Letter to the Editor

Comment on “Unilateral Global Bifurcation from Intervals for Fourth-Order Problems and Its Applications”

Ziyatkhan Aliyev^{1,2}

¹Baku State University, AZ1148 Baku, Azerbaijan

²Institute of Mathematics and Mechanics NAS of Azerbaijan, AZ1141 Baku, Azerbaijan

Correspondence should be addressed to Ziyatkhan Aliyev; z.aliyev@mail.ru

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In the recent paper W. Shen and T. He and G. Dai and X. Han established unilateral global bifurcation result for a class of nonlinear fourth-order eigenvalue problems. They show the existence of two families of unbounded continua of nontrivial solutions of these problems bifurcating from the points and intervals of the line trivial solutions, corresponding to the positive or negative eigenvalues of the linear problem. As applications of this result, these authors study the existence of nodal solutions for a class of nonlinear fourth-order eigenvalue problems with sign-changing weight. Moreover, they also establish the Sturm type comparison theorem for fourth-order problems with sign-changing weight. In the present comment, we show that these papers of above authors contain serious errors and, therefore, unfortunately, the results of these works are not true. Note also that the authors used the results of the recent work by G. Dai which also contain gaps.

We want to point out that the assertions of the papers [1, 2] cannot be true, as they contradict classical results of mathematical analysis, since nonlinear eigenvalue problems of fourth-order arise in many applications, (see [3, 4] and the references therein).

In the works [1, 2], based on the spectral theory of [5], the authors establish the unilateral global bifurcation result about the continuum of solutions for the following fourth-order eigenvalue problem:

$$\begin{aligned} u^{(4)}(t) &= \mu m(t) u(t) + g(t, u, \mu), \quad t \in (0, 1), \\ u(0) &= u(1) = u''(0) = u''(1) = 0, \end{aligned} \quad (1)$$

where m is a positive function [1] or sign-changing function [2] on $[0, 1]$, $g : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies the Carathéodory condition, and $g(t, 0, \mu) \equiv 0$. Let $I := (0, 1)$ and

$$M(I) := \{m \in C(\bar{I}) \mid \text{meas } \{t \in I, m(t) > 0\} \neq 0\}. \quad (2)$$

(In the paper [1] also studied global bifurcation for nonlinearizable and half-linearizable eigenvalue problems of fourth-order). It is also assumed that the function $g : I \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous and satisfies the following condition:

$$\lim_{s \rightarrow 0} \frac{g(t, s, \mu)}{|s|} = 0, \quad (3)$$

uniformly for $t \in [0, 1]$ and μ on bounded sets.

By (3), the linearization of (1) at $u = 0$ is the spectral problem

$$\begin{aligned} u^{(4)}(t) &= \mu m(t) u(t), \quad t \in (0, 1), \\ u(0) &= u(1) = u''(0) = u''(1) = 0. \end{aligned} \quad (4)$$

Let $E = \{u \in C^3(\bar{I}) \mid u(0) = u(1) = u''(0) = u''(1) = 0\}$ with the norm

$$\begin{aligned} \|u\| &= \max_{t \in \bar{I}} |u(t)| + \max_{t \in \bar{I}} |u'(t)| + \max_{t \in \bar{I}} |u''(t)| \\ &\quad + \max_{t \in \bar{I}} |u'''(t)|. \end{aligned} \quad (5)$$

Let $u \in E$ and t_* such that $u(t_*) = u''(t_*) = 0$. We note that t_* is a generalized simple zero if $u'(t_*) \neq 0$ or $u'''(t_*) \neq 0$. Otherwise, we note that t_* is a generalized double zero. If there is no generalized double zero of u , we note that u is a nodal solution [1, 2].

The linear problem (4) is investigated in [5] (see also [1, Lemma 3]) where, in particular, the following theorem is proved.

Theorem A. *Let $m \in M(I)$. The eigenvalue problem (4) has two sequences of simple real eigenvalues*

$$\begin{aligned} \lim_{k \rightarrow \infty} \mu_k^+ &\rightarrow +\infty, & 0 < \mu_1^+ < \mu_2^+ < \dots < \mu_k^+ < \dots, \\ \lim_{k \rightarrow \infty} \mu_k^- &\rightarrow -\infty, & 0 > \mu_1^- > \mu_2^- > \dots > \mu_k^- > \dots, \end{aligned} \quad (6)$$

and no other eigenvalues. Moreover, for each $k \in \mathbb{N}$ and each $v \in \{+, -\}$, the eigenfunction $v_{k,v}(t)$, corresponding to the eigenvalue μ_k^v , has exactly $k - 1$ generalized simple zeros in I .

