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Research Article

On a Five-Dimensional Nonautonomous Schistosomiasis Model with Latent Period

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A five-dimensional nonautonomous schistosomiasis model which include latent period is proposed and studied. By constructing several auxiliary functions and using some skills, we obtain some sufficient conditions for the extinction and permanence (uniform persistence) of infectious population of the model. New threshold values of integral form are obtained. For the corresponding autonomous schistosomiasis model, our results are consistent with the past results. For the periodic and almost periodic cases, some corollaries for the extinction and permanence of the disease are established. In order to illustrate our theoretical analysis, some numerical simulations are presented.

1. Introduction

Schistosomiasis is a chronic, parasitic disease caused by blood flukes (trematode worms) of the genus *Schistosoma*. More than 207 million people, 85% of them living in Africa, are infected worldwide, with an estimated 700 million people at risk in 74 endemic countries. Schistosomiasis is prevalent in tropical and subtropical areas, especially in poor communities without access to safe drinking water and adequate sanitation [1]. In the developing countries, schistosomiasis is frequently a serious health problem [2].

The first mathematical models for schistosomiasis were those developed by Macdonald in [3] and Hairston in [4]. Since then, a number of mathematical models have been proposed. Barbour [5] studied the prevalence of infection in the definitive host population and the snails, Chiyaka and Garira [6] were concerned with the combined effect of the inclusion the dynamics of the miracidia and cercariae populations and a density dependent infection term, Bhunu et al. [7] developed a mathematical model of the transmission dynamics of HIV/AIDS in the presence of schistosomiasis in the human-snail hosts, Yang and Xiao [8] presented a dynamic model of *Schistosoma japonicum* transmission that incorporated effects of the prepatent periods of the different stages of *Schistosoma* into Barbour's model, Qi and Cui [9] established a diffusive system of partial differential equations considering the diffusion of human and snail hosts, and Qi et al. [10] developed a schistosomiasis model of the prevalence of infection in the *Rattus norvegicus* and *Oncomelania* snail, in which the authors considered a schistosomiasis model on the *Rattus norvegicus* in Qianzhou and Zimuzhou, two islets in the center of Yangtze River near Nanjing, China. There have been no human residents or other livestock on the islets; thus the mammalian *Rattus norvegicus* is the only definitive host while snail serves as the unique intermediate host; the authors set up the compartmental model as follows:

$$\frac{dx_s}{dt} = A_x - \mu_x x_s - \beta_x x_s y_i,$$

$$\frac{dx_i}{dt} = \beta_x x_s y_i - \mu_x x_i - \alpha_x x_i,$$

$$\frac{dy_s}{dt} = A_y - \mu_y y_s - \beta_y x_i y_s,$$

$$\frac{dy_e}{dt} = \beta_y x_i y_s - \left(\mu_y + \theta\right) y_e,$$
$$\frac{dy_i}{dt} = \theta y_e - \mu_y y_i - \alpha_y y_i. \tag{1}$$

Here, x_s, x_i represent the number of susceptible and infected *Rattus norvegicus*, respectively. y_s, y_e, y_i are the number of susceptible, infected and preshedding, and infected and shedding *Oncomelania* snail, respectively. The other parameters interpretation of model (1) is shown in Parameters Section. All the parameters in the model are assumed to be nonnegative constants.

According to the concept of the next generation of matrix [11], the authors obtain the basic reproduction number [12] of system (1) as follows:

$$R_0 = \sqrt[3]{\frac{A_x A_y \theta \beta_x \beta_y}{\mu_x \mu_y \left(\mu_x + \alpha_x\right)\left(\mu_y + \alpha_y\right)\left(\mu_y + \theta\right)}}. \tag{2}$$

And they prove that the infection will go extinct if $R_0 \leq 1$ and schistosomiasis will be prevalent if $R_0 > 1$.

Nonautonomous phenomena, however, are dominating in real epidemic systems. Many diseases show seasonal behavior and taking into account the seasonal influence in epidemic models is very necessary. The nonautonomous phenomenon come from various sources, such as the variation of transmission rate and fluctuations in death and birth rates. Particularly, these coefficients of models usually vary with time when long-time dynamical behaviors are studied for an epidemic system. More and more realistic and interesting nonautonomous models can be found in papers [13–24] and the references cited therein.

Motivated by the ideas of the above aspects, system (1) is modified to take into account factors mentioned above where all coefficients vary with time to give a more appropriate result and better understanding of the prevalence of schistosomiasis. Thus, we refer to the following nonautonomous system for the study of schistosomiasis:

$$\begin{aligned}
\frac{dx_s(t)}{dt} &= A_x(t) - \mu_x(t) x_s(t) - \beta_x(t) x_s(t) y_i(t), \\
\frac{dx_i(t)}{dt} &= \beta_x(t) x_s(t) y_i(t) - \mu_x(t) x_i(t) \\
&\quad - \alpha_x(t) x_i(t), \\
\frac{dy_s(t)}{dt} &= A_y(t) - \mu_y(t) y_s(t) - \beta_y(t) x_i(t) y_s(t), \\
\frac{dy_e(t)}{dt} &= \beta_y(t) x_i(t) y_s(t) - \left(\mu_y(t) + \theta(t)\right) y_e(t), \\
\frac{dy_i(t)}{dt} &= \theta(t) y_e(t) - \mu_y(t) y_i(t) - \alpha_y(t) y_i(t)
\end{aligned} \tag{3}$$

with initial value

$$\begin{aligned}
x_s(0) &> 0, \\
x_i(0) &> 0, \\
y_s(0) &> 0, \\
y_e(0) &\geq 0, \\
y_i(0) &> 0.
\end{aligned} \tag{4}$$

In this paper, our purpose is to obtain the weaker conditions for the permanence and extinction of schistosomiasis. We will establish the sufficient conditions for the permanence and extinction of the disease and give some new threshold values of the integral form. When system (3) reduces to periodic or almost periodic case, the basic reproduction number is obtained (see Corollaries 18 and 19 given in Section 5). Our results show that the threshold value acts as a sharp threshold for the permanence and extinction of the disease.

This paper is organized as follows. In Section 2 we present preliminary setting and propositions which are used to analyze the long-time behavior of system (3) in the following sections. In Sections 3 and 4, we prove our main theorems on the extinction and permanence of infectious population of system (3). In Section 5, we derive some corollaries for the extinction and permanence of infectious population of system (3) for some special cases. In Section 6, we provide numerical examples to illustrate the validity of our analytical results.

2. Notations and Preliminaries

At first, we put the following assumptions for system (3).

(H1) Functions $A_x(t)$, $\mu_x(t)$, $\beta_x(t)$, $\alpha_x(t)$, $A_y(t)$, $\mu_y(t)$, $\beta_y(t)$, $\alpha_y(t)$, and $\theta(t)$ are positive, bounded, and continuous on $[0, +\infty)$.

(H2) There exist constants $\omega_i > 0$ $(i = 1, 2, 3, 4, 5, 6)$ such that

$$\begin{aligned}
\liminf_{t\to+\infty} \int_t^{t+\omega_1} A_x(s)\, ds &> 0, \\
\liminf_{t\to+\infty} \int_t^{t+\omega_2} A_y(s)\, ds &> 0, \\
\liminf_{t\to+\infty} \int_t^{t+\omega_3} \mu_x(s)\, ds &> 0, \\
\liminf_{t\to+\infty} \int_t^{t+\omega_4} \mu_y(s)\, ds &> 0, \\
\liminf_{t\to+\infty} \int_t^{t+\omega_5} \beta_x(s)\, ds &> 0, \\
\liminf_{t\to+\infty} \int_t^{t+\omega_6} \beta_y(s)\, ds &> 0.
\end{aligned} \tag{5}$$

For any continuous periodic function f with period $\omega > 0$, we denote by $\overline{f}$ the average value of $f(t)$ where $\overline{f} = (1/\omega)\int_0^\omega f(t)dt$.

For any continuous almost periodic function f, we denote by $m(f)$ the average value of $f(t)$ where $m(f) = \lim_{t\to+\infty}(1/t)\int_0^t f(t)\,dt$.

Remark 1. When system (3) degenerates into ω-periodic system, that is to say, all the coefficients of system (3) are all nonnegative and nonzero, continuous periodic functions with period $\omega > 0$, then (H2) is equivalent to the following cases:

$$\begin{aligned} &\overline{A}_x > 0,\\ &\overline{A}_y > 0,\\ &\overline{\mu}_x > 0,\\ &\overline{\mu}_y > 0,\\ &\overline{\beta}_x > 0,\\ &\overline{\beta}_y > 0. \end{aligned} \tag{6}$$

Remark 2. When system (3) degenerates into almost periodic system, that is to say, all the coefficients in it are all nonnegative and nonzero, continuous almost periodic functions, then (H2) is equivalent to the following cases:

$$\begin{aligned} &m(A_x) > 0,\\ &m(A_y) > 0,\\ &m(\mu_x) > 0,\\ &m(\mu_y) > 0,\\ &m(\beta_x) > 0,\\ &m(\beta_y) > 0. \end{aligned} \tag{7}$$

In what follows, we denote $N_x(t) := x_s(t) + x_i(t)$, $N_y(t) := y_s(t) + y_e(t) + y_i(t)$. We also define that $\underline{N}_x(t)$, $\overline{N}_x(t)$, $\underline{N}_y(t)$, $\overline{N}_y(t)$, $\underline{x}_s(t)$, $\underline{y}_s(t)$ are the solutions of the following equations with initial value $\underline{x}_s(0) = x_s(0) > 0$, $\underline{y}_s(0) = y_s(0) > 0$, $\underline{N}_x(0) = \overline{N}_x(0) = N_x(0) = x_s(0) + x_i(0)$, $\underline{N}_y(0) = \overline{N}_y(0) = N_y(0) = y_s(0) + y_e(0) + y_i(0)$, respectively:

$$\frac{dN_x(t)}{dt} = A_x(t) - (\mu_x(t) + \alpha_x(t))N_x(t), \tag{8}$$

$$\frac{dN_x(t)}{dt} = A_x(t) - \mu_x(t)N_x(t), \tag{9}$$

$$\frac{dN_y(t)}{dt} = A_y(t) - (\mu_y(t) + \alpha_y(t))N_y(t), \tag{10}$$

$$\frac{dN_y(t)}{dt} = A_y(t) - \mu_y(t)N_y(t), \tag{11}$$

$$\frac{dx_s(t)}{dt} = A_x(t) - \left(\mu_x(t) + \beta_x(t)\overline{N}_y(t)\right)x_s(t), \tag{12}$$

$$\frac{dy_s(t)}{dt} = A_y(t) - \left(\mu_y(t) + \beta_y(t)\overline{N}_x(t)\right)y_s(t). \tag{13}$$

Obviously, from system (3) and the comparison theorem, we have

$$\begin{aligned} &\underline{N}_x(t) \le N_x(t) \le \overline{N}_x(t),\\ &\underline{N}_y(t) \le N_y(t) \le \overline{N}_y(t),\\ &\underline{x}_s(t) \le x_s(t) \le \overline{N}_x(t),\\ &\underline{y}_s(t) \le y_s(t) \le \overline{N}_y(t). \end{aligned} \tag{14}$$

Furthermore, we have the following result.

