Research Article

Dynamic Analysis for a Fractional-Order Autonomous Chaotic System

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We introduce a discretization process to discretize a modified fractional-order optically injected semiconductor lasers model and investigate its dynamical behaviors. More precisely, a sufficient condition for the existence and uniqueness of the solution is obtained, and the necessary and sufficient conditions of stability of the discrete system are investigated. The results show that the system's fractional parameter has an effect on the stability of the discrete system, and the system has rich dynamic characteristics such as Hopf bifurcation, attractor crisis, and chaotic attractors.

1. Introduction

The idea of fractional-order calculus (FOC) has been well known since the development of the regular calculus. The significant progress on FOC has been witnessed because the FOC has a wide range of applications in diffusion. In the past decades, chaotic systems have become a focal point of renewed interest for many researchers. And we can find these nonlinear systems in various natural and man-made systems, which are known to have great sensitivity to initial conditions. And we can find chaos phenomenon in various natural and man-made systems, and these chaotic systems have great sensitivity to initial conditions. Because differential equations with fractional order can be applied in many areas of science and engineering, they attracted many researchers' attention and their complex behaviors have been widely studied in recent years. Nowadays, there is increasing interest in the subject of a fractional model which can give a more realistic interpretation of practical phenomena. Furthermore, many systems in interdisciplinary fields can be described by the fractional differential equations, such as turbulence, electromagnetism, signal processing, and quantum evolution of complex systems. It is also demonstrated that some fractional-order differential systems behave chaotically or hyperchaotically, such as fractional-order Chen system [1–3]

and fractional-order Lorenz system [4, 5]. The fractional-order equations are more suitable than integer-order ones to describe the biological, economic, and social systems where memory effects are important [6]. More recently, it has been found that some differential systems with fractional order possess chaotic behavior [7–16]. And many Lorenz-like or Lorenz-based chaotic systems were proposed and investigated. Some classical 3D autonomous chaotic systems have three particular fixed points: one saddle and two unstable saddle foci [17]. And the other 3D chaotic systems have two unstable saddle foci [18, 19]. Yang and Chen [20] found another 3D chaotic system with three fixed points: one saddle and two stable equilibria. However, an increasing number of three-dimensional chaotic systems have been found these years in many physical and engineering fields. Nonlinear dynamics of a semiconductor laser subject to optical injection is currently a hot research field due to its rich physics and complexity as well as its potential applications in communications [21, 22]. Momani et al. [23] applied the multistep generalized differential transform method to solve the fractional-order multiple chaotic FitzHugh-Nagumo (FHN) neurons model. They illustrated the algorithm by studying the dynamics of three coupled chaotic FHN neurons equations with different gap junctions under external electrical stimulation, and the fractional derivatives are described in

the Caputo sense. The numerical simulation results show that only few terms are required to deduce the approximate solutions which are found to be accurate and efficient.

Optically injected semiconductor lasers revealed amazingly rich behaviors like stable locking, coexistence of attractors, quasiperiodicity, instabilities, pulsations, and many routes to chaos such as period-doubling cascades, intermittency, breakup of tori, and homoclinic and heteroclinic tangencies [24–30], which deserve further systematic investigation. Wieczorek et al. [31] studied the dynamics and bifurcations of a single-mode semiconductor laser with optical injection and investigated the dependence of the dynamics on the injected field strength K and its detuning ω from the unperturbed laser resonant frequency and the linewidth enhancement factor α which seem to have the most significant influence on the dynamics. Chlouverakis and Adams [32] discussed the dynamics of a semiconductor laser subject to optical injection using the method of the Largest Lyapunov Exponent (LLE) and used this model as one of three examples to investigate the relation between the correlation (D_2) and the Kaplan-Yorke dimension (D_{KY}) of three-dimensional chaotic flows. Chu et al. [33] studied the dynamical properties of the semiconductor laser subject to optical injection and a new almost-Hamilton chaotic system with very high Lyapunov dimensions is constructed and investigated. The aim of this paper is to investigate the dynamical behavior of a discretization fractional order of a modified optically injected semiconductor lasers model.

The paper is organized as follows. In Section 2, we give some related preliminaries. We present the fractional-order semiconductor lasers model and introduce a discretization process to discretize a modified fractional-order optically injected semiconductor lasers model in Section 3. In Section 4, a sufficient condition for existence and uniqueness of the solution of the fractional-order system is investigated and the equilibrium points and their asymptotic stability in the fractional-order model and its discrete counterpart are also studied. In Section 5, we give some numerical simulations, which not only illustrate our results with the theoretical analysis, but also exhibit the complex dynamical behaviors of the discretization fractional order of a modified optically injected semiconductor lasers model. Finally, conclusions are given in Section 6.