Note that if m is positive on $[0, 1]$ then problem (4) has one sequence of positive eigenvalues (see [1]).

Let S_k^+ , $k \in \mathbb{N}$, denote the set of functions in E which have exactly $k - 1$ generalized simple zeros in I and are positive near $t = 0$ and set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$.

Let S be the closure of the set of nontrivial solution of problem (1).

One of the main results of the work [2] is the following theorem which plays an essential role in the study of problems considered in [1].

Theorem B ([2, Theorem 2.2] (see also [1, Lemma 1])). *Assume that (3) holds and $m \in M(I)$, then from each $(\mu_k^v, 0)$ it bifurcates two distinct unbounded continua $(C_k^v)^+$ and $(C_k^v)^-$ of S . Moreover, for $\sigma \in \{+, -\}$, we have that*

$$(C_k^v)^\sigma \subset (\{(\mu_k^v, 0)\} \cup (\mathbb{R} \times S_k^\sigma)). \quad (7)$$

In [1, p. 2] and [2, p. 9401] the authors write that “Clearly, the sets S_k^σ , $k \in \mathbb{N}$, $\sigma \in \{+, -\}$, are disjoint and open in E ” and further they use this assertion to prove Theorem B. It is obvious that these sets are disjoint, but they are not open in E . Hence, the statements of the Theorem B and [1, Theorems 10 and 19] are not true.

Now we will show that for $k > 1$ the set S_k^σ , $k \in \mathbb{N}$, $\sigma \in \{+, -\}$, are not open in E . For simplicity, consider the case $k = 2$ (this fact is shown for any $k > 2$ similarly).

Let $y(t) = \sin 2\pi t$ and $v(t) = \sin \pi t$, $t \in [0, 1]$. Note that $y(t) \in S_2^+$, $v(t) \in E$, and $\|v\| = 1 + \pi + \pi^2 + \pi^3 = c_0$. We take a sufficiently small number ε and consider the function $v_\varepsilon(t) = (\varepsilon/2c_0)v(t)$. Then we have $\|v_\varepsilon\| = \varepsilon/2 < \varepsilon$. Next, we consider the following equations:

$$\begin{aligned} y(t) + v_\varepsilon(t) &= 0, & t \in (0, 1), \\ y''(t) + v_\varepsilon''(t) &= 0, & t \in (0, 1). \end{aligned} \quad (8)$$

The solution of the first equation is $t_{*,1} = (1/\pi) \arccos(-\varepsilon/4c_0)$ and the solution of the second equation is $t_{*,2} = (1/\pi) \arccos(-\varepsilon/16c_0)$ and $t_{*,1} \neq t_{*,2}$. Hence $t_{*,1}$ is not a

generalized zero of the function $y_\varepsilon(t) = y(t) + v_\varepsilon(t)$ in $(0, 1)$, although y_ε is contained in ε neighborhood of the function y in E . This means that S_2^+ is not open in E . Hence S_2^- is also not open in E .

In Section 3 from [2] the authors established the Sturm type comparison theorem for fourth-order differential equations with sign-changing weight, which they used later in [2] and also in [1] (see [1, page 6, left column, line 6 from below]).

Lemma A (see [2, Lemma 3.1]). *Let $b_2(t) > b_1(t) > 0$ for $t \in I$ and $b_i(t) \in C(\bar{I})$, $i = 1, 2$. Also let $u_1, u_2 \in E$ be nontrivial solutions of the following differential equations:*

$$u^{(4)}(t) = b_i(t) u(t), \quad t \in I, \quad i = 1, 2, \quad (9)$$

respectively. If u_1 has k generalized simple zeros in I , then u_2 has at least $k + 1$ generalized simple zeros in I .

The proof of Lemma A contain gaps. Now we demonstrate this fact. Let $v := u_1''$. It follows by (9) that

$$\begin{aligned} u_1''(t) &= v(t), & t \in I, \\ v''(t) &= b_1(t) u_1(t). \end{aligned} \quad (10)$$

In the proof of Lemma A the authors claim that (see [2, p. 9403, formula (3.2)]) “By simple computation, one has that

$$u_1'(t) v'(t) = \frac{v^2(t)}{2} + \frac{b_1(t) u_1^2(t)}{2} + C, \quad (11)$$

for any constant C .”