Lemma 3. *Suppose that assumptions (H1) and (H2) hold. Then the following results hold.*

(a) *There exist constants $m_1 > 0$ and $M_1 > 0$, which are independent of the choice of initial value $N_x(0) > 0$, such that*

$$\begin{aligned} 0 < m_1 &< \liminf_{t\to+\infty}\underline{N}_x(t) \le \liminf_{t\to+\infty}N_x(t)\\ &\le \limsup_{t\to+\infty}N_x(t) \le \limsup_{t\to+\infty}\overline{N}_x(t) < M_1 < +\infty. \end{aligned} \tag{15}$$

(b) *There exist constants $m_2 > 0$ and $M_2 > 0$, which are independent of the choice of initial value $N_y(0) > 0$, such that*

$$\begin{aligned} 0 < m_2 &< \liminf_{t\to+\infty}\underline{N}_y(t) \le \liminf_{t\to+\infty}N_y(t)\\ &\le \limsup_{t\to+\infty}N_y(t) \le \limsup_{t\to+\infty}\overline{N}_y(t) < M_2 < +\infty. \end{aligned} \tag{16}$$

(c) *Each fixed solution $\underline{N}_x(t)$ of (8), $\overline{N}_x(t)$ of (9), $\underline{N}_y(t)$ of (10), and $\overline{N}_y(t)$ of (11) is bounded and globally uniformly attractive on R_+.*

(d) *When (8) is ω-periodic, then (8) has a unique nonnegative ω-periodic solution $\underline{N}_x(t)$ which is globally uniformly attractive. And (9)–(11) have similar results.*

(e) *When (8) is almost periodic, then (8) has a unique nonnegative almost periodic solution $\underline{N}_x(t)$ which is globally uniformly attractive. And (9)–(11) have similar results.*

(f) *If $\mu_x(t) > 0$ for all $t \ge 0$ and $0 < \liminf_{t\to\infty}(A_x(t)/(\mu_x(t)+\alpha_x(t))) \le \limsup_{t\to\infty}(A_x(t)/(\mu_x(t)+\alpha_x(t))) < \infty$, then for any solution $\underline{N}_x(t)$ of (8), we have*

$$\begin{aligned} \left(\frac{A_x}{\mu_x+\alpha_x}\right)^m &\le \liminf_{t\to+\infty}\underline{N}_x(t) \le \limsup_{t\to+\infty}\underline{N}_x(t)\\ &\le \left(\frac{A_x}{\mu_x+\alpha_x}\right)^M, \end{aligned} \tag{17}$$

where

$$\left(\frac{A_x}{\mu_x+\alpha_x}\right)^m = \liminf_{t\to+\infty}\frac{A_x(t)}{\mu_x(t)+\alpha_x(t)},$$
$$\left(\frac{A_x}{\mu_x+\alpha_x}\right)^M = \limsup_{t\to+\infty}\frac{A_x(t)}{\mu_x(t)+\alpha_x(t)}. \tag{18}$$

(g) *If $\mu_x(t) > 0$ for all $t \geq 0$ and $0 < \liminf_{t\to\infty}(A_x(t)/\mu_x(t)) \leq \limsup_{t\to\infty}(A_x(t)/\mu_x(t)) < \infty$, then, for any solution $\overline{N}_x(t)$ of (9), we have*

$$\left(\frac{A_x}{\mu_x}\right)^m \leq \liminf_{t\to+\infty}\overline{N}_x(t) \leq \limsup_{t\to+\infty}\overline{N}_x(t) \leq \left(\frac{A_x}{\mu_x}\right)^M, \tag{19}$$

where

$$\left(\frac{A_x}{\mu_x}\right)^m = \liminf_{t\to+\infty}\frac{A_x(t)}{\mu_x(t)},$$
$$\left(\frac{A_x}{\mu_x}\right)^M = \limsup_{t\to+\infty}\frac{A_x(t)}{\mu_x(t)}. \tag{20}$$

(h) *If $\mu_y(t) > 0$ for all $t \geq 0$ and $0 < \liminf_{t\to\infty}(A_y(t)/(\mu_y(t)+\alpha_y(t))) \leq \limsup_{t\to\infty}(A_y(t)/(\mu_y(t)+\alpha_y(t))) < \infty$, then, for any solution $\underline{N}_y(t)$ of (10), we have*

$$\left(\frac{A_y}{\mu_y+\alpha_y}\right)^m \leq \liminf_{t\to+\infty}\underline{N}_y(t) \leq \limsup_{t\to+\infty}\underline{N}_y(t) \leq \left(\frac{A_y}{\mu_y+\alpha_y}\right)^M, \tag{21}$$

where

$$\left(\frac{A_y}{\mu_y+\alpha_y}\right)^m = \liminf_{t\to+\infty}\frac{A_y(t)}{\mu_y(t)+\alpha_y(t)},$$
$$\left(\frac{A_y}{\mu_y+\alpha_y}\right)^M = \limsup_{t\to+\infty}\frac{A_y(t)}{\mu_y(t)+\alpha_y(t)}. \tag{22}$$

(i) *If $\mu_y(t) > 0$ for all $t \geq 0$ and $0 < \liminf_{t\to\infty}(A_y(t)/\mu_y(t)) \leq \limsup_{t\to\infty}(A_y(t)/\mu_y(t)) < \infty$, then, for any solution $\overline{N}_y(t)$ of (11), we have*

$$\left(\frac{A_y}{\mu_y}\right)^m \leq \liminf_{t\to+\infty}\overline{N}_y(t) \leq \limsup_{t\to+\infty}\overline{N}_y(t) \leq \left(\frac{A_y}{\mu_y}\right)^M, \tag{23}$$

where

$$\left(\frac{A_y}{\mu_y}\right)^m = \liminf_{t\to+\infty}\frac{A_y(t)}{\mu_y(t)},$$
$$\left(\frac{A_y}{\mu_y}\right)^M = \limsup_{t\to+\infty}\frac{A_y(t)}{\mu_y(t)}. \tag{24}$$

Lemma 4. *Suppose that assumptions (H1) and (H2) hold. Then the solution $(x_s(t), x_i(t), y_s(t), y_e(t), y_i(t))$ of system (3) with initial value (3) exists and is uniformly bounded. Furthermore, for any $t > 0$, we have*

$$\begin{aligned} x_s(t) &> 0,\\ x_i(t) &> 0,\\ y_s(t) &> 0,\\ y_e(t) &\geq 0,\\ y_i(t) &> 0. \end{aligned} \tag{25}$$

For any $p > 0$, $q > 0$, and $t > 0$ we define

$$\begin{aligned} G_1(p,t) &= \left(1+\frac{1}{p}\right)\theta(t) - \alpha_y(t),\\ G_2(p,q,t) &= \frac{1}{q}\beta_x(t)\overline{N}_x(t) + \mu_y(t) + \alpha_y(t)\\ &\quad - \frac{1}{p}\theta(t) - \mu_x(t) - \alpha_x(t),\\ G_2'(p,q,t) &= \frac{1}{q}\beta_x(t)\underline{x}_s(t) + \mu_y(t) + \alpha_y(t) - \frac{1}{p}\theta(t)\\ &\quad - \mu_x(t) - \alpha_x(t),\\ W_1(p,t) &= py_e(t) - y_i(t),\\ W_2(q,t) &= qy_i(t) - x_i(t), \end{aligned} \tag{26}$$

where $(x_s, x_i, y_s, y_e, y_i)$ is any solution of system (3).

We use the following two results in order to investigate the long-time behavior of system (3).

Lemma 5. *If there exist positive constants $p > 0$ and $T_1 > 0$ such that $G_1(p,t) < 0$ for all $t \geq T_1$, then there exists $T_2 \geq T_1$ such that either $W_1(p,t) > 0$ for all $t \geq T_2$ or $W_1(p,t) \leq 0$ for all $t \geq T_2$.*

Proof. Suppose that there does not exist $T_2 \geq T_1$ such that $W_1(p,t) > 0$ for all $t \geq T_2$ or $W_1(p,t) \leq 0$ for all $t \geq T_2$ hold. Then there necessarily exists $s_1 \geq T_1$ such that $W_1(p,s_1) = 0$ and $(dW_1(p,t)/dt)|_{t=s_1} < 0$. Hence we have

$$py_e(s_1) = y_i(s_1), \tag{27}$$

$$\begin{aligned}
&\left.\frac{dW_1(p,t)}{dt}\right|_{t=s_1} \\
&\quad = p\beta_y(s_1)x_i(s_1)y_s(s_1) \\
&\qquad - p\left(\mu_y(s_1)+\theta(s_1)\right)y_e(s_1)-\theta(s_1)y_e(s_1) \\
&\qquad + \left(\mu_y(s_1)+\alpha_y(s_1)\right)y_i(s_1) \\
&\quad > \left(\mu_y(s_1)+\alpha_y(s_1)\right)y_i(s_1) \\
&\qquad - \left[\mu_y(s_1)+\left(1+\frac{1}{p}\right)\theta(s_1)\right]py_e(s_1).
\end{aligned} \tag{28}$$

Substituting (27) into (28), we have

$$\begin{aligned}
0 &> -\left\{\left(1+\frac{1}{p}\right)\theta(s_1)-\alpha_y(s_1)\right\}py_e(s_1) \\
&= -py_e(s_1)G_1(p,s_1).
\end{aligned} \tag{29}$$

From Lemma 4, we have $G_1(p,s_1) > 0$, which is a contradiction. □

Lemma 6. *If there exist positive constants $p > 0$, $q_1 > 0$, $q_2 > 0$, and $T_1 > 0$ such that $G_1(p,t) < 0$, $G_2(p,q_1,t) < 0$, $G_2'(p,q_2,t) > 0$ for all $t \geq T_1$, then there exists $T_2 \geq T_1$ such that for every $t \geq T_2$, we have either $W_2(q_i,t) > 0$ or $W_2(q_i,t) \leq 0$, $i = 1,2$.*

Proof. From Lemma 5, we know that there exists $T_2' \geq T_1$ such that either $W_1(p,t) > 0$ for all $t \geq T_2'$ or $W_1(p,t) \leq 0$ for all $t \geq T_2'$.

