit its proof.

Theorem 4. Suppose that (5) and (35) hold, where $\epsilon_1 \in (0, 1/3)$. If $\|u_N\| > \tilde{\Delta}_k$, then the optimal solution of (18)–(19) must be on the boundary of trust region, where u_N is defined in Subalgorithm 3.1.

In order to propose a generalized dogleg method of (18)–(19), we set

$$u(t) = tu_N + (1-t)u_{cp} \quad (45)$$

and calculate t^* such that $\|u(t^*)\| = \tilde{\Delta}_k$. Denote

$$\begin{aligned}d &= \|u_N - u_{cp}\|^2, \\ e &= (u_N - u_{cp})^T u_{cp}, \\ f &= \|u_{cp}\|^2 - \tilde{\Delta}_k^2.\end{aligned}\quad (46)$$

If $\|u_{cp}\| < \tilde{\Delta}_k$, then there exist two real roots t_1 and t_2 :

$$\begin{aligned}t_1 &= \frac{-e + \sqrt{e^2 - df}}{d}, \\ t_2 &= \frac{-e - \sqrt{e^2 - df}}{d},\end{aligned}\quad (47)$$

where $0 < t_1 < 1$ and $t_2 < 0$ (see [14, 15]).

Based on the preceding theorems and analysis, we now give a generalized dogleg algorithm for solving (18)–(19).

Algorithm 5.

Step 1. Compute u_N by Subalgorithm 3.1.

Step 2. If $\|u_N\| \leq \tilde{\Delta}_k$, then $u_k = u_N$, and stop.

Step 3. If $\xi_a < 0$, then compute b_τ as defined in (32), and go to Step 4. Otherwise, go to Step 5.

Step 4. If $b_\tau \geq 0$ or $\tilde{c}_k^T \tilde{g}_k \leq 0$, go to Step 5. Otherwise, compute $u_{cp} = -\tau_{cp}g_k$, where τ_{cp} is defined by (43). If $\tau_{cp} = \tau_\Delta$, where

τ_Δ is defined by (27), then $u_k = u_{cp}$, and stop. If $\tau_{cp} = 0$, go to Step 5. Otherwise, go to Step 6.

Step 5. Set $\tilde{b}_k = \tilde{c}_k = 0$ and compute

$$\begin{aligned} c_{ag} &= \tilde{g}_k^T \tilde{B}_k \tilde{g}_k - \tilde{a}_k^T \tilde{g}_k \tilde{g}_k^T \tilde{g}_k, \\ \tilde{\tau}_{cp} &= \frac{\tilde{g}_k^T \tilde{g}_k}{c_{ag}}, \\ u_d &= -\tilde{\tau}^* \tilde{g}_k, \end{aligned} \quad (48)$$

where

$$\tilde{\tau}^* = \begin{cases} \tau_\Delta, & \text{if } c_{ag} \leq 0, \\ \min \{ \tau_\Delta, \tilde{\tau}_{cp} \}, & \text{if } c_{ag} > 0. \end{cases} \quad (49)$$

If $\tilde{\tau}^* = \tau_\Delta$, then $u_k = u_d$, and stop. Otherwise, calculate $u_{cp} = -\tilde{\tau}_{cp} \tilde{g}_k$, go to Step 6.

Step 6. Calculate t_1 and t_2 as defined in (47); then,

$$t^* = \arg \min \{ \tilde{\psi}_k(u(0)), \tilde{\psi}_k(u(t_1)), \tilde{\psi}_k(u(t_2)) \}, \quad (50)$$

where $u(t)$ is defined by (45). $u_k = u(t^*)$, and stop.

Then, we give the predicted decent bound in each iteration, which is the lower bound of the predicted reduction in each iteration:

$$\text{pred}_k(u) = \tilde{\psi}_k(0) - \tilde{\psi}_k(u), \quad (51)$$

where $\tilde{\psi}_k(u)$ is defined by (18).

Theorem 6. Suppose that (5) and (35) hold, where $\epsilon_1 \in (0, 1/3)$. If u_k is obtained by Algorithm 5, then

$$\text{pred}(u_k) \geq \frac{1}{2} \frac{\|\tilde{g}_k\|}{\|\tilde{B}_k\|} \min \left\{ \|\tilde{g}_k\|, \frac{\xi \tilde{\Delta}_k}{\|\tilde{B}_k^{-1}\|} \right\}, \quad (52)$$

where $\xi = (1 - \epsilon_1)/(1 + \epsilon_1)^2$.