2. Preliminaries

The Caputo definition of fractional derivative [34] is given as follows:

$$D^\alpha f(t) = I^{l-\alpha} f^{(l)}(t), \quad \alpha > 0, \quad (1)$$

where l is the least integer which is not less than α and I^θ is the Riemann-Liouville integral operator of order θ which is given by

$$I^\theta u(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-\tau)^{(\theta-1)} u(\tau) d\tau, \quad \theta > 0, \quad (2)$$

where $\Gamma(\theta)$ is Euler's Gamma function. The operator D^α is termed "Caputo differential operator of order α ." In [35, 36],

the geometric and physical interpretations of the fractional derivatives were investigated.

The stability conditions and their applications to fractional-order differential equations were reported in [37–40]. The local stability of the equilibrium points of a linearized fractional-order system can be obtained from the following Matignon result [37]:

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}, \quad (i = 1, 2, 3), \quad (3)$$

where λ_1, λ_2 , and λ_3 are the eigenvalues of the system. Then, we consider the following nonlinear autonomous fractional-order system:

$$D^\alpha X(t) = f(X(t)), \quad X(0) = X_0, \quad (4)$$

where $X(t) = (x_1, x_2, x_3)^T \in R^3$ and $f : R^3 \rightarrow R^3$ is a nonlinear vector function in terms of X . The Jacobian matrix evaluated at the equilibrium point $X^* = (x_1^*, x_2^*, x_3^*)$ is

$$J(X^*) = \left(\frac{\partial f_i}{\partial x_j} \right)_{ij} \Big|_{X=X^*}. \quad (5)$$

Hence, we have the following lemma [41].

Lemma 1. *If all the eigenvalues λ_1, λ_2 , and λ_3 of the equilibrium point X^* of system (4) satisfy Matignon's condition (3), then X^* is locally asymptotically stable.*

3. Fractional-Order Semiconductor Lasers Model and Its Discretization

Chlouverakis and Adams [32] present a modified optically injected semiconductor lasers model as follows:

$$\begin{aligned} \frac{\partial E_x}{\partial t} &= K + \frac{1}{2}E_x + \left(\omega - \frac{1}{2}\alpha n\right)E_y, \\ \frac{\partial E_y}{\partial t} &= -\left(\omega - \frac{1}{2}\alpha n\right)E_x + \frac{1}{2}E_y n, \\ \frac{\partial n}{\partial t} &= -2\Gamma n - (1 + 2Bn)(E_x^2 + E_y^2 - 1), \end{aligned} \quad (6)$$

where the complex electric field $E = E_x + iE_y$ and n is the normalized population inversion, K is the injection strength, ω is the detuning, and α is the linewidth enhancement factor. And the parameters $B = \omega_R/(2\Gamma_0)$ and $\Gamma = \Gamma_N/(2\omega_R) + B$, Γ_0 is the inverse photon lifetime, Γ_N is the inverse electron lifetime, and ω_R is the angular relaxation oscillation frequency. For simplification, in this paper, we established a dimensionless modified optically injected semiconductor lasers model by dimensionless method:

$$\begin{aligned} \dot{x} &= a + z(x - by), \\ \dot{y} &= z(bx - cy), \\ \dot{z} &= 1 - x^2 - y^2, \end{aligned} \quad (7)$$

FIGURE 1: (a) Bifurcation diagram for specific values set ($a = 0.4$, $b = 3$) versus c , (b) Lyapunov-exponent spectrum against c corresponding to the bifurcation diagram of (a), and (c) Kaplan-Yorke dimension versus the control parameter c corresponding to the bifurcation diagram of (a).

where $(x, y, z) \in R^3$ are the state variables and $(a, b, c) \in R^3$ are real parameters. The bifurcation diagram of $|y|_{\max}$ versus control parameter c , the Lyapunov-exponent spectrum, and the Kaplan-Yorke dimension for specific values set ($a = 0.4$, $b = 3$) versus the bifurcation parameter c on the open interval $(-8, 1)$ are shown in Figures 1(a)–1(c), respectively.

System (7) is dissipative and exhibits a chaotic attractor when the parameter values $a = 0.4$, $b = 3$, and $c = 0.73$. Figure 2 shows the chaotic attractor and the Poincaré map in $x - z\{y = 0\}$ plane of chaotic system (7).

Next, we will investigate the following fractional order of a modified optically injected semiconductor lasers model:

$$\begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= a + z(x - by), \\ \frac{d^\alpha y}{dt^\alpha} &= z(bx - cy), \\ \frac{d^\alpha z}{dt^\alpha} &= 1 - x^2 - y^2, \end{aligned} \quad (8)$$

where α is the fractional order satisfying $0 < \alpha \leq 1$ and d^α/dt^α is in the sense of the Caputo fractional derivative.