For the first case, if $W_1(p,t) > 0$ for all $t \geq T_2'$, we will show that there exists $T_3 \geq T_1$ such that either $W_2(q_1,t) > 0$ for all $t \geq T_3$ or $W_2(q_1,t) \leq 0$ for all $t \geq T_3$. Suppose that there does not exist $T_3 \geq T_1$ such that $W_2(q_1,t) > 0$ for all $t \geq T_3$ or $W_2(q_1,t) \leq 0$ for all $t \geq T_3$ hold. Then there necessarily exists $s_2 \geq T_1$ such that $W_2(q_1,s_2) = 0$ and $(dW_2(q_1,t)/dt)|_{t=s_2} < 0$. Hence we have

$$q_1y_i(s_2) = x_i(s_2), \tag{30}$$

$$\begin{aligned}
&\left.\frac{dW_2(q_1,t)}{dt}\right|_{t=s_2} = q_1\theta(s_2)y_e(s_2)-\left(\mu_y(s_2)\right. \\
&\quad \left.+\alpha_y(s_2)\right)q_1y_i(s_2)-\beta_x(s_2)x_s(s_2)y_i(s_2) \\
&\quad + \left(\mu_x(s_2)+\alpha_x(s_2)\right)x_i(s_2) \geq \frac{q_1}{p}\theta(s_2)y_i(s_2) \\
&\quad - \left(\mu_y(s_2)+\alpha_y(s_2)\right)q_1y_i(s_2)-\beta_x(s_2)x_s(s_2) \\
&\quad \cdot y_i(s_2)+\left(\mu_x(s_2)+\alpha_x(s_2)\right)x_i(s_2) = q_1y_i(s_2) \\
&\quad \cdot \left\{\frac{1}{p}\theta(s_2)-\mu_y(s_2)-\alpha_y(s_2)\right. \\
&\quad \left.-\frac{1}{q_1}\beta_x(s_2)x_s(s_2)\right\}+\left(\mu_x(s_2)+\alpha_x(s_2)\right)x_i(s_2).
\end{aligned} \tag{31}$$

Substituting (30) into (31), we have

$$\begin{aligned}
0 &> q_1y_i(s_2)\left\{\frac{1}{p}\theta(s_2)+\mu_x(s_2)+\alpha_x(s_2)-\mu_y(s_2)\right. \\
&\quad \left.-\alpha_y(s_2)-\frac{1}{q_1}\beta_x(s_2)x_s(s_2)\right\} \geq -q_1y_i(s_2) \\
&\quad \cdot G_2(p,q_1,s_2).
\end{aligned} \tag{32}$$

From Lemma 4, we have $G_2(p,q_1,s_2) > 0$, which is a contradiction.

For the second case, if $W_1(p,t) \leq 0$ for all $t \geq T_2'$, we will prove that there exists $T_4 \geq T_1$ such that either $W_2(q_2,t) > 0$ for all $t \geq T_4$ or $W_2(q_2,t) \leq 0$ for all $t \geq T_4$. Suppose that there does not exist $T_4 \geq T_1$ such that $W_2(q_2,t) > 0$ for all $t \geq T_4$ or $W_2(q_2,t) \leq 0$ for all $t \geq T_4$ hold. Then there necessarily exists $s_3 \geq T_1$ such that $W_2(q_2,s_3) = 0$ and $(dW_2(q_2,t)/dt)|_{t=s_3} > 0$. Hence we have

$$q_2y_i(s_3) = x_i(s_3), \tag{33}$$

$$\begin{aligned}
&\left.\frac{dW_2(q_2,t)}{dt}\right|_{t=s_3} = q_2\theta(s_3)y_e(s_3)-\left(\mu_y(s_3)\right. \\
&\quad \left.+\alpha_y(s_3)\right)q_2y_i(s_3)-\beta_x(s_3)x_s(s_3)y_i(s_3) \\
&\quad + \left(\mu_x(s_3)+\alpha_x(s_3)\right)x_i(s_3) \leq \frac{q_2}{p}\theta(s_3)y_i(s_3) \\
&\quad - \left(\mu_y(s_3)+\alpha_y(s_3)\right)q_2y_i(s_3)-\beta_x(s_3)x_s(s_3) \\
&\quad \cdot y_i(s_3)+\left(\mu_x(s_3)+\alpha_x(s_3)\right)x_i(s_3) = -q_2y_i(s_3) \\
&\quad \cdot \left\{\frac{1}{q_2}\beta_x(s_3)x_s(s_3)+\mu_y(s_3)+\alpha_y(s_3)\right. \\
&\quad \left.-\frac{1}{p}\theta(s_3)\right\}+\left(\mu_x(s_3)+\alpha_x(s_3)\right)x_i(s_3).
\end{aligned} \tag{34}$$

Substituting (33) into (34), we have

$$\begin{aligned}
0 &< -q_2y_i(s_3)\left\{\frac{1}{q_2}\beta_x(s_3)x_s(s_3)+\mu_y(s_3)+\alpha_y(s_3)\right. \\
&\quad \left.-\frac{1}{p}\theta(s_3)-\mu_x(s_3)-\alpha_x(s_3)\right\} \leq -q_2y_i(s_3) \\
&\quad \cdot G_2'(p,q_2,s_3).
\end{aligned} \tag{35}$$

From Lemma 4, we have $G_2'(p,q_2,s_3) < 0$, which is a contradiction.

If we let $T_2 = \max\{T_3, T_4\}$, then the results of the lemma hold. □

3. Extinction of Infectious Population

In this section, we obtain sufficient conditions for the extinction of the infectious population of system (3). The definition of the extinction is as follows.

Definition 7. We say that the infective $x_i(t)$ of system (3) is extinct if

$$\lim_{t\to+\infty} x_i(t) = 0. \tag{36}$$

Similarly, we can define the extinction of $y_i(t)$. Next, one of the main results of this paper is investigated.

Theorem 8. *If there exist positive constants $\lambda > 0$, $p > 0$, $q > 0$, and $T_1 > 0$ such that*

$$R_1(\lambda, p) := \limsup_{t\to+\infty} \int_t^{t+\lambda} \left\{\frac{1}{p}\theta(s) - \mu_y(s) - \alpha_y(s)\right\} ds < 0, \tag{37}$$

$$R_1'(\lambda, q) := \limsup_{t\to+\infty} \int_t^{t+\lambda} \left\{\frac{1}{q}\beta_x(s)\overline{N}_x(s) - \mu_x(s) - \alpha_x(s)\right\} ds < 0, \tag{38}$$

$$R_1^*(\lambda, p, q) := \limsup_{t\to+\infty} \int_t^{t+\lambda} \left\{pq\beta_y(s)\overline{N}_y(s) - \mu_y(s) - \theta(s)\right\} ds < 0 \tag{39}$$

and $G_1(p,t) < 0$, $G_2(p,q,t) < 0$ for all $t \geq T_1$, then $\lim_{t\to+\infty} x_i(t) = 0$ and $\lim_{t\to+\infty} y_i(t) = 0$.

Proof. Since there exist $p > 0$, $T_1 > 0$ such that $G_1(p,t) < 0$ holds for all $t \geq T_1$, from Lemma 5, we only have to consider the following two cases:

(i) $py_e(t) \leq y_i(t)$ for all $t \geq T_2$.

(ii) $py_e(t) > y_i(t)$ for all $t \geq T_2$.

First we consider case (i). Since we have $py_e(t) \leq y_i(t)$ for all $t \geq T_2$, it follows from the fifth equation of system (3) that

$$\begin{aligned}\frac{dy_i(t)}{dt} &= \theta(t)y_e(t) - \mu_y(t)y_i(t) - \alpha_y(t)y_i(t)\\ &\leq \left\{\frac{1}{p}\theta(t) - \mu_y(t) - \alpha_y(t)\right\} y_i(t)\end{aligned} \tag{40}$$

for all $t \geq T_2$. Hence we have

$$y_i(t) \leq y_i(T_2)\exp\left(\int_{T_2}^t \left\{\frac{1}{p}\theta(s) - \mu_y(s) - \alpha_y(s)\right\}\right) \tag{41}$$

for all $t \geq T_2$. Now it follows from (37) that there exist constants $\delta_1 > 0$ and $T_3 > T_2$ such that

$$\int_t^{t+\lambda} \left\{\frac{1}{p}\theta(s) - \mu_y(s) - \alpha_y(s)\right\} ds < -\delta_1 \tag{42}$$

for all $t \geq T_3$. From (41) and (42) we have $\lim_{t\to+\infty} y_i(t) = 0$. Then the second equation of the corresponding limit system (3) is $dx_i(t)/dt = -(\mu_x(t) + \alpha_x(t))x_i(t)$. From (i) and (H2) we can easily see that $\lim_{t\to+\infty} x_i(t) = 0$.

Next we consider case (ii). Because there exist $p > 0, q > 0$ such that $G_2(p,q,t) < 0$ for all $t \geq T_1$, then from the proof of Lemma 6, we know that there exists $T_2' \geq \max\{T_1, T_2\}$ such that either $W_2(q,t) > 0$ for all $t \geq T_2'$ or $W_2(q,t) \leq 0$ for all $t \geq T_2'$. Thus we also have two cases to discuss.

On the one hand, if $W_2(q,t) \leq 0$ holds for all $t \geq T_2'$, that is, $y_i(t) \leq (1/q)x_i(t)$ holds for all $t \geq T_2'$, then from the second equation of system (3) we obtain

$$\begin{aligned}\frac{dx_i(t)}{dt} &= \beta_x(t)x_s(t)y_i(t) - \left(\mu_x(t) + \alpha_x(s)\right)x_i(t)\\ &\leq \left\{\frac{1}{q}\beta_x(t)\overline{N}_x(t) - \mu_x(t) - \alpha_x(t)\right\} x_i(t)\end{aligned} \tag{43}$$

for all $t \geq T_2'$. Hence, we get

$$\begin{aligned}x_i(t) &\leq x_i\left(T_2'\right)\\ &\quad\cdot \exp\left(\int_{T_2'}^t \left\{\frac{1}{q}\beta_x(s)\overline{N}_x(s) - \mu_x(s) - \alpha_x(s)\right\} ds\right)\end{aligned} \tag{44}$$

for all $t \geq T_2'$. From (38) we see that there exist constants $\delta_2 > 0$ and $T_4 > T_2'$ such that

$$\int_t^{t+\lambda} \left\{\frac{1}{q}\beta_x(s)\overline{N}_x(s) - \mu_x(s) - \alpha_x(s)\right\} ds < -\delta_2 \tag{45}$$

for all $t \geq T_4$. From (44) and (45), we have $\lim_{t\to+\infty} x_i(t) = 0$.

On the other hand, if $W_2(q,t) > 0$ for all $t \geq T_2'$ hold, then we have $x_i(t) < qy_i(t) < pqy_e(t)$ for all $t \geq T_2'$. From the fourth equation of system (3), we have

$$\begin{aligned}\frac{dy_e(t)}{dt} &= \beta_y(t)x_i(t)y_s(t) - \left(\mu_y(t) + \theta(t)\right)y_e(t)\\ &< \left\{pq\beta_y(t)\overline{N}_y(t) - \mu_y(t) - \theta(t)\right\} y_e(t)\end{aligned} \tag{46}$$

for all $t \geq T_2'$. Hence we have

$$\begin{aligned}y_e(t) &< y_e\left(T_2'\right)\\ &\quad\cdot \exp\left(\int_{T_2'}^t \left\{pq\beta_y(s)\overline{N}_y(s) - \mu_y(s) - \theta(s)\right\}\right).\end{aligned} \tag{47}$$

From (39), we see that there exist constants $\delta_3 > 0$ and $T_5 > T_2'$ such that

$$\int_t^{t+\lambda} \left\{pq\beta_y(s)\overline{N}_y(s) - \mu_y(s) - \theta(s)\right\} ds < -\delta_3 \tag{48}$$

for all $t \geq T_5$. From (47) and (48), we have $\lim_{t\to+\infty} y_e(t) = 0$. Thus the fifth equation of the corresponding limit system (3) is $dy_i(t)/dt = -(\mu_y(t) + \alpha_y(t))y_i(t)$. From (i) and (H2), we can easily get that $\lim_{t\to+\infty} y_i(t) = 0$. In a similar way, we can also prove $\lim_{t\to+\infty} x_i(t) = 0$. □

Remark 9. We assume that all coefficients of system (3) are given by identically constant functions; then system (3) becomes an autonomous system. Conditions (37)–(39) are as follows:

$$\begin{gathered}\frac{1}{p}\theta-\mu_y-\alpha_y<0,\\ \frac{1}{q}\beta_x\frac{A_x}{\mu_x}-\mu_x-\alpha_x<0,\\ pq\beta_y\frac{A_y}{\mu_y}-\mu_y-\theta<0.\end{gathered}\tag{49}$$

These inequalities show that $\theta\beta_x A_x/\mu_x(\mu_y+\alpha_y)(\mu_x+\alpha_x)<pq<\mu_y(\mu_y+\theta)/\beta_y A_y$, which implies $R_0<1$; here R_0 is given in (2). This shows that the result is consistent with the literature [10].