This theorem is similar to that in [9] and its proof is omitted.

4. New Quasi-Newton Algorithm and Its Global Convergence

In this section, we propose a quasi-Newton method with a fractional model for linearly equality constrained optimization and prove its convergence under some reasonable conditions. In order to solve problem (1)-(2), we consider the fractional model approximation for $f(x)$ about $u = 0$; that is

$$m_k(u) = f_k + \tilde{\psi}_k(u), \quad (53)$$

where $x = x_k + s = x_k + Q_2 u$, $f_k = f(x_k)$, and $\tilde{\psi}_k(u)$ is defined as (18). Thus, $m_k(0) = f_k$, $\tilde{g}_k$, and $\tilde{B}_k$ are the corresponding gradient and Hessian approximations of the function at the k th iteration. We choose u_k to minimize $m_k(u)$. There is a unique minimizer if and only if $\tilde{B}_k$ is positive definite. In the following, we give our algorithm. If the current iteration is the feasible point x_k , then an equivalent form of (1)-(2) is to solve the reduced unconstrained problem

$$\min_{u \in \mathbb{R}^{n-m}} \tilde{f}(u) = f(x_k + Q_2 u). \quad (54)$$

In the following, we consider the choice of the parameter vectors $\tilde{a}_k$, $\tilde{b}_k$, and $\tilde{c}_k$. We choose these vectors such that (53) satisfies the following conditions:

$$m_k(0) = \tilde{f}_k, \quad (55)$$

$$\nabla m_k(0) = \tilde{g}_k,$$

$$m_k(-u_{k-1}) = \tilde{f}_{k-1}, \quad (56)$$

$$\nabla m_k(-u_{k-1}) = \tilde{g}_{k-1},$$

where

$$x_k = x_{k-1} + s_{k-1} = x_{k-1} + Q_2 u_{k-1},$$

$$\tilde{f}_k = \tilde{f}(0) = f(x_k) = f_k, \quad (57)$$

$$\tilde{f}_{k-1} = \tilde{f}(-u_{k-1}) = f(x_{k-1}) = f_{k-1},$$

$$\tilde{g}_{k-1} = Q_2^T g_{k-1}.$$

Obviously, (55) holds. Then, from (56), we have

$$f_{k-1} = f_k - \frac{(1 - \tilde{c}_k^T u_{k-1}) \tilde{g}_k^T u_{k-1}}{\tilde{v}_{k-1}} + \frac{(1 - \tilde{c}_k^T u_{k-1})^2 u_{k-1}^T \tilde{B}_k u_{k-1}}{2 \tilde{v}_{k-1}^2}, \quad (58)$$

$$\tilde{g}_{k-1} = \frac{1}{\tilde{v}_{k-1}^3} \tilde{Q}_{k-1} [\tilde{v}_{k-1} \tilde{g}_k - (1 - \tilde{c}_k^T u_{k-1}) \tilde{B}_k u_{k-1}], \quad (59)$$

where $\tilde{v}_{k-1} = (1 + \tilde{a}_k^T u_{k-1})(1 + \tilde{b}_k^T u_{k-1})$ and

$$\begin{aligned} \tilde{Q}_{k-1} &= (1 - \tilde{c}_k^T u_{k-1}) \tilde{v}_{k-1} I - [\tilde{c}_k \tilde{v}_{k-1} + (1 - \tilde{c}_k^T u_{k-1}) \\ &\quad \cdot (\tilde{a}_k (1 + \tilde{b}_k^T u_{k-1}) + \tilde{b}_k (1 + \tilde{a}_k^T u_{k-1}))] u_{k-1}^T. \end{aligned} \quad (60)$$

If we choose

$$\begin{aligned} \tilde{a}_k &= k_1 \tilde{g}_{k-1}, \\ \tilde{b}_k &= k_2 \tilde{B}_{k-1} u_{k-1}, \\ \tilde{c}_k &= k_3 \tilde{g}_k, \end{aligned} \quad (61)$$

then these unknown parameters k_1 , k_2 , and k_3 can be obtained from (58)-(59). In the following, we give the derivation process of k_1 , k_2 , and k_3 . First, we define some notations:

$$\begin{aligned}
 \alpha_k &= \tilde{g}_{k-1}^T u_{k-1}, \\
 \tilde{\alpha}_k &= \tilde{g}_{k-1}^T \xi_1, \\
 \hat{\alpha}_k &= \tilde{g}_{k-1}^T \xi_2, \\
 \beta_k &= u_{k-1}^T \tilde{B}_{k-1} u_{k-1}, \\
 \tilde{\beta}_k &= u_{k-1}^T \tilde{B}_{k-1} \xi_1, \\
 \hat{\beta}_k &= u_{k-1}^T \tilde{B}_{k-1} \xi_2, \\
 \zeta_k &= u_{k-1}^T \tilde{B}_k u_{k-1}, \\
 \tilde{\zeta}_k &= u_{k-1}^T \tilde{B}_k \xi_1, \\
 \hat{\zeta}_k &= u_{k-1}^T \tilde{B}_k \xi_2, \\
 \gamma_k &= \tilde{g}_k^T u_{k-1}, \\
 \tilde{\gamma}_k &= \tilde{g}_k^T \xi_1, \\
 \hat{\gamma}_k &= \tilde{g}_k^T \xi_2,
 \end{aligned} \tag{62}$$

where the vectors ξ_1 and ξ_2 are chosen to satisfy

$$\tilde{\alpha}_k = \tilde{\gamma}_k = \hat{\zeta}_k = \hat{\gamma}_k = 0. \tag{63}$$

For convenience, we omit the index k of $\alpha_k, \tilde{\alpha}_k, \hat{\alpha}_k, \beta_k, \dots, \tilde{\gamma}_k$ and $\hat{\gamma}_k$.

On one hand, from (58) we have

$$1 - \tilde{c}_k^T u_{k-1} = \eta \tilde{v}_{k-1}, \tag{64}$$

where

$$\eta = \frac{\gamma + \sqrt{\gamma^2 + 2\zeta(\tilde{f}_{k-1} - \tilde{f}_k)}}{\zeta}. \tag{65}$$

If the sequence $\{\tilde{f}_k\}$ is monotonically decreasing and $\tilde{B}_k$ is positive definite, then we know that $\eta > 0$ and (64) becomes

$$1 - k_3\gamma = \eta(1 + k_1\alpha)(1 + k_2\beta). \tag{66}$$

On the other hand, by left-multiplying ξ_1^T on (59) and combining with (63), we have

$$\begin{aligned}
 (1 - k_3\gamma)(k_2(\tilde{\beta}\zeta - \beta\tilde{\zeta}) - \tilde{\zeta}) \\
 - (1 + k_1\alpha)(1 + k_2\beta)k_2\tilde{\beta}\gamma = 0.
 \end{aligned} \tag{67}$$

Then, from (66), we have

$$k_2 = \frac{\eta\tilde{\zeta}}{\tilde{\gamma}}, \tag{68}$$

where

$$\dot{\gamma} = \eta\tilde{\beta}\zeta - \eta\beta\tilde{\zeta} - \tilde{\beta}\gamma. \tag{69}$$

Similarly, by left multiplying ξ_2^T on (59), from (63) and (66) we have

$$k_1 = \frac{-((\tilde{\alpha}\beta + \tilde{\eta}\tilde{\beta})k_2 + \tilde{\alpha})}{(\alpha\tilde{\alpha}\beta + \tilde{\eta}(\tilde{\alpha}\beta + \alpha\tilde{\beta}))k_2 + \alpha\tilde{\alpha} + \tilde{\alpha}\tilde{\eta}}, \tag{70}$$

where $\tilde{\eta} = \gamma - \eta\zeta$. Substituting (68) into the above equation, we have

$$k_1 = -\frac{i}{\dot{\gamma}}, \tag{71}$$

where

$$\begin{aligned}
 i &= \tilde{\eta}(\tilde{\alpha}\tilde{\beta} - \eta^2\tilde{\zeta}\tilde{\beta}), \\
 \dot{\gamma} &= \alpha\tilde{\alpha} + \eta\tilde{\alpha}\tilde{\beta}\tilde{\eta}^2.
 \end{aligned} \tag{72}$$

And then from (66), we have

$$k_3 = \frac{1 - \eta(1 + k_1\alpha)(1 + k_2\beta)}{\gamma}. \tag{73}$$

Now we give the new quasi-Newton algorithm based on the fractional model (53).

Algorithm 7.