And the discretization process of the fractional-order modified optically injected semiconductor lasers model is given as follows.

Assume that $x(0) = x_0$, $y(0) = y_0$, and $z(0) = z_0$ are the initial conditions of system (8). So, the discretization of system (8) with piecewise constant arguments is given as

$$\begin{aligned} D^\alpha x(t) &= a + z\left(\left[\frac{t}{s}\right]s\right)\left(x\left(\left[\frac{t}{s}\right]s\right) - by\left(\left[\frac{t}{s}\right]s\right)\right), \\ D^\alpha y(t) &= z\left(\left[\frac{t}{s}\right]s\right)\left(bx\left(\left[\frac{t}{s}\right]s\right) - cy\left(\left[\frac{t}{s}\right]s\right)\right), \\ D^\alpha z(t) &= 1 - \left(x\left(\left[\frac{t}{s}\right]s\right)\right)^2 - \left(y\left(\left[\frac{t}{s}\right]s\right)\right)^2. \end{aligned} \quad (9)$$

First, we let $t \in [0, s)$, so $t/s \in [0, 1)$. Thus, we obtain

$$\begin{aligned} D^\alpha x(t) &= a + z_0(x_0 - by_0), \\ D^\alpha y(t) &= z_0(bx_0 - cy_0), \\ D^\alpha z(t) &= 1 - x_0^2 - y_0^2, \end{aligned} \quad (10)$$

and the solution of (10) is reduced to

$$\begin{aligned} x_1(t) &= x_0 + I^\alpha(a + z_0(x_0 - by_0)) \\ &= x_0 + \frac{t^\alpha}{\alpha\Gamma(\alpha)}(a + z_0(x_0 - by_0)), \end{aligned}$$

FIGURE 2: (a) Time history and (b) Poincaré map in $x - z\{y = 0\}$ plane of chaotic system (7).

$$\begin{aligned}
 y_1(t) &= y_0 + I^\alpha (z_0 (bx_0 - cy_0)) \\
 &= y_0 + \frac{t^\alpha}{\alpha\Gamma(\alpha)} (z_0 (bx_0 - cy_0)), \\
 z_1(t) &= z_0 + I^\alpha (1 - x_0^2 - y_0^2) \\
 &= z_0 + \frac{t^\alpha}{\alpha\Gamma(\alpha)} (1 - x_0^2 - y_0^2).
 \end{aligned} \tag{11}$$

Second, let $t \in [s, 2s)$, so $t/s \in [1, 2)$. Hence, we get

$$\begin{aligned}
 D^\alpha x(t) &= a + z_1 (x_1 - by_1), \\
 D^\alpha y(t) &= z_1 (bx_1 - cy_1), \\
 D^\alpha z(t) &= 1 - x_1^2 - y_1^2,
 \end{aligned} \tag{12}$$

which have the following solutions:

$$\begin{aligned}
 x_2(t) &= x_1(s) + I^\alpha (a + z_1(s) (x_1(s) - by_1(s))) \\
 &= x_1(s) \\
 &\quad + \frac{(t-s)^\alpha}{\alpha\Gamma(\alpha)} (a + z_1(s) (x_1(s) - by_1(s))), \\
 y_2(t) &= y_1(s) + I^\alpha (z_1(s) (bx_1(s) - cy_1(s))) \\
 &= y_1(s) + \frac{(t-s)^\alpha}{\alpha\Gamma(\alpha)} (z_1(s) (bx_1(s) - cy_1(s))), \\
 z_2(t) &= z_1(s) + I^\alpha (1 - x_1^2(s) - y_1^2(s)) \\
 &= z_1(s) + \frac{(t-s)^\alpha}{\alpha\Gamma(\alpha)} (1 - x_1^2(s) - y_1^2(s)).
 \end{aligned} \tag{13}$$

Thus, after repeating the discretization process n times, we obtain

$$\begin{aligned}
 x_{n+1}(t) &= x_n(ns) + I^\alpha (a + z_n(ns) (x_n(ns) - by_n(ns))) \\
 &= x_n(ns) \\
 &\quad + \frac{(t - ns)^\alpha}{\alpha\Gamma(\alpha)} (a + z_n(ns) (x_n(ns) - by_n(ns))), \\
 y_{n+1}(t) &= y_n(ns) + I^\alpha (z_n(ns) (bx_n(ns) - cy_n(ns))) \\
 &= y_n(ns) \\
 &\quad + \frac{(t - ns)^\alpha}{\alpha\Gamma(\alpha)} (z_n(ns) (bx_n(ns) - cy_n(ns))), \\
 z_{n+1}(t) &= z_n(ns) + I^\alpha (1 - x_n^2(ns) - y_n^2(ns)) \\
 &= z_n(ns) + \frac{(t - ns)^\alpha}{\alpha\Gamma(\alpha)} (1 - x_n^2(ns) - y_n^2(ns)),
 \end{aligned} \tag{14}$$