Remark 10. It is obvious that if there exist positive constants $p>0$, $q>0$, and $T_1>0$ such that

$$\begin{gathered}\limsup_{t\to+\infty}\left\{\frac{1}{p}\theta(t)-\mu_y(t)-\alpha_y(t)\right\}<0,\\ \limsup_{t\to+\infty}\left\{\frac{1}{q}\beta_x(t)\overline{N}_x(t)-\mu_x(t)-\alpha_x(t)\right\}<0,\\ \limsup_{t\to+\infty}\left\{pq\beta_y(t)\overline{N}_y(t)-\mu_y(t)-\theta(t)\right\}<0\end{gathered}\tag{50}$$

and $G_1(p,t)<0$, $G_2(p,q,t)<0$ for all $t\geq T_1$, then $\lim_{t\to+\infty}x_i(t)=0$ and $\lim_{t\to+\infty}y_i(t)=0$.

4. Permanence of Infectious Population

In this section, we obtain sufficient conditions for the permanence of infectious population of system (3).

The definition of the permanence is as follows.

Definition 11. We say that the infective $x_i(t)$ of system (3) is permanent if there exist positive constants $x_{i1}>0$ and $x_{i2}>0$, which are independent of the choice of initial value satisfying (3), such that

$$0<x_{i1}\leq\liminf_{t\to+\infty}x_i(t)\leq\limsup_{t\to+\infty}x_i(t)\leq x_{i2}.\tag{51}$$

Similarly, we can give the definition of the permanence of $y_i(t)$.

Let the function

$$\begin{aligned}b(t,u_1,u_2)&:=3\sqrt[3]{\beta_x(t)\beta_y(t)\theta(t)u_1(t)u_2(t)}\\&-\left[(\mu_x(t)+\alpha_x(t))+(\mu_y(t)+\alpha_y(t))\right.\\&\left.+(\mu_y(t)+\theta(t))\right]\end{aligned}\tag{52}$$

and let $\underline{N}_x(t)$ be some fixed solution of (8) with initial value $\underline{N}_x(0)>0$ and $\underline{N}_y(t)$ be some fixed solution of (10) with initial value $\underline{N}_y(0)>0$.

We have the following theorem.

Theorem 12. *If there is a constant $\lambda_1>0$ such that*

$$R_0^*=\liminf_{t\to+\infty}\int_t^{t+\lambda_1}b\left(s,\underline{N}_x(s),\underline{N}_y(s)\right)ds>0,\tag{53}$$

then the infective $x_i(t)$ and $y_i(t)$ of system (3) are permanent.

Proof. Since $\underline{N}_x(t)\leq N_x(t)$, $\underline{N}_y(t)\leq N_y(t)$ for all $t\geq 0$, then, from system (3), we have

$$\begin{aligned}\frac{dx_i(t)}{dt}&\geq\beta_x(t)\left(\underline{N}_x(t)-x_i(t)\right)y_i(t)-\mu_x(t)x_i(t)\\&\quad-\alpha_x(t)x_i(t),\\ \frac{dy_e(t)}{dt}&\geq\beta_y(t)\left(\underline{N}_y(t)-y_e(t)-y_i(t)\right)x_i(t)\\&\quad-\left(\mu_y(t)+\theta(t)\right)y_e(t),\\ \frac{dy_i(t)}{dt}&=\theta(t)y_e(t)-\mu_y(t)y_i(t)-\alpha_y(t)y_i(t).\end{aligned}\tag{54}$$

Let us consider the following system:

$$\begin{aligned}\frac{d\overline{x}_i(t)}{dt}&=\beta_x(t)\left(\underline{N}_x(t)-\overline{x}_i(t)\right)\overline{y}_i(t)-\mu_x(t)\overline{x}_i(t)\\&\quad-\alpha_x(t)\overline{x}_i(t),\\ \frac{d\overline{y}_e(t)}{dt}&=\beta_y(t)\left(\underline{N}_y(t)-\overline{y}_e(t)-\overline{y}_i(t)\right)\overline{x}_i(t)\\&\quad-\left(\mu_y(t)+\theta(t)\right)\overline{y}_e(t),\\ \frac{d\overline{y}_i(t)}{dt}&=\theta(t)\overline{y}_e(t)-\mu_y(t)\overline{y}_i(t)-\alpha_y(t)\overline{y}_i(t).\end{aligned}\tag{55}$$

Firstly, we prove that the number R_0^* is independent of the choice of $\underline{N}_x(t)$ and $\underline{N}_y(t)$. In fact, (c) of Lemma 3 implies that, for any sufficiently small $\epsilon>0$ ($<\min(m_1,m_2)$), any solution $\underline{N}_x^*(t)$ of (8) with initial value $\underline{N}_x^*(0)>0$, and any solution $\underline{N}_y^*(t)$ of (10) with initial value $\underline{N}_y^*(0)>0$, there exists $T>0$ such that as $t\geq T$

$$\begin{gathered}\underline{N}_x(t)-\epsilon\leq\underline{N}_x^*(t)\leq\underline{N}_x(t)+\epsilon,\\ \underline{N}_y(t)-\epsilon\leq\underline{N}_y^*(t)\leq\underline{N}_y(t)+\epsilon,\\ \underline{N}_x(t)\geq m_1,\\ \underline{N}_y(t)\geq m_2.\end{gathered}\tag{56}$$

Hence,

$$\begin{aligned}&b\left(t,\underline{N}_x(t)-\epsilon,\underline{N}_y(t)-\epsilon\right)\leq b\left(t,\underline{N}_x^*(t),\underline{N}_y^*(t)\right)\\&\quad\leq b\left(t,\underline{N}_x(t)+\epsilon,\underline{N}_y(t)+\epsilon\right).\end{aligned}\tag{57}$$

For $t \geq T$, since the inequality $\sqrt[3]{a_1 + a_2 + a_3} \leq \sqrt[3]{a_1} + \sqrt[3]{a_2} + \sqrt[3]{a_3}$ holds for all $a_i \geq 0$ $(i = 1, 2, 3)$ and from Taylor's theorem of binary function, we obtain

$$\begin{aligned}
&\liminf_{t\to+\infty} \int_t^{t+\lambda_1} b\left(s, \underline{N}_x(s) + \epsilon, \underline{N}_y(s) + \epsilon\right) ds \leq R_0^* \\
&\quad + \left(\sqrt[3]{(M_1 + M_2)\epsilon} + \sqrt[3]{\epsilon^2}\right) \\
&\quad \cdot 3\lambda_1 \sup_{t\geq 0} \sqrt[3]{\beta_x(t)\beta_y(t)\theta(t)}, \\
&\liminf_{t\to+\infty} \int_t^{t+\lambda_1} b\left(s, \underline{N}_x(s) - \epsilon, \underline{N}_y(s) - \epsilon\right) ds \geq R_0^* \\
&\quad - \epsilon\lambda_1 \left(\sqrt[3]{\frac{M_2}{m_1^2}} + \sqrt[3]{\frac{M_1}{m_2^2}}\right) \sup_{t\geq 0} \sqrt[3]{\beta_x(t)\beta_y(t)\theta(t)}.
\end{aligned} \tag{58}$$

By the arbitrariness of ϵ, we finally obtain

$$\liminf_{t\to+\infty} \int_t^{t+\lambda_1} b\left(s, \underline{N}_x^*(s), \underline{N}_y^*(s)\right) ds = R_0^*. \tag{59}$$

This shows that R_0^* is independent of the choice of $\underline{N}_x(t)$ and $\underline{N}_y(t)$. From inequalities (15) and (16), we can set T largely enough such that

$$m_1 < \underline{N}_x(t) \leq N_x(t) \leq \overline{N}_x(t) < M_1, \tag{60}$$

$$m_2 < \underline{N}_y(t) \leq N_y(t) \leq \overline{N}_y(t) < M_2 \tag{61}$$

for all $t \geq T$. From (53), for sufficiently small $\eta > 0$, there exists $T_1 \geq T$ such that

$$\int_t^{t+\lambda_1} b\left(s, \underline{N}_x(s), \underline{N}_y(s)\right) ds > \eta \tag{62}$$

for all $t \geq T_1$. We define

$$\begin{aligned}
\beta_x^+ &:= \sup_{t\geq 0} \beta_x(t), \\
\mu_x^+ &:= \sup_{t\geq 0} \mu_x(t), \\
\beta_y^+ &:= \sup_{t\geq 0} \beta_y(t), \\
\mu_y^+ &:= \sup_{t\geq 0} \mu_y(t), \\
\alpha_x^+ &:= \sup_{t\geq 0} \alpha_x(t), \\
\theta^+ &:= \sup_{t\geq 0} \theta(t), \\
\alpha_y^+ &:= \sup_{t\geq 0} \alpha_y(t).
\end{aligned} \tag{63}$$

From (60)–(63), we see that, for positive constants $\eta_1 < \eta$, there exist small $\epsilon_i > 0$, $(i = 1, 2, 3)$ and $T_2 \geq T_1$ such that

$$\begin{aligned}
&\int_t^{t+\lambda_1} b\left(s, \underline{N}_x(s) - \epsilon_1 - \kappa_1\epsilon_2, \underline{N}_y(s) - \kappa_2\epsilon_1 - \kappa_3\epsilon_2 \right. \\
&\quad \left. - \epsilon_3\right) ds > \eta_1,
\end{aligned} \tag{64}$$

$$\underline{N}_x(t) - \epsilon_1 - \kappa_1\epsilon_2 > m_1, \tag{65}$$

$$\underline{N}_y(ts) - \kappa_2\epsilon_1 - \kappa_3\epsilon_2 - \epsilon_3 > m_2 \tag{66}$$

hold for all $t \geq T_2$, where $\kappa_1 := \beta_x^+ M_1\omega_3$, $\kappa_2 := \beta_y^+ M_2\omega_4$, $\kappa_3 := 1 + \kappa_1\kappa_2$. From (H2), we can choose ϵ_1, ϵ_2 such that both the following two equations hold for all $t \geq T_2$.

$$\int_t^{t+\omega_3} \left\{\beta_x(s) M_1\epsilon_2 - \left(\mu_x(s) + \alpha_x(s)\right)\epsilon_1\right\} ds < -\eta_1, \tag{67}$$

$$\begin{aligned}
&\int_t^{t+\omega_4} \left\{\beta_y(s) M_2\left(\epsilon_1 + \kappa_1\epsilon_2\right) - \left(\mu_y(s) + \theta(s)\right)\epsilon_3\right\} ds \\
&\quad < -\eta_1.
\end{aligned} \tag{68}$$