Step 0. Choose $x_0 \in R^n$, $\epsilon_1 \in (0, 1/3)$, $\epsilon > 0$, $\Delta_{\max} > 0$, $0 < \iota_1 < \iota_2 < 1$, $0 < \delta_1 < 1 < \delta_2$, $0 < \iota < 1$, $\tilde{B}_0 = I$, and the initial trust region radius $\Delta_0 \in (0, \Delta_{\max}]$. Compute Q_2 as defined in (15). Set $k = 0$.

Step 1 (stopping criterion). Compute $f_k = f(x_k)$, $g_k = \nabla f(x_k)$, and $\tilde{g}_k = Q_2^T g_k$. If $\|\tilde{g}_k\| \leq \epsilon$, then $x_* = x_k$, and stop. If $k = 0$, go to Step 3.

Step 2. Compute $y_{k-1} = \tilde{g}_k - \tilde{g}_{k-1}$. Update $\tilde{B}_k$ by

$$\tilde{B}_k = \tilde{B}_{k-1} - \frac{\tilde{B}_{k-1} u_{k-1} u_{k-1}^T \tilde{B}_{k-1}}{u_{k-1}^T \tilde{B}_{k-1} u_{k-1}} + \frac{z_{k-1} z_{k-1}^T}{z_{k-1}^T u_{k-1}}, \tag{74}$$

where

$$\begin{aligned}
 z_{k-1} &= \theta y_{k-1} + (1 - \theta) \tilde{B}_{k-1} u_{k-1}, \quad \theta \in [0, 1], \\
 \theta &= \begin{cases} 1, & \text{if } y_{k-1}^T u_{k-1} \geq 0.2 u_{k-1}^T \tilde{B}_{k-1} u_{k-1}, \\ \frac{0.8 u_{k-1}^T \tilde{B}_{k-1} u_{k-1}}{u_{k-1}^T \tilde{B}_{k-1} u_{k-1} - y_{k-1}^T u_{k-1}}, & \text{otherwise.} \end{cases}
 \end{aligned} \tag{75}$$

Step 3. If $k \leq 1$, then set $\tilde{a}_k = \tilde{b}_k = \tilde{c}_k = 0$, $\tilde{B}_k = Q_2^T B_k Q_2$, and $d_k = -\tilde{B}_k^{-1} \tilde{g}_k$, compute α_k such that Wolfe-Powell conditions are satisfied, and set $x_{k+1} = x_k + s_k = x_k + \alpha_k d_k$ and $u_k = Q_2^T s_k$, $k = k + 1$, and go to Step 1.

Step 4. By the parameters $\tilde{g}_{k-1}$, $\tilde{g}_k$, and $\tilde{B}_k u_{k-1}$ get ξ_1 and ξ_2 .

Step 5. Compute $\alpha, \tilde{\alpha}, \dots, \gamma, \tilde{\gamma}$ and $\tilde{\gamma}$ as defined in (62). If $\gamma = 0$, then set $\tilde{b}_k = \tilde{c}_k = 0$. Calculate

$$\begin{aligned} \rho_k &= (\tilde{f}_{k-1} - \tilde{f}_k)^2 - (\tilde{g}_{k-1}^T u_{k-1}) (\tilde{g}_k^T u_{k-1}), \\ \dot{\beta} &= \begin{cases} \frac{(\tilde{f}_{k-1} - \tilde{f}_k) + \sqrt{\rho_k}}{-\tilde{g}_{k-1}^T u_{k-1}}, & \text{if } \rho_k \geq 0, \\ 1, & \text{otherwise} \end{cases} \end{aligned} \quad (76)$$

and set

$$\tilde{a}_k = \frac{1 - \dot{\beta}}{\tilde{g}_{k-1}^T u_{k-1}} \tilde{g}_{k-1}. \quad (77)$$

Otherwise, compute $\dot{\gamma}$ and $\ddot{\gamma}$ as defined in (72) and (69). If $\dot{\gamma} = 0$ or $\ddot{\gamma} = 0$, then set $\tilde{a}_k = \tilde{b}_k = \tilde{c}_k = 0$. Otherwise, calculate $\tilde{a}_k$, $\tilde{b}_k$, and $\tilde{c}_k$, where k_1, k_2 , and k_3 in (61) are determined by (71), (68), and (73).

Step 6. If $\|\tilde{a}_k\| > \epsilon_1 / \tilde{\Delta}_k$, then $\tilde{a}_k = \epsilon_1 a_k / \tilde{\Delta}_k \|a_k\|$. Update $\tilde{b}_k$ and $\tilde{c}_k$ with the same way such that (5) are satisfied.