where $t \in [ns, (n+1)s)$. For $t \rightarrow (n+1)s$, system (14) is reduced to

$$\begin{aligned}
 x_{n+1} &= x_n + I^\alpha (a + z_n (x_n - by_n)) \\
 &= x_n + \frac{s^\alpha}{\alpha\Gamma(\alpha)} (a + z_n (x_n - by_n)), \\
 y_{n+1} &= y_n + I^\alpha (z_n (bx_n - cy_n)) \\
 &= y_n + \frac{s^\alpha}{\alpha\Gamma(\alpha)} (z_n (bx_n - cy_n)), \\
 z_{n+1} &= z_n + I^\alpha (1 - x_n^2 - y_n^2) \\
 &= z_n + \frac{s^\alpha}{\alpha\Gamma(\alpha)} (1 - x_n^2 - y_n^2).
 \end{aligned} \tag{15}$$

4. Dynamical Behaviors of the Fractional-Order Semiconductor Lasers Model

4.1. Existence and Uniqueness of the Solution. The fractional order of the modified optically injected semiconductor lasers system can be written as

$$D^\alpha X(t) = F(X(t)), \quad t \in (0, T], \quad X(0) = X_0, \quad (16)$$

where

$$\begin{aligned} X &= \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \\ X_0 &= \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \\ F(X) &= \begin{bmatrix} a + z(x - by) \\ z(bx - cy) \\ 1 - x^2 - y^2 \end{bmatrix}. \end{aligned} \quad (17)$$

Define the supremum norm as

$$\|N\| = \sup_{t \in (0, T]} |N(t)|; \quad (18)$$

then, the norm of the matrix $M = [m_{ij}[t]]$ is defined by

$$\|M\| = \sum_{i,j} \sup_{t \in (0, T]} |m_{ij}[t]|. \quad (19)$$

We investigate the existence and uniqueness of the solution in the region $\Omega \times (0, T]$, where

$$\Omega = \{(x, y, z) : \max(|x|, |y|, |z|) \leq \xi\}. \quad (20)$$

Hence, we can get the solution of system (8) as follows:

$$X = X_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} F(X(\tau)) d\tau, \quad (21)$$

so

$$\begin{aligned} H(X_1) - H(X_2) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} (F(X_1(\tau)) - F(X_2(\tau))) d\tau. \end{aligned} \quad (22)$$

Thus, we get the following inequality:

$$\begin{aligned} |H(X_1) - H(X_2)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t |(t - \tau)^{\alpha-1} (F(X_1(\tau)) - F(X_2(\tau)))| d\tau, \end{aligned} \quad (23)$$

which yields

$$\begin{aligned} \|H(X_1) - H(X_2)\| &\leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \max\{-\xi e^{-\xi}, c\xi e^{-c\xi}, 0\} \|X_1 - X_2\|, \end{aligned} \quad (24)$$

where

$$L = \frac{T^\alpha}{\Gamma(\alpha + 1)} \max\{-\xi e^{-\xi}, c\xi e^{-c\xi}, 0\}; \quad (25)$$

the mapping $X = H(X)$ is a contraction mapping if $L < 1$, and the following theorem gives the sufficient condition for existence and uniqueness of the solution of system (8).

Theorem 2. *The sufficient condition for existence and uniqueness of the solution of system (8) in the region $\Omega \times (0, T]$ with initial conditions $X(0) = X_0$ and $t \in (0, T]$ is*

$$L = \frac{T^\alpha}{\Gamma(\alpha + 1)} \max\{-\xi e^{-\xi}, c\xi e^{-c\xi}, 0\} < 1. \quad (26)$$

4.2. Stability of the Fixed Points of the Discrete System. In order to find the equilibrium points of system (8), we let

$$\begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= 0, \\ \frac{d^\alpha y}{dt^\alpha} &= 0, \\ \frac{d^\alpha z}{dt^\alpha} &= 0, \end{aligned} \quad (27)$$

and from direct calculation we know that the system has at most five nonnegative equilibrium points:

$$\begin{aligned} E_0 &= \left(\frac{c}{\sqrt{c^2 + b^2}}, \frac{b}{\sqrt{c^2 + b^2}}, \frac{a\sqrt{c^2 + b^2}}{b^2 - c} \right), \\ E_1 &= \left(-\frac{c}{\sqrt{c^2 + b^2}}, -\frac{b}{\sqrt{c^2 + b^2}}, -\frac{a\sqrt{c^2 + b^2}}{b^2 - c} \right). \end{aligned} \quad (28)$$