First we claim that $\limsup_{t\to+\infty} \overline{y}_i(t) > \epsilon_2$. In fact, if it is not true, then there exists $T_3 \geq T_2$ such that

$$\overline{y}_i(t) \leq \epsilon_2 \tag{69}$$

for all $t \geq T_3$. Suppose that $\overline{x}_i(t) \geq \epsilon_1$ for all $t \geq T_3$. Then, from (60) and (69) we have

$$\begin{aligned}
\overline{x}_i(t) &= \overline{x}_i(T_3) + \int_{T_3}^t \left\{\beta_x(s)\left(\underline{N}_x(s) - \overline{x}_i(s)\right)\overline{y}_i(s)\right. \\
&\quad \left. - \left(\mu_x(s) + \alpha_x(s)\right)\overline{x}_i(s)\right\} ds \leq \overline{x}_i(T_3) \\
&\quad + \int_{T_3}^t \left\{\beta_x(s) M_1\epsilon_2 - \left(\mu_x(s) + \alpha_x(s)\right)\epsilon_1\right\} ds
\end{aligned} \tag{70}$$

for all $t \geq T_3$. Thus, from (67), we have $\lim_{t\to+\infty} \overline{x}_i(t) = -\infty$, which contradicts with (b) of Lemma 3. Therefore we see that there exist $s_1 \geq T_3$ such that $\overline{x}_i(s_1) < \epsilon_1$. Suppose that there exists $s_2 \geq s_1$ such that $\overline{x}_i(s_2) > \epsilon_1 + \kappa_1\epsilon_2$. Then, we see that there necessarily exists $s_3 \in (s_1, s_2)$ such that $\overline{x}_i(s_3) = \epsilon_1$ and $\overline{x}_i(t) > \epsilon_1$ for all $t \in (s_3, s_2]$. It is easy to see that there exists an integer n that satisfies $s_2 \in [s_3 + n\omega_3, s_3 + (n+1)\omega_3)$. Then, from (67), we have

$$\begin{aligned}
\epsilon_1 + \kappa_1\epsilon_2 &< \overline{x}_i(s_2) = \overline{x}_i(s_3) \\
&\quad + \int_{s_3}^{s_2} \left\{\beta_x(s)\left(\underline{N}_x(s) - \overline{x}_i(s)\right)\overline{y}_i(s)\right. \\
&\quad \left. - \left(\mu_x(s) + \alpha_x(s)\right)\overline{x}_i(s)\right\} ds < \epsilon_1 + \left\{\int_{s_3}^{s_3+n\omega_3}\right. \\
&\quad \left. + \int_{s_3+n\omega_3}^{s_2}\right\} \left\{\beta_x(s) M_1\epsilon_2 - \left(\mu_x(s) + \alpha_x(s)\right)\epsilon_1\right\} ds \\
&\leq \epsilon_1 + \int_{s_3+n\omega_3}^{s_2} \beta_x(s) M_1\epsilon_2 ds < \epsilon_1 + \kappa_1\epsilon_2,
\end{aligned} \tag{71}$$

which is a contradiction. Therefore, we see that

$$\overline{x}_i(t) \le \epsilon_1 + \kappa_1\epsilon_2 \tag{72}$$

for all $t \ge s_1$. In a similar way, from (68), we can show that there exists $\widetilde{s_1} \ge s_1$ such that

$$\overline{y}_e(t) \le \kappa_2\epsilon_1 + \kappa_1\kappa_2\epsilon_2 + \epsilon_3 \tag{73}$$

for all $t \ge \widetilde{s_1}$.

For $t \ge 0$, we define a differentiable function $V(t) = \overline{x}_i(t)\overline{y}_e(t)\overline{y}_i(t)$. When $t \ge \widetilde{s_1}$, from (69)–(73), we have

$$\begin{aligned}
&\left.\frac{dV(t)}{dt}\right|_{(55)} \\
&= \beta_x(t)\left(\underline{N}_x(t) - \overline{x}_i(t)\right)\overline{y}_e(t)\overline{y}_i^2(t) \\
&\quad + \beta_y(t)\left(\underline{N}_y(t) - \overline{y}_e(t) - \overline{y}_i(t)\right)\overline{x}_i^2(t)\overline{y}_i(t) \\
&\quad + \theta(t)\overline{x}_i(t)\overline{y}_e^2(t) \\
&\quad - \left(\mu_x(t) + 2\mu_y(t) + \theta(t) + \alpha_x(t) + \alpha_y(t)\right)V(t) \\
&\ge \beta_x(t)\left(\underline{N}_x(t) - \epsilon_1 - \kappa_1\epsilon_2\right)\overline{y}_e(t)\overline{y}_i^2(t) \\
&\quad + \beta_y(t)\left(\underline{N}_y(s) - \kappa_2\epsilon_1 - \kappa_3\epsilon_2 - \epsilon_3\right)\overline{x}_i^2(t)\overline{y}_i(t) \\
&\quad + \theta(t)\overline{x}_i(t)\overline{y}_e^2(t) \\
&\quad - \left(\mu_x(t) + 2\mu_y(t) + \theta(t) + \alpha_x(t) + \alpha_y(t)\right)V(t).
\end{aligned} \tag{74}$$

Since the inequality $a + b + c \ge 3\sqrt[3]{abc}$ holds for all $a, b, c \ge 0$ and from (65) and (66), we have

$$\begin{aligned}
&\left.\frac{dV(t)}{dt}\right|_{(55)} \ge b\left(t, \underline{N}_x(t) - \epsilon_1 - \kappa_1\epsilon_2, \underline{N}_y(t) - \kappa_2\epsilon_1\right. \\
&\quad \left. - \kappa_3\epsilon_2 - \epsilon_3\right)V(t).
\end{aligned} \tag{75}$$

Integrating the above inequality from $\widetilde{s_1}$ to t, we have

$$V(t) \ge V(\widetilde{s_1})\exp\left(\int_{\widetilde{s_1}}^{t} b\left(s, \underline{N}_x(s) - \epsilon_1 - \kappa_1\epsilon_2, \underline{N}_y(s) - \kappa_2\epsilon_1 - \kappa_3\epsilon_2 - \epsilon_3\right)ds\right) \tag{76}$$

for all $t \ge \widetilde{s_1}$. By (62), we obtain $\lim_{t\to+\infty}V(t) = \infty$. This contradicts with the boundedness of $\overline{x}_i(t)$, $\overline{y}_e(t)$, and $\overline{y}_i(t)$. From this contradiction, we finally conclude $\limsup_{t\to+\infty}\overline{y}_i(t) > \epsilon_2$.

Next, we prove

$$\liminf_{t\to+\infty}\overline{y}_i(t) \ge \overline{y}_{i1}, \tag{77}$$

where $\overline{y}_{i1} > 0$ is a constant given in the following lines. For convenience sake, we let ω^* be the least common multiple of ω_3, ω_4, and λ_1. If we define

$$\begin{aligned}
&\liminf_{t\to+\infty} b\left(t, \underline{N}_x(t) - \epsilon_1 - \kappa_1\epsilon_2, \underline{N}_y(t) - \kappa_2\epsilon_1 - \kappa_3\epsilon_2\right. \\
&\quad \left. - \epsilon_3\right) := \underline{m}
\end{aligned} \tag{78}$$

for $\epsilon > 0$, there exists $\widetilde{T_3}(\ge T_2)$ such that

$$\begin{aligned}
&b\left(t, \underline{N}_x(t) - \epsilon_1 - \kappa_1\epsilon_2, \underline{N}_y(t) - \kappa_2\epsilon_1 - \kappa_3\epsilon_2 - \epsilon_3\right) \\
&\quad > \underline{m} - \epsilon
\end{aligned} \tag{79}$$

for all $t \ge \widetilde{T_3}$. Then, from inequalities (64) and (67)-(68) and (H2), we see that there exist constants $\widetilde{T_4}(\ge \widetilde{T_3})$, $\lambda_2(> \max(\omega_5, \omega_6))$, here we choose λ_2 is an integral multiple of ω^*, and $\eta_2 > 0$ such that

$$\int_t^{t+\lambda_3}\left\{\beta_x(s)M_1\epsilon_2 - \left(\mu_x(s) + \alpha_x(s)\right)\epsilon_1\right\}ds < -M_1, \tag{80}$$

$$\begin{aligned}
&\int_t^{t+\lambda_3}\left\{\beta_y(s)M_2\left(\epsilon_1 + \kappa_1\epsilon_2\right) - \left(\mu_y(s) + \theta(s)\right)\epsilon_3\right\}ds \\
&\quad < -M_2,
\end{aligned} \tag{81}$$

$$\begin{aligned}
&\int_t^{t+\lambda_3} b\left(s, \underline{N}_x(s) - \epsilon_1 - \kappa_1\epsilon_2, \underline{N}_y(s) - \kappa_2\epsilon_1 - \kappa_3\epsilon_2\right. \\
&\quad \left. - \epsilon_3\right)ds > \eta_2,
\end{aligned} \tag{82}$$

$$\int_t^{t+\lambda_3}\beta_x(s)\,ds > \eta_2, \tag{83}$$

$$\int_t^{t+\lambda_3}\beta_y(s)\,ds > \eta_2 \tag{84}$$

for all $t \ge \widetilde{T_4}$, $\lambda_3 \ge \lambda_2$, and λ_3 is an integral multiple of ω. Let $K_0 > 0$ be a positive integer satisfying

$$\begin{aligned}
&e^{-(\mu_x^+ + \alpha_x^+ + \mu_y^+ + \theta^+)\lambda_2}\eta_2^2 m_1 m_2 \nu_2 \nu_3 e^{K_0\eta_2 + (\underline{m} - \epsilon)\omega^*} \\
&\quad > \left(\epsilon_1 + \kappa_1\epsilon_2\right)\left[\epsilon_3 + \left(\epsilon_1 + \kappa_1\epsilon_2\right)\kappa_2\right],
\end{aligned} \tag{85}$$

where $\nu_2 := \epsilon_2 e^{-3(\mu_y^+ + \alpha_y^+)\lambda_2}$, $\nu_3 := e^{-2(\mu_x^+ + \alpha_x^+)\lambda_2}\eta_2 m_1 \nu_2$. Since we have proved $\limsup_{t\to+\infty}y_i(t) > \epsilon_2$, there are only two possibilities as follows:

(i) There exists $\widetilde{T_5} \ge \widetilde{T_4}$ such that $\overline{y}_i(t) \ge \epsilon_2$ for all $t \ge \widetilde{T_5}$.

(ii) $\overline{y}_i(t)$ oscillates about ϵ_2 for large $t \ge \widetilde{T_4}$.