Step 7. By the parameters $\tilde{a}_k$, $\tilde{b}_k$, $\tilde{c}_k$, $\tilde{g}_k$, and $\tilde{\Delta}_k$, solve the subproblem (18)-(19) by Algorithm 5 to get u_k . Set $s_k = Q_2 u_k$.

Step 8. Compute

$$\rho_k = \frac{\text{Ared}(u_k)}{\text{Pred}(u_k)}, \quad (78)$$

where

$$\text{Ared}(u_k) = \text{Ared}(s_k) = f(x_k) - f(x_k + s_k), \quad (79)$$

$$\text{Pred}(u_k) = m_k(0) - m_k(u_k) = -\tilde{\psi}_k(u_k). \quad (80)$$

Step 9. Update the trust region radius:

$$\tilde{\Delta}_{k+1} = \begin{cases} \delta_1 \tilde{\Delta}_k, & \text{if } \rho_k \leq \iota_1, \\ \min\{\delta_2 \tilde{\Delta}_k, \Delta_{\max}\}, & \text{if } \rho_k \geq \iota_2, \|u_k\| = \tilde{\Delta}_k, \\ \tilde{\Delta}_k, & \text{otherwise.} \end{cases} \quad (81)$$

Step 10. If $\rho_k \geq \iota_1$, then $x_{k+1} = x_k + s_k$. Set $k = k + 1$, and go to Step 1. Otherwise, $x_{k+1} = x_k$, $k = k + 1$, and go to Step 6.

Next we present the global convergence theorem which says the reduced gradients converge to zero.

Theorem 8. Assume that (5) and (35) hold, where $\epsilon_1 \in (0, 1/3)$. If f is continuously differentiable and bounded below in some set containing all iterations generated by Algorithm 7, the sequences $\{\|\tilde{g}_k\|\}$ and $\{\|\tilde{B}_k\|\}$ are uniformly bounded. Then,

$$\liminf_{k \rightarrow \infty} \|\tilde{g}_k\| = 0. \quad (82)$$

Proof. Assume that the theorem is false and there is $\bar{\epsilon} > 0$ such that $\tilde{g}_k \geq \bar{\epsilon}$ for all k . From the assumption, we can assume that

$$\begin{aligned} \|\tilde{a}_k\| &\leq \bar{a}, \\ \|\tilde{b}_k\| &\leq \bar{b}, \\ \|\tilde{c}_k\| &\leq \bar{c}, \\ \|\tilde{g}_k\| &\leq \bar{g}, \\ 0 &< \|\tilde{B}_k\| \leq \bar{B}, \\ 0 &< \|\tilde{B}_k^{-1}\| \leq \bar{B} \end{aligned} \quad (83)$$

hold for all k . From (52) and the assumptions in the theorem, we have

$$\text{pred}(s_k) \geq \frac{\bar{\epsilon}}{2\bar{B}} \min\left\{\bar{\epsilon}, \frac{\xi \tilde{\Delta}_k}{\bar{B}}\right\} \geq c_1 \min\{\tilde{\Delta}_k, c_2\}, \quad (84)$$

where c_1 and c_2 are some positive constants. Then, from Step 10 of Algorithm 7, we have

$$f_k - f_{k+1} \geq \iota_1 \text{pred}(s_k) \geq \iota_1 c_1 \min\{\tilde{\Delta}_k, c_2\}. \quad (85)$$

Since $f(x)$ is bounded from below and $f_{k+1} < f_k$ for all k , we have that $\sum_{k=1}^{\infty} \min\{\tilde{\Delta}_k, c_2\}$ is convergent, and $\tilde{\Delta}_k \rightarrow 0$ as $k \rightarrow \infty$.

On the other hand, when $\tilde{\Delta}_k \rightarrow 0$, $\|u_k\| \rightarrow 0$. From Step 6 of Algorithm 7, we have

$$\begin{aligned} |\tilde{a}_k^T u_k| &\leq \|\tilde{a}_k\| \tilde{\Delta}_k \leq \epsilon_1, \\ |\tilde{b}_k^T u_k| &\leq \epsilon_1, \end{aligned} \quad (86)$$

where $\epsilon_1 \in (0, 1/3)$. Thus, we have

$$\begin{aligned} \frac{1}{1 - \tilde{a}_k^T u_k} &= 1 + \tilde{a}_k^T u_k + o(\|u_k\|), \\ \frac{1}{1 - \tilde{b}_k^T u_k} &= 1 + \tilde{b}_k^T u_k + o(\|u_k\|). \end{aligned} \quad (87)$$

TABLE 1: Test functions.