By considering the Jacobian matrices for these fixed points and calculating their eigenvalues, we can investigate the local stability of these points based on the roots of the model's characteristic equation [42]. The Jacobian matrix of system (15) is given by

$$\begin{aligned} J(x_n, y_n, z_n) &= \begin{pmatrix} 1 + \frac{s^\alpha z_n}{\alpha \Gamma(\alpha)} & -\frac{s^\alpha b z_n}{\alpha \Gamma(\alpha)} & \frac{s^\alpha (x_n - b y_n)}{\alpha \Gamma(\alpha)} \\ \frac{s^\alpha b z_n}{\alpha \Gamma(\alpha)} & 1 - \frac{s^\alpha c z_n}{\alpha \Gamma(\alpha)} & \frac{s^\alpha (b x_n - c y_n)}{\alpha \Gamma(\alpha)} \\ -\frac{2s^\alpha x_n}{\alpha \Gamma(\alpha)} & -\frac{2s^\alpha y_n}{\alpha \Gamma(\alpha)} & 1 \end{pmatrix}. \end{aligned} \quad (29)$$

Theorem 3. *For system (15), the following statements hold true:*

- (i) E_0 is unstable if $ac > 0$.
- (ii) E_0 is locally asymptotically stable if and only if $0 < a < \alpha \Gamma(\alpha) / s^\alpha - 2s^\alpha / \alpha \Gamma(\alpha)$.

Proof. The Jacobian matrix evaluated at the fixed point E_0 is

$$J(E_0) = \begin{pmatrix} 1 - \frac{s^\alpha a}{\alpha \Gamma(\alpha)} & 0 & \frac{s^\alpha}{\alpha \Gamma(\alpha)} \\ 0 & 1 + \frac{s^\alpha ca}{\alpha \Gamma(\alpha)} & 0 \\ \frac{2s^\alpha}{\alpha \Gamma(\alpha)} & 0 & 1 \end{pmatrix}. \quad (30)$$

The eigenvalues of $J(E_0)$ are $\lambda_2 = 1 + s^\alpha ca/\alpha \Gamma(\alpha)$ and $\lambda_{1,3} = 1 - s^\alpha a/2\alpha \Gamma(\alpha) \pm (s^\alpha/2\alpha \Gamma(\alpha))\sqrt{a^2 + 8}$. Since $0 < \alpha \leq 1$ and $s > 0$, $s^\alpha/\alpha \Gamma(\alpha)$, $ac > 0$, then $\lambda_2 > 1$ which implies that the fixed point E_0 is unstable. If $0 < a < \alpha \Gamma(\alpha)/s^\alpha - 2s^\alpha/\alpha \Gamma(\alpha)$, the eigenvalues of $J(E_0)$ are $\lambda_{1,3} = 1 - s^\alpha a/2\alpha \Gamma(\alpha) \pm (s^\alpha/2\alpha \Gamma(\alpha))\sqrt{a^2 + 8} < 1$ and $\lambda_2 = 1 + s^\alpha ca/\alpha \Gamma(\alpha) < 1$, so the stability conditions $|\lambda_2| < 1$ and $|\lambda_{1,3}| < 1$ are satisfied if $0 < a < \alpha \Gamma(\alpha)/s^\alpha - 2s^\alpha/\alpha \Gamma(\alpha)$. $\square$

4.2.1. The Case $b = 0$

Theorem 4. For system (15), the following statements hold true:

- (i) E_1 is unstable if $ac < 0$.
- (ii) E_1 is locally asymptotically stable if and only if $a < -2\sqrt{2}$.

Proof. The Jacobian matrix evaluated at the fixed point E_1 is

$$J(E_1) = \begin{pmatrix} 1 + \frac{s^\alpha a}{\alpha \Gamma(\alpha)} & 0 & -\frac{s^\alpha}{\alpha \Gamma(\alpha)} \\ 0 & 1 - \frac{s^\alpha ca}{\alpha \Gamma(\alpha)} & 0 \\ \frac{2s^\alpha}{\alpha \Gamma(\alpha)} & 0 & 1 \end{pmatrix}. \quad (31)$$

The eigenvalues of $J(E_1)$ are $\lambda_2 = 1 - s^\alpha ca/\alpha \Gamma(\alpha)$ and $\lambda_{1,3} = 1 + s^\alpha a/2\alpha \Gamma(\alpha) \pm (s^\alpha/2\alpha \Gamma(\alpha))\sqrt{a^2 - 8}$. Since $0 < \alpha \leq 1$ and $s > 0$, $s^\alpha/\alpha \Gamma(\alpha)$, $ac < 0$, then $\lambda_2 > 1$ which implies that the fixed point E_0 is unstable. If $a < -2\sqrt{2}$, the eigenvalues of $J(E_1)$ are $\lambda_2 = 1 - s^\alpha ca/\alpha \Gamma(\alpha) < 1$ and $\lambda_{1,3} = 1 + s^\alpha a/2\alpha \Gamma(\alpha) \pm (s^\alpha/2\alpha \Gamma(\alpha))\sqrt{a^2 - 8} < 1$, so the stability conditions $|\lambda_2| < 1$ and $|\lambda_{1,3}| < 1$ are satisfied if $a < -2\sqrt{2}$. $\square$