In case (i), we have $\liminf_{t\to+\infty}\overline{y}_i(t) \ge \epsilon_2 := \overline{y}_{i1}$. In case (ii), then there necessarily exist two constants $t_1, t_2 \ge \widetilde{T_4}(t_2 \ge t_1)$ such that

$$\begin{aligned}
&\overline{y}_i(t_1) = \overline{y}_i(t_2) = \epsilon_2, \\
&\overline{y}_i(t) < \epsilon_2, \quad \forall t \in (t_1, t_2).
\end{aligned} \tag{86}$$

Suppose that $t_2 - t_1 \leq (K_0 + 3)\lambda_2$. Then, from the third equation of system (55), we have

$$\frac{d\overline{y}_i(t)}{dt} \geq -\left(\mu_y^+ + \alpha_y^+\right)\overline{y}_i(t). \tag{87}$$

Hence, we obtain

$$\begin{aligned}\overline{y}_i(t) &\geq \overline{y}_i(t_1)\exp\left(\int_{t_1}^{t} -\left(\mu_y^+ + \alpha_y^+\right)ds\right)\\ &\geq \epsilon_2 e^{-(\mu_y^+ + \alpha_y^+)(K_0+3)\lambda_2} := \overline{y}_{i1}\end{aligned} \tag{88}$$

for all $t \in (t_1, t_2)$. Suppose that $t_2 - t_1 > (K_0 + 3)\lambda_2$. Then, from (87), we have

$$\overline{y}_i(t) \geq \epsilon_2 e^{-(\mu_y^+ + \alpha_y^+)(K_0+3)\lambda_2} = \overline{y}_{i1} \tag{89}$$

for all $t \in (t_1, t_1 + (K_0 + 3)\lambda_2)$. Now, we are in a position to show that $\overline{y}_i(t) \geq \overline{y}_{i1}$ for all $t \in [t_1 + (K_0 + 3)\lambda_2, t_2)$. Suppose that $\overline{x}_i(t) \geq \epsilon_1$ for all $t \in [t_1, t_1 + \lambda_2]$. Then, from (80), we have

$$\begin{aligned}&\overline{x}_i(t_1 + \lambda_2)\\ &\quad\leq \overline{x}_i(t_1)\\ &\qquad + \int_{t_1}^{t_1+\lambda_2}\left\{\beta_x(s)M_1\epsilon_2 - \left(\mu_x(s) + \alpha_x(s)\right)\epsilon_1\right\}ds\\ &\quad< M_1 - M_1 = 0,\end{aligned} \tag{90}$$

which is a contradiction. Therefore, there exists $s_4 \in [t_1, t_1 + \lambda_2]$ such that $x_i(s_4) < \epsilon_1$. Then, similar to the proof that $\limsup_{t\to+\infty}\overline{y}_i(t) > \epsilon_2$, we can show that $\overline{x}_i(t) \leq \epsilon_1 + \kappa_1\epsilon_2$ for all $t \geq s_4$. Similarly, from (75), we can show that there exists $\widetilde{s_4} \in [t_1, t_1 + \lambda_2]$ such that $\overline{y}_e(t) \leq \epsilon_3 + \kappa_2\epsilon_1 + \kappa_1\kappa_2\epsilon_2$ for all $t \geq \widetilde{s_4}$. Thus we have

$$\begin{aligned}\overline{x}_i(t) &\leq \epsilon_1 + \kappa_1\epsilon_2,\\ \overline{y}_e(t) &\leq \epsilon_3 + \kappa_2\epsilon_1 + \kappa_1\kappa_2\epsilon_2\end{aligned} \tag{91}$$

for all $t \geq t_1 + \lambda_2 \geq \max(s_4, \widetilde{s_4})$. From (87), we have

$$\overline{y}_i(t) \geq \nu_2 = \epsilon_2 e^{-3(\mu_y^+ + \alpha_y^+)\lambda_2} > \overline{y}_{i1} \tag{92}$$

for all $t \in [t_1, t_1 + 3\lambda_2]$. Thus, from (65), (91), and (92), we have

$$\begin{aligned}\frac{d\overline{x}_i(t)}{dt} &= \beta_x(t)\left(\underline{N}_x(t) - \overline{x}_i(t)\right)\overline{y}_i(t)\\ &\quad - \left(\mu_x(t) + \alpha_x(t)\right)\overline{x}_i(t)\\ &\geq \beta_x(t)m_1\nu_2 - \left(\mu_x^+ + \alpha_x^+\right)\overline{x}_i(t)\end{aligned} \tag{93}$$

for all $t \in [t_1 + \lambda_2, t_1 + 3\lambda_2]$. Hence, from (83), we obtain

$$\begin{aligned}&\overline{x}_i(t_1 + 3\lambda_2)\\ &\quad\geq e^{-(\mu_x^+ + \alpha_x^+)(t_1+3\lambda_2)}\left\{\overline{x}_i(t_1 + 2\lambda_2)e^{(\mu_x^+ + \alpha_x^+)(t_1+2\lambda_2)}\right.\\ &\qquad\left. + \int_{t_1+2\lambda_2}^{t_1+3\lambda_2} e^{(\mu_x^+ + \alpha_x^+)s}\beta_x(s)m_1\nu_2\,ds\right\}\\ &\quad\geq e^{-(\mu_x^+ + \alpha_x^+)(t_1+3\lambda_2)}\int_{t_1+2\lambda_2}^{t_1+3\lambda_2} e^{(\mu_x^+ + \alpha_x^+)s}\beta_x(s)m_1\nu_2\,ds\\ &\quad\geq e^{-(\mu_x^+ + \alpha_x^+)\lambda_2}\eta_2 m_1\nu_2,\end{aligned} \tag{94}$$

$$\begin{aligned}&\overline{x}_i(t) \geq e^{-(\mu_x^+ + \alpha_x^+)t}\left\{\overline{x}_i(t_1 + \lambda_2)e^{(\mu_x^+ + \alpha_x^+)(t_1+\lambda_2)}\right.\\ &\qquad\left. + \int_{t_1+\lambda_2}^{t} e^{(\mu_x^+ + \alpha_x^+)s}\beta_x(s)m_1\nu_2\,ds\right\}\\ &\qquad\geq e^{-(\mu_x^+ + \alpha_x^+)(t_1+3\lambda_2)}\int_{t_1+\lambda_2}^{t_1+2\lambda_2} e^{(\mu_x^+ + \alpha_x^+)s}\beta_x(s)m_1\nu_2\,ds\\ &\qquad\geq e^{-2(\mu_x^+ + \alpha_x^+)\lambda_2}\eta_2 m_1\nu_2 = \nu_3\end{aligned} \tag{95}$$

for all $t \in [t_1 + 2\lambda_2, t_1 + 3\lambda_2]$. Thus, from (66), (91), (92), and (95), we have

$$\begin{aligned}\frac{d\overline{y}_e(t)}{dt} &= \beta_y(t)\left(\underline{N}_y(t) - \overline{y}_e(t) - \overline{y}_i(t)\right)\overline{x}_i(t)\\ &\quad - \left(\mu_y(t) + \theta(t)\right)\overline{y}_e(t)\\ &\geq \beta_y(t)m_2\nu_3 - \left(\mu_y(t) + \theta(t)\right)\overline{y}_e(t)\end{aligned} \tag{96}$$

for all $t \in [t_1 + 2\lambda_2, t_1 + 3\lambda_2]$. Hence, from (84), we obtain

$$\begin{aligned}&\overline{y}_e(t_1 + 3\lambda_2)\\ &\quad\geq e^{-(\mu_y^+ + \theta^+)(t_1+3\lambda_2)}\left\{\overline{y}_e(t_1 + 2\lambda_2)e^{(\mu_y^+ + \theta^+)(t_1+2\lambda_2)}\right.\\ &\qquad\left. + \int_{t_1+2\lambda_2}^{t_1+3\lambda_2}\beta_y(s)m_2\nu_3 e^{(\mu_y^+ + \theta^+)s}\,ds\right\}\\ &\quad\geq e^{-(\mu_y^+ + \theta^+)(t_1+3\lambda_2)}\int_{t_1+2\lambda_2}^{t_1+3\lambda_2}\beta_y(s)m_2\nu_3 e^{(\mu_y^+ + \theta^+)s}ds\\ &\quad\geq e^{-(\mu_y^+ + \theta^+)\lambda_2}\eta_2 m_2\nu_3.\end{aligned} \tag{97}$$

We claim that $\overline{y}_i(t) > \overline{y}_{i1}$ for all $t \in (t_1 + (K_0 + 3)\lambda_2, t_2)$. If it is not true, there exists a $t_0 > 0$ such that $t_0 \in (t_1 + (K_0 + 3)\lambda_2, t_2)$, $\overline{y}_i(t_0) = \overline{y}_{i1}$ and $\overline{y}_i(t) \geq \overline{y}_{i1}$ for all $t \in (t_1, t_0)$. Then there exists $m \in N$ such that $t_0 \in [t_1 + (K_0 + 3)\lambda_2 + m\omega, t_1 + (K_0 + 3)\lambda_2 + (m+1)\omega)$. Let $V(t) = \overline{x}_i(t)\overline{y}_e(t)\overline{y}_i(t)$. From (74), (75), (91), and (92), the derivative of $V(t)$ along solutions of (55) satisfies

$$\left.\frac{dV(t)}{dt}\right|_{(55)} \geq b\left(t, \underline{N}_x(t) - \epsilon_1 - \kappa_1\epsilon_2, \underline{N}_y(t) - \kappa_2\epsilon_1 - \kappa_3\epsilon_2 - \epsilon_3\right) V(t) \tag{98}$$

for all $t \in [t_1 + \lambda_2, t_2]$. Integrating the above inequality from $t_1 + 3\lambda_2$ to t_0, we have

$$\begin{aligned}
&\overline{x}_i(t_0)\overline{y}_e(t_0)\overline{y}_i(t_0) \geq \overline{x}_i(t_1 + 3\lambda_2)\overline{y}_e(t_1 + 3\lambda_2)\overline{y}_i(t_1 + 3\lambda_2) \\
&\cdot \exp\left(\int_{t_1+3\lambda_2}^{t_0} b\left(t, \underline{N}_x(t) - \epsilon_1 - \kappa_1\epsilon_2, \underline{N}_y(t) - \kappa_2\epsilon_1 - \kappa_3\epsilon_2 - \epsilon_3\right) dt\right) \geq e^{-(\mu_x^+ + \alpha_x^+ + \mu_y^+ + \theta^+)\lambda_2}\eta_2^2 m_1 m_2 \nu_2 \nu_3 \overline{y}_{i1} \\
&\cdot \exp\left(\left\{\int_{t_1+3\lambda_2}^{t_1+(K_0+3)\lambda_2} + \int_{t_1+(K_0+3)\lambda_2}^{t_1+(K_0+3)\lambda_2+m\omega} + \int_{t_1+(K_0+3)\lambda_2+m\omega}^{t_0}\right\} b\left(t, \underline{N}_x(t) - \epsilon_1 - \kappa_1\epsilon_2, \underline{N}_y(t) - \kappa_2\epsilon_1 - \kappa_3\epsilon_2 - \epsilon_3\right) dt\right) \\
&\geq e^{-(\mu_x^+ + \alpha_x^+ + \mu_y^+ + \theta^+)\lambda_2}\eta_2^2 m_1 m_2 \nu_2 \nu_3 \overline{y}_{i1} e^{K_0\eta_2 + (\underline{m} - \epsilon)\omega^*} > (\epsilon_1 + \kappa_1\epsilon_2)\left[\epsilon_3 + (\epsilon_1 + \kappa_1\epsilon_2)\kappa_2\right]\overline{y}_{i1}.
\end{aligned} \tag{99}$$

Thus, $\overline{x}_i(t_0)\overline{y}_e(t_0) > (\epsilon_1 + \kappa_1\epsilon_2)[\epsilon_3 + (\epsilon_1 + \kappa_1\epsilon_2)\kappa_2]$, which contradicts with (91).