Pro.	Function name
1	HS 9
2	HS 48
3	HS 49
4	HS 50
5	Chen 3.3.1
6	Chen 3.3.2
7	Cube
8	Penalty-I
9	Beale
10	Conic
11	Extended Powell
12	Broyden Tridiagonal
13	Rosenbrock
14	Chained Freudenstein and Roth
15	Extended Trigonometric
16	Penalty-III
17	Troesch
18	Cragg and Levy

By computing, we obtain

$$\frac{(1 + \tilde{c}_k^T u_k) \tilde{g}_k^T u_k}{(1 - \tilde{a}_k^T u_k)(1 - \tilde{b}_k^T u_k)} = g_k^T u_k + O(\|u_k\|^2),$$

$$\frac{(1 + \tilde{c}_k^T u_k)^2 u_k^T \tilde{B}_k u_k}{(1 - \tilde{a}_k^T u_k)^2 (1 - \tilde{b}_k^T u_k)^2} = u_k^T \tilde{B}_k u_k + o(\|u_k\|^2), \quad (88)$$

$$\text{Ared}(u_k) = -g_k^T u_k - u_k^T \tilde{B}_k u_k + o(\|u_k\|^2).$$

Then, from (80), we have

$$\text{Pred}(u_k) = \text{Ared}(u_k) + O(\|\tilde{\Delta}_k\|^2), \quad (89)$$

which indicates that

$$\lim_{k \rightarrow \infty} \rho_k = 1. \quad (90)$$

By the updating in Step 9 of Algorithm 7, we have $\tilde{\Delta}_{k+1} \geq \tilde{\Delta}_k$, which is a contradiction to $\tilde{\Delta}_k \rightarrow 0$. The theorem is proved. $\square$

5. Numerical Tests

In this section, Algorithm 7 (abbreviated as FTR) is tested with some test problems which are chosen from [16, 17]. These test problems are listed in Table 1. We choose linear constrained problems HS9, HS48, HS49, HS50, Chen 3.3.1,

and Chen 3.3.2. Moreover, in order to test Algorithm 7 more generally, we designed some problems where the objective functions are Pro. 7–18 (see [14, 18]) and the linear equality constraints are Pro. 1–6. If $\tilde{b}_k = \tilde{c}_k = 0$ in Algorithm 7, we can obtain the conic model algorithm and call this algorithm CTR. We solve the following 18 test problems by FTR and CTR and compare their results.

All the computations are carried out in Matlab R2012b on a microcomputer in double precision arithmetic. These tests use the same stopping criterion $\|\tilde{g}_k\| \leq 10^{-4}$. The columns in the tables have the following meanings: Pro. denotes the numbers of the test problems; n is the dimension of the test problems; Iter is the number of iterations; nf and ng are the numbers of function and gradient evaluations, respectively; $\|\tilde{g}\|$ is the Euclidean norm of the final reduced gradient; CPU(s) denotes the total iteration time of the algorithm in seconds. The parameters in these algorithms are

$$\begin{aligned} B_0 &= I, \\ \epsilon_1 &= 0.33, \\ \Delta_0 &= 1, \\ \Delta_{\max} &= 10, \\ l &= 0.15, \\ \mu &= 0.85, \\ \iota_1 &= 0.25, \\ \iota_2 &= 0.75, \\ \delta_2 &= 4. \end{aligned} \quad (91)$$

The numerical comparison for 18 small-scale test problems is listed in Table 2. We can see that FTR is better than CTR for 15 tests in the number of iterations and the remaining 3 tests are similar. Because FTR needs some extra algebra computation for some parameters, FTR takes more time than CTR for small problems.

The numerical results of some large-scale problems are presented in Table 3. From Table 3, we find that for large-scale problems the CPU time of FTR is approximately the same as that of CTR but it has fewer number of iterations. From the above comparison, we see that FTR is slightly more effective and robust for these large-scale test problems.

The fractional model in Algorithm 7 is the extension of conic model. By using more information of function and gradient from the previous iterations and choosing parameters flexibly, the fractional model can be more approximate to the original problem. And the global convergence of the proposed quasi-Newton trust region algorithm is also proved. Numerical experiment shows the algorithm is effective and robust, including for large-scale test problems. The theoretical results and the numerical results lead us to believe that the method is worthy of further study. For example, we can consider using fractional model to solve the nonlinear equality constrained optimization problem.