4.2.2. The Case $b \neq 0$. Next, we will discuss the stability of the fixed point E_1 . The Jacobian matrix of system (15) evaluated at E_1 is

$$J(E_1) = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}, \quad (32)$$

where

$$\begin{aligned} \gamma_{11} &= 1 + \frac{s^\alpha a \sqrt{c^2 + b^2}}{\alpha (b^2 - c) \Gamma(\alpha)}, \\ \gamma_{12} &= -\frac{s^\alpha ba \sqrt{c^2 + b^2}}{\alpha (b^2 - c) \Gamma(\alpha)}, \\ \gamma_{13} &= -\frac{s^\alpha (b^2 - c)}{\alpha \Gamma(\alpha) \sqrt{c^2 + b^2}}, \\ \gamma_{21} &= \frac{s^\alpha b \sqrt{c^2 + b^2}}{\alpha (b^2 - c) \Gamma(\alpha)}, \\ \gamma_{22} &= 1 - \frac{s^\alpha ca \sqrt{c^2 + b^2}}{\alpha (b^2 - c) \Gamma(\alpha)}, \\ \gamma_{23} &= 0, \\ \gamma_{31} &= \frac{-2s^\alpha c}{\alpha \sqrt{c^2 + b^2} \Gamma(\alpha)}, \\ \gamma_{32} &= \frac{-2s^\alpha b}{\alpha \sqrt{c^2 + b^2} \Gamma(\alpha)}, \\ \gamma_{33} &= 1. \end{aligned} \quad (33)$$

The characteristic equation of E_1 is given by

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0, \quad (34)$$

where

$$\begin{aligned} A_1 &= 2 - V_1, \\ A_2 &= -3 + V_2 - V_3 - V_4 + V_5, \\ A_3 &= -1 - V_6 + V_2 - V_7 - V_8, \\ V_1 &= \frac{s^\alpha ca \sqrt{c^2 + b^2}}{\alpha (b^2 - c) \Gamma(\alpha)}, \\ V_2 &= \frac{2s^{2\alpha} c (b^2 - c)}{\alpha^2 \Gamma^2(\alpha) (c^2 + b^2)}, \\ V_3 &= \frac{s^\alpha a \sqrt{c^2 + b^2}}{\alpha (b^2 - c) \Gamma(\alpha)}, \\ V_4 &= \frac{s^{2\alpha} b^2 a (c^2 + b^2)}{\alpha^2 (b^2 - c)^2 \Gamma^2(\alpha)}, \\ V_5 &= \frac{s^{2\alpha} ca^2 (c^2 + b^2)}{\alpha^2 (b^2 - c)^2 \Gamma^2(\alpha)}, \\ V_6 &= \frac{-2s^{3\alpha} (b^2 + 2ac^2)}{\alpha^3 \sqrt{c^2 + b^2} \Gamma^3(\alpha)}, \end{aligned}$$

$$V_7 = \frac{s^{2\alpha} (b^2 - ac) a (c^2 + b^2)}{\alpha^2 (b^2 - c)^2 \Gamma(\alpha)},$$

$$V_8 = \frac{s^\alpha (1 - c) a \sqrt{c^2 + b^2}}{\alpha (b^2 - c) \Gamma(\alpha)}.$$
(35)

According to Jury's criterion [42], the fixed point E_1 is locally asymptotically stable if

$$\begin{aligned} 1 + A_1 + A_2 + A_3 &> 0, \\ 1 - A_1 + A_2 - A_3 &> 0, \\ 1 - A_2 + A_1 A_3 - A_3^2 &> 0, \\ 1 + A_2 - A_1 A_3 - A_3^2 &> 0. \end{aligned}$$
(36)