Therefore, $\overline{y}_i(t) \geq \overline{y}_{i1}$ is valid for all $t \in [t_1, t_2]$, which implies $\liminf_{t\to+\infty}\overline{y}_i(t) \geq \overline{y}_{i1} > 0$. According to the comparison theorem, we have $\liminf_{t\to+\infty} y_i(t) \geq \overline{y}_{i1} > 0$. Since $\limsup_{t\to+\infty} y_i(t) \leq \limsup_{t\to+\infty} N_y(t) \leq M_2 < +\infty$, the infective $y_i(t)$ of system (3) is permanent.

Furthermore, from system (3), there exists $x_{i1} > 0$ such that $\liminf_{t\to+\infty} x_i(t) \geq x_{i1} > 0$. Since $\limsup_{t\to+\infty} x_i(t) \leq \limsup_{t\to+\infty} N_x(t) \leq M_1 < +\infty$, the infective $x_i(t)$ of system (3) is also permanent. This completes the proof. □

Remark 13. For the corresponding autonomous system of system (3), condition (53) can become as follows:

$$\begin{aligned}
&3\sqrt[3]{\beta_x\beta_y\theta\frac{A_x}{\mu_x + \alpha_x}\frac{A_y}{\mu_y + \alpha_y}} \\
&> (\mu_x + \alpha_x) + (\mu_y + \alpha_y) + (\mu_y + \theta) \\
&\geq 3\sqrt[3]{(\mu_x + \alpha_x)(\mu_y + \alpha_y)(\mu_y + \theta)}.
\end{aligned} \tag{100}$$

This shows that $\sqrt[3]{\beta_x\beta_y\theta(A_x/\mu_x)(A_y/\mu_y)} > \sqrt[3]{(\mu_x + \alpha_x)(\mu_y + \alpha_y)(\mu_y + \theta)}$, which implies $R_0 > 1$; here R_0 is given in (2). This means that the result is consistent with the literature [10].

Remark 14. It is easy to prove that if $\liminf_{t\to+\infty} b(t, \underline{N}_x(t), \underline{N}_y(t)) > 0$, then the infective $y_i(t)$ of system (3) is permanent.

5. Some Corollaries

As consequences of Theorems 8 and 12, we have a series of corollaries as follows.

Corollary 15. *Suppose that $\mu_x(t) > 0$, $\mu_y(t) > 0$ for all $t \geq 0$ and $0 < \liminf_{t\to\infty}(A_x(t)/(\mu_x(t) + \alpha_x(t))) \leq \limsup_{t\to\infty}(A_x(t)/\mu_x(t)) < \infty$, $0 < \liminf_{t\to\infty}(A_y(t)/(\mu_y(t) + \alpha_y(t))) \leq \limsup_{t\to\infty}(A_y(t)/\mu_y(t)) < \infty$. If there exist positive constants $\lambda > 0$, $p > 0$, $q > 0$, and $T_1 > 0$ such that*

$$\begin{aligned}
&\limsup_{t\to+\infty}\int_t^{t+\lambda}\left\{\frac{1}{p}\theta(s) - \mu_y(s) - \alpha_y(s)\right\} ds < 0, \\
&\limsup_{t\to+\infty}\int_t^{t+\lambda}\left\{\frac{1}{q}\beta_x(s)\left(\frac{A_x}{\mu_x}\right)^M - \mu_x(s) - \alpha_x(s)\right\} ds \\
&\quad < 0, \\
&\limsup_{t\to+\infty}\int_t^{t+\lambda}\left\{pq\beta_y(s)\left(\frac{A_y}{\mu_y}\right)^M - \mu_y(s) - \theta(s)\right\} ds \\
&\quad < 0
\end{aligned} \tag{101}$$

and $G_1(p,t) < 0$, $G_2(p,q,t) < 0$ for all $t \geq T_1$, then the two infectious populations $x_i(t)$ and $y_i(t)$ of system (3) are extinct.

Corollary 16. *If there is a constant $\lambda_1 > 0$ such that*

$$\begin{aligned}
&\liminf_{t\to+\infty}\int_t^{t+\lambda_1} b\left(s, \left(\frac{A_x}{\mu_x + \alpha_x}\right)^m, \left(\frac{A_y}{\mu_y + \alpha_y}\right)^m\right) ds \\
&\quad > 0,
\end{aligned} \tag{102}$$

then the infective $y_i(t)$ of system (3) is permanent.

Corollary 17. *If there exists a constant $\lambda_1 > 0$ such that $\overline{R}_{01} > 1$, where*

$$\overline{R}_{01} = \frac{\left(3\sqrt[3]{\beta_x\beta_y\theta\underline{N}_x\underline{N}_y}\right)_0}{\left((\mu_x + \alpha_x) + (\mu_y + \alpha_y) + (\mu_y + \theta)\right)^0}. \tag{103}$$

Here,

$$\left(3\sqrt[3]{\beta_x\beta_y\theta\underline{N}_x\underline{N}_y}\right)_0 = \liminf_{t\to+\infty}\int_t^{t+\lambda_1} 3\sqrt[3]{\beta_x(t)\beta_y(t)\theta(t)\underline{N}_x(t)\underline{N}_y(t)}\,dt,$$

$$\left((\mu_x+\alpha_x)+(\mu_y+\alpha_y)+(\mu_y+\theta)\right)^0 = \limsup_{t\to+\infty}\int_t^{t+\lambda_1}(\mu_x(t)+\alpha_x(t))+(\mu_y(t)+\alpha_y(t)) + (\mu_y(t)+\theta(t))\,dt, \tag{104}$$

$\underline{N}_x(t)$ is some fixed solution of (8) with initial value $\underline{N}_x(0) > 0$ and $\underline{N}_y(t)$ is some fixed solution of (10) with initial value $\underline{N}_y(0) > 0$. Then the infective $y_i(t)$ of system (3) is permanent.

Corollary 18. *Suppose system (3) is ω-periodic; then the infective $y_i(t)$ of system (3) is permanent provided that*

$$\overline{R}_{02} = \frac{\overline{\left(3\sqrt[3]{\beta_x\beta_y\theta\underline{N}_x\underline{N}_y}\right)}}{\overline{\overline{\left((\mu_x+\alpha_x)+(\mu_y+\alpha_y)+(\mu_y+\theta)\right)}}} > 1. \tag{105}$$

Here, $\underline{N}_x(t)$ is the globally uniformly attractive nonnegative ω-periodic solution of (8) and $\underline{N}_y(t)$ is the globally uniformly attractive nonnegative ω-periodic solution of (10).

Corollary 19. *Suppose system (3) is almost periodic; then the infective $y_i(t)$ of system (3) is permanent provided that*

$$\overline{R}_{03} = \frac{m\left(3\sqrt[3]{\beta_x\beta_y\theta\underline{N}_x\underline{N}_y}\right)}{m\left((\mu_x+\alpha_x)+(\mu_y+\alpha_y)+(\mu_y+\theta)\right)} > 1. \tag{106}$$

Here, $\underline{N}_x(t)$ is the globally uniformly attractive nonnegative almost periodic solution of (8) and $\underline{N}_y(t)$ is the globally uniformly attractive nonnegative almost periodic solution of (10).

Corollary 20. *Suppose that $\mu_x(t) > 0$, $\mu_y(t) > 0$ for all $t \geq 0$ and $0 < \liminf_{t\to\infty}(A_x(t)/(\mu_x(t)+\alpha_x(t))) \leq \limsup_{t\to\infty}(A_x(t)/\mu_x(t)) < \infty$, $0 < \liminf_{t\to\infty}(A_y(t)/(\mu_y(t)+\alpha_y(t))) \leq \limsup_{t\to\infty}(A_y(t)/\mu_y(t)) < \infty$. If there exist positive constants $\lambda > 0$, $p > 0$, $q > 0$, and $T_1 > 0$ such that*

$$p > \frac{(\theta)^0}{(\mu_y+\alpha_y)_0},\quad q > \frac{(\beta_x\overline{N}_x)^0}{(\mu_x+\alpha_x)_0},\quad pq > \frac{(\mu_y+\theta)^0}{(\beta_y\overline{N}_y)_0} \tag{107}$$

and $G_1(p,t) < 0$, $G_2(p,q,t) < 0$ for all $t \geq T_1$, then the two infectious populations $x_i(t)$ and $y_i(t)$ of system (3) are extinct.

Here,

$$\begin{aligned}
(\theta)^0 &= \limsup_{t\to+\infty}\int_t^{t+\lambda}\theta(s)\,ds,\\
(\mu_y+\theta)^0 &= \limsup_{t\to+\infty}\int_t^{t+\lambda}(\mu_y(s)+\theta(s))\,ds,\\
(\beta_x\overline{N}_x)^0 &= \limsup_{t\to+\infty}\int_t^{t+\lambda}\beta_x(s)\overline{N}_x(s)\,ds,\\
(\beta_y\overline{N}_y)_0 &= \liminf_{t\to+\infty}\int_t^{t+\lambda}\beta_y(s)\overline{N}_y(s)\,ds,\\
(\mu_y+\alpha_y)_0 &= \liminf_{t\to+\infty}\int_t^{t+\lambda}(\mu_y(s)+\alpha_y(s))\,ds,\\
(\mu_x+\alpha_x)_0 &= \liminf_{t\to+\infty}\int_t^{t+\lambda}(\mu_x(s)+\alpha_x(s))\,ds.
\end{aligned} \tag{108}$$

$\overline{N}_x(t)$ is some fixed solution of (9) with initial value $\overline{N}_x(0) > 0$ and $\overline{N}_y(t)$ is some fixed solution of (11) with initial value $\overline{N}_y(0) > 0$.

Corollary 21. *Suppose system (3) is ω-periodic; we suppose that $\mu_x(t) > 0$, $\mu_y(t) > 0$ for all $t \geq 0$ and $0 < \liminf_{t\to\infty}(A_x(t)/(\mu_x(t)+\alpha_x(t))) \leq \limsup_{t\to\infty}(A_x(t)/\mu_x(t)) < \infty$, $0 < \liminf_{t\to\infty}(A_y(t)/(\mu_y(t)+\alpha_y(t))) \leq \limsup_{t\to\infty}(A_y(t)/\mu_y(t)) < \infty$. Then if there exist positive constants $\lambda > 0$, $p > 0$, $q > 0$, and $T_1 > 0$ such that*

$$p > \frac{\overline{(\theta)}}{\overline{\overline{(\mu_y+\alpha_y)}}},\quad q > \frac{\overline{(\beta_x\overline{N}_x)}}{\overline{\overline{(\mu_x+\alpha_x)}}},\quad pq < \frac{\overline{(\mu_y+\theta)}}{\overline{\overline{(\beta_y\overline{N}_y)}}} \tag{109}$$

and $G_1(p,t) < 0$, $G_2(p,q,t) < 0$ for all $t \geq T_1$, then the two infectious populations $x_i(t)$ and $y_i(t)$ of system (3) are extinct. Here, $\overline{N}_x(t)$ is the globally uniformly attractive nonnegative ω-periodic solution of (9) and $\overline{N}_y(t)$ is the globally uniformly attractive nonnegative ω-periodic solution of (11).