TABLE 2: The numerical results of Algorithm 7 for some test problems.

Pro.	n	Starting point	Algorithm	Iter	nf/ng	$\ g\ $	CPU (s)
1	6	(0, 0, 0, 0, 0, 0)	CTR	6	7/8	7.559746(-5)	0.084639
			FTR	5	6/6	1.693174(-5)	0.161438
2	5	(3, 5, -3, 2, -2)	CTR	13	14/9	1.637781(-5)	0.094287
			FTR	8	9/9	2.148387(-6)	0.098268
3	5	(2, 2, -9, 3, 3)	CTR	27	28/21	5.815109(-5)	0.110429
			FTR	25	26/17	4.996278(-5)	0.149000
4	10	(0, -6, 6, 0, 0, 0, -6, 6, 0, 0)	CTR	12	13/13	3.889891(-5)	0.100359
			FTR	8	9/9	2.397640(-5)	0.152117
5	8	(0, 3, -1, -2, 0, 3, -1, -2)	CTR	16	17/14	7.255223(-5)	0.099120
			FTR	16	17/17	1.249467(-5)	0.129272
6	8	(7/3, 11/3, -1, -1, 7/3, 11/3, -1, -1)	CTR	22	23/19	3.128858(-6)	0.101571
			FTR	21	22/19	2.865070(-7)	0.167233
7	6	(3, 4, 3, 4, 3, 4)	CTR	8	9/9	2.208537(-7)	0.093060
			FTR	8	9/9	1.765690(-8)	0.139521
8	6	(3, 4, 3, 4, 3, 4)	CTR	14	15/11	8.493218(-7)	0.095873
			FTR	12	13/10	1.155176(-5)	0.166608
9	6	(6, 8, 6, 8, 6, 8)	CTR	21	22/18	1.208648(-6)	0.094048
			FTR	19	20/18	3.705893(-5)	0.170549
10	6	-(1.5, 2, 1.5, 2, 1.5, 2)	CTR	13	14/10	1.789828(-6)	0.104145
			FTR	9	10/7	4.017095(-6)	0.158877
11	8	(0.3, 0.4, ..., 0.3, 0.4)	CTR	15	16/13	3.013621(-5)	0.087370
			FTR	7	8/8	7.252083(-5)	0.121116
12	6	(0.3, 0.4, ..., 0.3, 0.4)	CTR	26	27/16	5.317180(-5)	0.1160001
			FTR	24	25/14	5.917188(-5)	0.187447
13	6	(1.5, 2, 1.5, 2, 1.5, 2)	CTR	6	7/7	8.707409(-5)	0.081667
			FTR	6	7/7	5.998420(-5)	0.115520
14	6	(6, 8, 6, 8, 6, 8)	CTR	34	35/20	3.185767(-5)	0.106515
			FTR	22	23/23	4.845919(-6)	0.158603
15	8	(0.3, 0.4, ..., 0.3, 0.4)	CTR	10	11/9	1.738762(-5)	0.092062
			FTR	9	10/8	5.005195(-5)	0.146342
16	6	(0.3, 0.4, ..., 0.3, 0.4)	CTR	13	14/12	3.409181(-5)	0.083445
			FTR	12	13/12	9.332787(-5)	0.164178
17	6	(0.3, 0.4, ..., 0.3, 0.4)	CTR	29	30/17	1.845528(-6)	0.088997
			FTR	19	20/17	5.983264(-6)	0.159851
18	8	(0.3, 0.4, ..., 0.3, 0.4)	CTR	12	13/9	7.722466(-7)	0.087020
			FTR	9	10/9	4.353866(-7)	0.149733

TABLE 3: The numerical results of Algorithm 7 for some test problems ($n = 3000$).