Thus, conditions (36) imply that the fixed point E_0 is locally asymptotically stable if the system's parameters belong to the set

$$\{(a, b, c, \alpha, s) : -[1 + V_6 - V_2 + V_7 + V_8] \in \Omega_i \forall i = 1, 2, 3\},$$
(37)

where

$$\begin{aligned} \Omega_1 &= [V_1 - V_2 + V_3 + V_4 - V_5, -4 + V_1 + V_2 - V_3 - V_4 + V_5], \\ \Omega_2 &= \left[2 - V_1, -\frac{\sqrt{(2 - V_1)^2 - 4(-4 + V_2 - V_3 - V_4 + V_5)}}{2}, 2 \right. \\ &\quad \left. - V_1 - \frac{\sqrt{(2 - V_1)^2 - 4(-4 + V_2 - V_3 - V_4 + V_5)}}{2} \right], \\ \Omega_3 &= \left[-2 + V_1, -\frac{\sqrt{(2 - V_1)^2 - 4(-4 + V_2 - V_3 - V_4 + V_5)}}{2}, -2 \right. \\ &\quad \left. + V_1 + \frac{\sqrt{(2 - V_1)^2 - 4(-4 + V_2 - V_3 - V_4 + V_5)}}{2} \right]. \end{aligned}$$
(38)

Lemma 5 (an explicit criterion of Hopf bifurcation in maps [43]). *Let X^* be a fixed point of the n th-order discrete-time dynamical system; the characteristic equation of the Jacobian matrix $A = (a_{ij})_{n \times n}$ is given as*

$$P_v(\lambda) = \lambda^n + s_1 \lambda^{n-1} + \dots + s_{n-1} \lambda + s_n, \quad (39)$$

where $s_j = s_j(v, k)$, $j = 1, \dots, n$, v is the bifurcation parameter, and k is the control parameter or another parameter to be determined. Consider the sequence of determinants $\Delta_0^\pm(v, k) = 1$, $\Delta_1^\pm(v, k), \dots, \Delta_n^\pm(v, k)$, where

$$\Delta_j^\pm(v, k) = \begin{vmatrix} 1 & s_1 & s_2 & \dots & s_{j-1} \\ 0 & 1 & s_1 & \dots & s_{j-2} \\ 0 & 0 & 1 & \dots & s_{j-3} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix} \pm \begin{vmatrix} s_{n-j+1} & s_{n-j+2} & \dots & s_{n-1} & s_n \\ s_{n-j+1} & s_{n-j+3} & \dots & s_n & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ s_{n-1} & s_n & \dots & 0 & 0 \\ s_n & 0 & \dots & 0 & 0 \end{vmatrix},$$
(40)

$j = 1, \dots, n.$

However, the following conditions hold:

- (H1) *Eigenvalue assignment* $\Delta_{n-1}^-(v_0, k) = 0$, $P_{v_0}(1) > 0$, $(-1)^n P_{v_0}(-1) > 0$, $\Delta_{n-1}^-(v_0, k) > 0$, $\Delta_{n-1}^+(v_0, k) = 0$, $\Delta_j^\pm(v_0, k) > 0$, $j = n-3, n-5, \dots, 1$ (or 2) when n is even (or odd, resp.).
- (H2) *Transversality condition* $d\Delta_{n-1}^-(v_0, k)/dv \neq 0$.
- (H3) *Nonresonance condition* $\cos(2\pi/m) \neq \varphi$ or resonance condition $\cos(2\pi/m) = \varphi$, where $m = 3, 4, 5, \dots$ and $\varphi = 1 - 0.5P_{v_0}(1)\Delta_{n-3}^-(v_0, k)/\Delta_{n-2}^+(v_0, k)$; then Hopf bifurcation occurs at v_0 .

5. Numerical Simulations

Even if it is possible to obtain the analytical conditions, the processing is very difficult. So, in order to analyze the stability of the interior fixed points, we investigate the global dynamical behavior of system (15) through numerical simulations. And we show some attractors of system (15), such as asymptotic behaviors near the equilibrium points, periodic orbits, and strange attractors. In the following, we will use the numerical simulations to show the influence of different fractional-order parameters q and s on the stability of system (15).

We fixed $a = 0.4$ and $b = 3$ and vary the parameters q, s , and c . Let $s = 0.01$. Figures 3(a)–3(c) show the portraits of system (15) with $q = 0.87$, $q = 0.93$, and $q = 0.98$, respectively. The numerical results show that with the increase of fractional-order parameter q the discrete system (15) is consistent with the fractional-order system (8).

When we fixed $s = 0.01$, Figures 3(d)–3(f) depict the attractors of system (15) with $q = 0.7$, $q = 0.8$, and $q = 0.87$, respectively. And when $s = 0.001$, we present the phase portraits of system (15) with $q = 0.9$, $q = 0.93$, and $q = 0.97$, as shown in Figures 3(g)–3(i), respectively. From

FIGURE 3: Continued.

FIGURE 3: Trajectory of system (15) in phase space when $a = 0.4$ and $b = 3$.

Figures 3(d)–3(i), we notice that the chaotic behavior of system (15) gradually becomes stable with the reduction of the parameter s and the gradual increase in the fractional-order parameter q .