Corollary 22. *Suppose system (3) is almost periodic; also suppose that $\mu_x(t) > 0$, $\mu_y(t) > 0$ for all $t \geq 0$ and $0 < \liminf_{t\to\infty}(A_x(t)/(\mu_x(t)+\alpha_x(t))) \leq \limsup_{t\to\infty}(A_x(t)/\mu_x(t)) < \infty$, $0 < \liminf_{t\to\infty}(A_y(t)/(\mu_y(t)+\alpha_y(t))) \leq \limsup_{t\to\infty}(A_y(t)/\mu_y(t)) < \infty$. Then if there exist positive*

constants $\lambda > 0$, $p > 0$, $q > 0$, *and* $T_1 > 0$ *such that*

$$p > \frac{m(\theta)}{m(\mu_y + \alpha_y)},$$
$$q > \frac{m(\beta_x \overline{N}_x)}{m(\mu_x + \alpha_x)}, \tag{110}$$
$$pq < \frac{m(\mu_y + \theta)}{m(\beta_y \overline{N}_y)}$$

and $G_1(p,t) < 0$, $G_2(p,q,t) < 0$ *for all* $t \geq T_1$, *then the two infectious populations* $x_i(t)$ *and* $y_i(t)$ *of system (3) are extinct. Here* $\overline{N}_x(t)$ *is the globally uniformly attractive nonnegative almost periodic solution of (9) and* $\overline{N}_y(t)$ *is the globally uniformly attractive nonnegative almost periodic solution of (11).*

6. Numerical Simulation

Numerical verification of the results is necessary for completeness of the analytical study. In this section, we will present some numerical simulations to substantiate and discuss our analytical findings of system (3) by the means of the software Matlab.

In order to testify the validity of our results, in system (3), fix $A_x(t) = 0.5 + 0.1\sin(t)$, $A_y(t) = 0.6 + 0.1\sin(t)$, $\mu_x(t) = 0.2 + 0.18\sin(t)$, $\mu_y(t) = 0.3 + 0.28\sin(t)$, $\alpha_x(t) = 0.4 + 0.3\sin(t)$, $\alpha_y(t) = 0.5 + 0.2\sin(t)$, $\theta(t) = 0.28 + 0.2\sin(t)$, $\beta_x(t) = 0.2 + 0.2\sin(t)$, $\beta_y(t) = 0.1 + 0.1\sin(t)$. It is easy to verify that (i) and (H2) hold. From (8)–(11), we have

$$\lim_{t\to\infty} \underline{N}_x(t) = \frac{0.5 + 0.1\sin(t)}{0.6 + 0.48\sin(t)},$$
$$\lim_{t\to\infty} \overline{N}_x(t) = \frac{0.5 + 0.1\sin(t)}{0.2 + 0.18\sin(t)},$$
$$\lim_{t\to\infty} \underline{N}_y(t) = \frac{0.6 + 0.1\sin(t)}{0.8 + 0.48\sin(t)}, \tag{111}$$
$$\lim_{t\to\infty} \overline{N}_y(t) = \frac{0.6 + 0.1\sin(t)}{0.3 + 0.28\sin(t)}.$$

Then system (3) becomes periodic with period 2π. We choose $\lambda = 2\pi$, $p = 3$, and $q = 1$; then we have

$$G_1(p,t) = -0.1267 + 0.0667\sin(t) < 0,$$
$$G_2(p,q,t) = \frac{(0.2 + 0.2\sin(t))(0.5 + 0.1\sin(t))}{0.2 + 0.18\sin(t)} - \frac{67}{75} - \frac{7}{15}\sin(t). \tag{112}$$

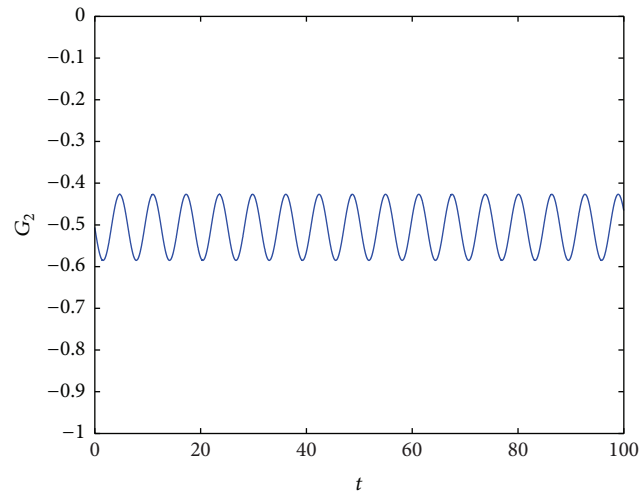

Figure 1: The figure shows that movement paths of $G_2(p,q,t)$ as functions of time t.

Obviously, it is easy to see that $G_2(p,q,t) < 0$ for all $t > 0$ from Figure 1. Furthermore, we can get that

$$3 = p > \frac{\overline{(\theta)}}{\overline{(\mu_y + \alpha_y)}} \simeq 0.35,$$
$$1 = q > \frac{\overline{(\beta_x \overline{N}_x)}}{\overline{(\mu_x + \alpha_x)}} \simeq 0.8279, \tag{113}$$
$$3 = pq < \frac{\overline{(\mu_y + \theta)}}{\overline{(\beta_y \overline{N}_y)}} \simeq 3.2402.$$

Hence, from Corollary 21, we get that the disease will be extinct (see Figure 2). Increasing the infective rate to $\beta_x(t) = 0.8 + 0.1\sin(t)$, $\beta_y(t) = 0.5 + 0.1\sin(t)$, from Corollary 18, we can get $\overline{R}_{02} = \overline{(3\sqrt[3]{\beta_x \beta_y \theta \underline{N}_x \underline{N}_y})} / \overline{((\mu_x + \alpha_x) + (\mu_y + \alpha_y) + (\mu_y + \theta))} = 4.1087 > 1$, which means the condition of Corollary 18 is satisfied. The conclusion of Corollary 18 is verified (see Figure 3).

Parameters

A_x: Per capita birth rate of *Rattus norvegicus*
μ_x: Per capita natural death rate of *Rattus norvegicus*
α_x: Per capita disease induced death rate of *Rattus norvegicus*
β_x: Per capita contact transmission rate from infected snails to susceptible *Rattus norvegicus*
A_y: Per capita birth rate of snail host

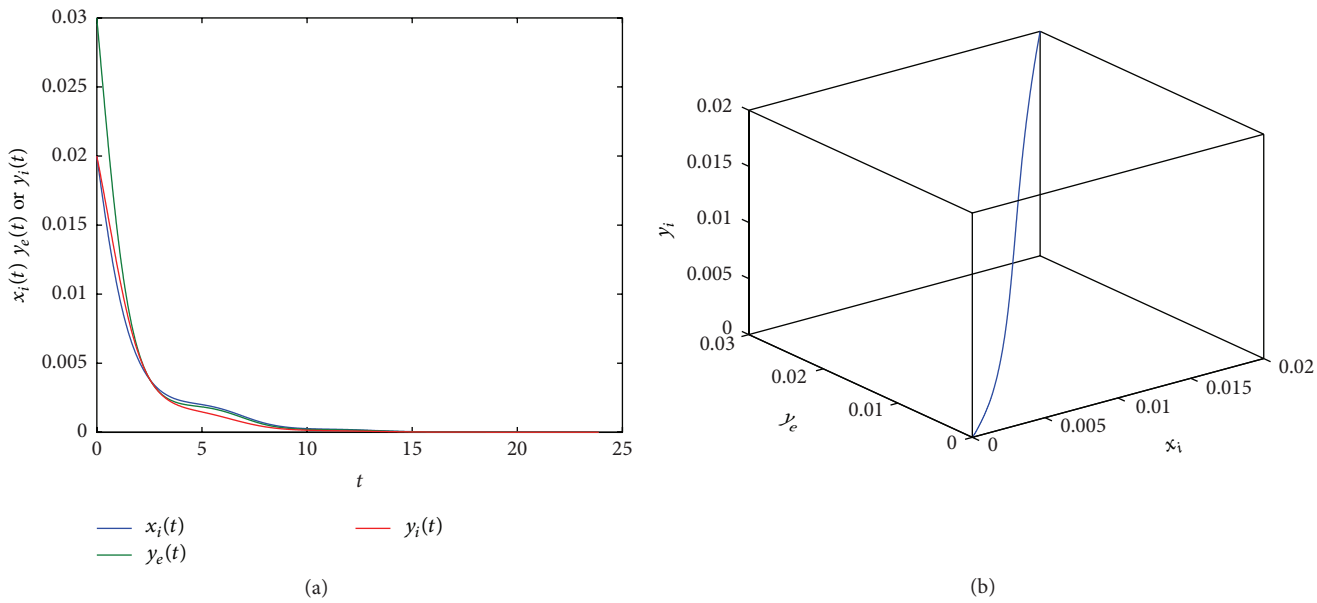

Figure 2: (a) shows movement paths of x_i, y_e, and y_i as functions of time t. The graph of the trajectory in (x_i, y_e, y_i)-space is shown in (b). The disease is extinct.

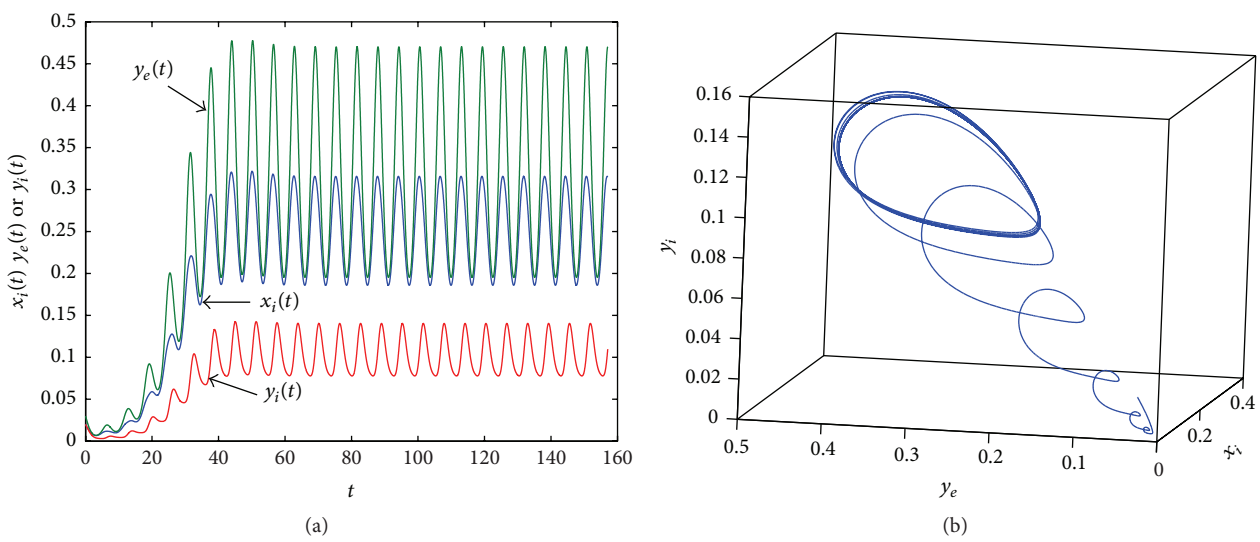

Figure 3: (a) shows movement paths of x_i, y_e, and y_i as functions of time t. The graph of the trajectory in (x_i, y_e, y_i)-space is shown in (b). System (3) is permanent.

μ_y: Per capita natural death rate of snail host
α_y: Per capita disease induced death rate of snail host
β_y: Per capita contact transmission rate from infected *Rattus norvegicus* to susceptible snails
θ: Per capita transition rate from infected and preshedding snails to shedding snails.

Competing Interests

The authors declare that they have no competing interests.

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