Pro.	Starting point	Algorithm	Iter	nf/ng	$\ g\ $	CPU (s)
7	(6, 8, ..., 6, 8)	CTR	25	26/26	3.737202(-9)	28.5976
		FTR	21	22/22	3.849243(-9)	26.2795
8	(6, 8, ..., 6, 8)	CTR	12	13/12	8.930053(-6)	13.8295
		FTR	12	13/11	3.113077(-7)	14.7506
9	(6, 8, ..., 6, 8)	CTR	25	26/26	3.388081(-6)	28.4904
		FTR	24	25/25	1.176732(-6)	30.7385
10	(6, 8, ..., 6, 8)	CTR	12	13/13	1.226943(-7)	14.9229
		FTR	10	11/11	6.842114(-6)	14.9369
11	(6, 8, ..., 6, 8)	CTR	18	19/19	2.718913(-8)	21.4796
		FTR	4	5/5	5.738665(-5)	8.48251

Competing Interests

The authors have no competing interests regarding this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant no. 11071117 and 71301060), the Natural Science Foundation of Jiangsu Province (BK20141409), Funding of Jiangsu Innovation Program for Graduate Education (KYZZ-0089) ("the Fundamental Research Funds for the Central Universities"), and the Humanistic and Social Science Foundation of Ministry of Education of China (12YJA630122).

References

- [1] W. C. Davidon, "Conic approximations and collinear scalings for optimizers," *SIAM Journal on Numerical Analysis*, vol. 17, no. 2, pp. 268–281, 1980.
- [2] R. Schnabel, "Conic methods for unconstrained minimization and tensor methods for nonlinear equations," in *Mathematical Programming: The State of the Art*, A. Bachem, M. Grötschel, and B. Korte, Eds., pp. 417–438, Springer, Heidelberg, Germany, 1982.
- [3] D. C. Sorensen, "Newton's method with a model trust region modification," *SIAM Journal on Numerical Analysis*, vol. 19, no. 2, pp. 409–426, 1982.
- [4] W. Sun and Y. X. Yuan, "A conic trust-region method for nonlinearly constrained optimization," *Annals of Operations Research*, vol. 103, pp. 175–191, 2001.
- [5] C. X. Xu and X. Y. Yang, "Convergence of conic quasi-Newton trust region methods for unconstrained minimization," *Mathematical Application*, vol. 11, no. 2, pp. 71–76, 1998.
- [6] Y. X. Yuan, "A review of trust region algorithms for optimization," in *Proceedings of the International Congress on Industrial and Applied Mathematics (ICIAM '00)*, vol. 99, pp. 271–282, 2000.
- [7] D. M. Gay, "Computing optimal locally constrained steps," *SIAM Journal on Scientific and Statistical Computing*, vol. 2, no. 2, pp. 186–197, 1981.
- [8] J.-M. Peng and Y.-X. Yuan, "Optimality conditions for the minimization of a quadratic with two quadratic constraints," *SIAM Journal on Optimization*, vol. 7, no. 3, pp. 579–594, 1997.
- [9] H. L. Zhu, Q. Ni, and M. L. Zeng, "A quasi-Newton trust region method based on a new fractional model," *Numerical Algebra, Control and Optimization*, vol. 5, no. 3, pp. 237–249, 2015.
- [10] W. Y. Sun, J. Y. Yuan, and Y. X. Yuan, "Conic trust region method for linearly constrained optimization," *Journal of Computational Mathematics*, vol. 21, no. 3, pp. 295–304, 2003.
- [11] X. P. Lu and Q. Ni, "A trust region method with new conic model for linearly constrained optimization," *Or Transactions*, vol. 12, pp. 32–42, 2008.
- [12] Q. Ni, *Optimization Method and Program Design*, Science Press, Beijing, China, 2009.
- [13] L. W. Zhang and Q. Ni, "Trust region algorithm of new conic model for nonlinearly equality constrained optimization," *Journal on Numerical Methods and Computer Applications*, vol. 31, no. 4, pp. 279–289, 2010.
- [14] M. F. Zhu, Y. Xue, and F. S. Zhang, "A quasi-Newton type trust region method based on the conic model," *Numerical Mathematics*, vol. 17, no. 1, pp. 36–47, 1995 (Chinese).
- [15] X. P. Lu and Q. Ni, "A quasi-Newton trust region method with a new conic model for the unconstrained optimization," *Applied Mathematics and Computation*, vol. 204, no. 1, pp. 373–384, 2008.
- [16] W. Hock and K. Schittkowski, *Test Examples for Nonlinear Programming Codes*, Springer, Berlin, Germany, 1981.
- [17] X. Y. Chen, *Research on the geometric algorithms for programs with constraints of linear equalities [M.S. thesis]*, Fujian Normal University, 2012.
- [18] J. J. More, B. S. Garbow, and K. E. Hillstom, "Testing unconstrained optimization software," *ACM Transactions on Mathematical Software*, vol. 7, no. 1, pp. 17–41, 1981.

Hindawi

Submit your manuscripts at
<http://www.hindawi.com>