Multiplying the equations in (10) by v' and u_1' , respectively, and adding both sides we obtain

$$(u_1'(t) v'(t))' = v(t) v'(t) + b_1(t) u_1(t) u_1'(t). \quad (12)$$

Integrating this relation from 0 to t , we have

$$u_1'(t) v'(t) = \frac{v^2(t)}{2} + \int_0^t b_1(s) \left(\frac{u_1^2(s)}{2} \right)' ds + C. \quad (13)$$

Formula (13) shows that the formula (11) is not true, except in the case $b_1(t) \equiv \text{const}$.

We show that the statement of theorem A ([1, Lemma 3]) that the eigenfunction $v_k^v(t)$, $k \in \mathbb{N}$, corresponding to the eigenvalue μ_k^v , has exactly $k - 1$ generalized simple zeros in I is not true. The proof of this statement has been achieved as follows (see proof of Proposition 3.3 from [5]). Let $t_{k,1}^v$ be the first zero of $v_k^v(t)$, $k > 1$, in I . Set $I_1 = (0, t_{k,1}^v)$. For any $\phi \in X_1 \equiv W_0^{1,2}(I_1) \cap W^{2,2}(I_1)$, let $\tilde{\phi}$ be the extension by zero of ϕ on I . It is obvious that $\tilde{\phi} \in X \equiv W_0^{1,2}(I) \cap W^{2,2}(I)$. By Definition 2.1 from [5], we have

$$\begin{aligned} \int_{I_1} (v_k^v)'' \phi'' dt &= \int_I (v_k^v)'' \tilde{\phi}'' dt = \mu_k^v \int_I m(t) v_k^v \tilde{\phi} dt \\ &= \mu_k^v \int_{I_1} m(t) v_k^v \phi dt. \end{aligned} \quad (14)$$

Hence the restriction of v_k^ν in I_1 is a nonnegative solution of the following problem:

$$\begin{aligned} u^{(4)}(t) &= \mu_k^\nu m(t) u(t), \quad t \in (0, t_{k,1}^\nu), \\ u(0) &= u(1) = u''(0) = u''(1) = 0. \end{aligned} \quad (15)$$

In fact, the restriction of v_k^ν in I_1 is a classical solution of problem (15). Indeed, Proposition 2.1 from [5] yields v_k^ν in I is a classical solution of problem (1). Hence v_k^ν satisfies $(v_k^\nu)^{(4)} = \lambda_k^\nu m(t) v_k^\nu$ in I_1 .

It remains to show that $(v_k^\nu)''(t_{k,1}^\nu) = 0$. Let us choose $\phi \in C_0^2(\bar{I}_1) \subset X_1$. Substituting in (14) and integrating by parts we obtain

$$(v_k^\nu)''(t_{k,1}^\nu) \phi'(t_{k,1}^\nu) = 0. \quad (16)$$

Since ϕ is arbitrary, it follows from last equality that $(v_k^\nu)''(t_{k,1}^\nu) = 0$.

Note that this proof is not true. Indeed, if, for any $\phi \in X_1$, one lets $\tilde{\phi}$ be the extension by zero of ϕ on I , then it does not follow that $\tilde{\phi} \in X$. By the embedding theorem (see [6]), we have $X_1 \hookrightarrow C^{1,\alpha}(\bar{I}_1)$ with $0 < \alpha < 1/2$, and so $\phi \in C^{1,\alpha}(\bar{I}_1)$. If we take a function ϕ such that $\phi'(t_{k,1}^\nu - 0) \neq 0$, then by definition of $\tilde{\phi}$ it follows that $\tilde{\phi}'(t_{k,1}^\nu - 0) \neq 0$. But, then again, by definition of $\tilde{\phi}$ we have $\tilde{\phi}'(t_{k,1}^\nu + 0) = 0$. From this we conclude that the function $\tilde{\phi}$ has not a derivative at the point $t_{k,1}^\nu$. Hence, $\tilde{\phi} \notin X$ by the embedding $X \hookrightarrow C^{1,\alpha}(\bar{I})$. Thus the equality (14) is not true for all $\phi \in X_1$.

Competing Interests

There are no competing interests related to this paper.

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