Figure 3(j) is the strange attractor of system (15) with $q = 0.98$ and $s = 0.001$. Obviously, it confirms the above result; this is because Figure 3(j) shows that the reversibility of the above result, namely, the stable behavior of system (15), is destabilizing when s is small and decreasing the fractional-order parameter q .

We let the time series length be equal to 3000. Figure 4(a) shows the curve of $\log C(r, m) - \log r$ when the parameter m varies in $[1, 18]$, and the error bars are purely statistical. The curves attain a constant slope (the same for all curves) as $r \rightarrow 0$ and the slope should be equal to the information dimension. The slope curve of the time series is shown in Figure 4(b). In Figure 4(c), with the various embedding dimension, correlation exponent is on the increase. When $m \geq 8$, the sharp increase to very big values indicates that $D_2 = 1.1466$. When embedding dimension increases to 7, the curve above it appears to converge and we compute the Kolmogorov entropy $K = 0.3096$. From Figure 4(d), we found that the curves appear to converge to this curve at least up to the accuracy that we could hope to achieve in such numerical experiments. In general, results were found to be more sensitive to the number of iterations than to the averaging number.

From the above numerical experiments, we get the following results:

- (1) When embedding dimension varies from 6 to 8, the correlation integral begins to gradually saturate. A correlation exponent of the different system verified by the number of the time series needs specific analysis.
- (2) If embedding dimension does not exceed the saturated values of the minimum embedding dimension, then the estimation of the correlation exponents is not up to the precision with which we could hope to extract the metric entropy.

- (3) The constancy of the slopes will break down for r smaller than some value r_0 . With the incensement of the embedding dimension, the distances between points increase, and r_0 shifts to higher values of r . When the embedding dimension is to achieve a certain value, the Kolmogorov entropy reaches saturation and begins to converge.

However, when the bifurcation diagram loses its continuity, this means that the state of the system is either quasiperiodic or chaotic, as shown in Figures 5(a) and 5(b). And, according to Lemma 5, we select $a = 0.4, b = 3, q = 0.98$, and $s = 0.01$ and draw the phase diagram of system (15) with $c = -0.8, c = -1$, and $c = -1.9$, as shown in Figures 6–8, respectively. From Figures 6–8, we know that system (15) loses its stability with the gradual decrease of bifurcation parameter c . That is to say, by choosing an appropriate bifurcation parameter, we can prove that the Hopf bifurcation occurs when the bifurcation parameter passes through the critical value.

6. Conclusion

In this paper, we introduced a fractional order of modified optically injected semiconductor lasers model and discretized this system by using a new discretization technique. More precisely, a sufficient condition for existence and uniqueness of the solution of the proposed fractional-order system is investigated, and we also studied the local stability of the equilibrium of the discrete fractional-order semiconductor lasers system. Moreover, the results showed that the fractional parameter q has an effect on the stability of the discrete system. And the chaotic behavior of system (15) will be stabilized when reducing the value of parameter s and increasing the fractional-order parameter q . Meanwhile, the stable behavior of system (15) is destabilizing when decreasing the fractional-order parameter q . We also found that the discrete system exhibits much richer dynamical behaviors than its corresponding fractional-order counterpart, and the existence of Hopf bifurcation of discrete system has been

FIGURE 4: At $N = 3000$. (a) Correlation integral curve. Points pertaining to the same value of m are connected by lines. The values of m are $m = 1$ (top curve), 4, 6, ..., 18 (bottom curve); (b) slope curve of the time series. An ordinate value is the slope of $\log C(r, m)$ and $\log r$; (c) correlation exponent curve $D_2 = 1.1466$; (d) the Kolmogorov entropy of system (15) attractor is equal to $D_2 = 0.3096$.

FIGURE 5: (a) Phase trajectory in 3D space with $a = 0.4$, $b = 3$, $c = -1.9$, $q = 0.96$, and $s = 0.001$; (b) interior phase trajectory of (a).

verified. Finally, the numerical simulations are given to show the chaotic attractors of the fractional-order system and the richer dynamics of its discrete counterpart. In addition, we proposed a method to estimate the Kolmogorov entropy and

use it to verify the state of the system. Future work on the topic may be extended to apply the chaos control (OGY method, feedback control method, and adaptive control method) or bifurcation control (the linear and nonlinear feedback

FIGURE 6: Phase diagram of system (15) with $a = 0.4$, $b = 3$, and $c = -0.8$.

FIGURE 7: Phase diagram of system (15) with $a = 0.4$, $b = 3$, and $c = -1$.

FIGURE 8: Phase diagram of system (15) with $a = 0.4$, $b = 3$, and $c = -1.9$.

method, washout-filter method, and frequency domain analysis and approximation method) on the proposed systems and include examples to demonstrate actual solutions of the systems.

Competing Interests

The authors have declared that no competing interests exist.

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