Research Article

Global Dynamics of Rational Difference Equations

$$x_{n+1} = (x_n + x_{n-1})/(q + y_n y_{n-1}) \text{ and } y_{n+1} = (y_n + y_{n-1})/(p + x_n x_{n-1})$$

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Global dynamics of a system of nonlinear difference equations was investigated, which had five kinds of equilibria including isolated points and a continuum of nonhyperbolic equilibria along the coordinate axes. The local stability of these equilibria was analyzed which led to nine regions in the parameters space. The solution of the system converged to the equilibria or the boundary point $(+\infty, 0)$ or $(0, +\infty)$ in each region depending on nonnegative initial conditions. These results completely described the behavior of the system.

1. Introduction

In this paper, we focus on the global dynamics of the following system:

$$\begin{aligned} x_{n+1} &= \frac{x_n + x_{n-1}}{q + y_n y_{n-1}}, \\ y_{n+1} &= \frac{y_n + y_{n-1}}{p + x_n x_{n-1}}, \end{aligned} \quad (1)$$

$$n = 0, 1, \dots,$$

where the parameters p and q are positive and the initial conditions (x_{-1}, y_{-1}) and (x_0, y_0) are nonnegative.

In [1], the stability of (1) was investigated. If $p > 2$ and $q > 2$, the equilibrium $(0, 0)$ of (1) is globally asymptotically stable. If $p < 2$ and $q < 2$, the equilibria $(0, 0)$ and $(\sqrt{2-p}, \sqrt{2-q})$ of (1) are locally unstable. The global dynamics of (1) was considered only for the case $p > 2$ and $q > 2$.

System (1) can be regarded as a generalization of the equation

$$x_{n+1} = \frac{x_n + x_{n-1}}{\beta + x_n x_{n-1}}, \quad n = 0, 1, \dots, \quad (2)$$

with the parameter β being positive and initial conditions x_{-1}, x_0 being nonnegative, which was studied in [2] on the stability of the equilibria, nonexistence of prime period-two solutions, and global dynamics of the equation. More accurate results were obtained in our forthcoming work that every positive solution $\{x_n\}$ of (2) converged to its equilibria, $\bar{x} = 0$ for $\beta \geq 2$ and $\bar{x}_+ = \sqrt{2-\beta}$ for $0 < \beta < 2$.

The above equations and systems are also the special cases of a general equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_n x_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad (3)$$

$$n = 0, 1, \dots,$$

with nonnegative parameters and proper initial conditions. Several global asymptotic results for some special cases of (3) were obtained in [3–13].

As for the definition of stability and the method of linearized stability, see [1–21]. For other types of equations and systems, see [14–19, 22–39]. As for the definition of basin of attraction and the stable manifold and so on, see [35–39].

In this article, we try to determine a complete picture of the behavior of (1). First, we part completely the regions of parameters by equilibria. Second, by the theory of linearized

stability, we describe the local stability of these equilibria for five cases and derive nine regions in (p, q) plane. At last, we present the main results on global dynamics of (1) in these regions.

It is the first time that the parameters spaces are divided into nine regions and complex dynamics of (1) are derived according to the initial conditions for each region. It is also the first time that we determine the details that the equilibrium is nonhyperbolic or a saddle point if it is unstable. It is worth pointing out that the system has a continuum of nonhyperbolic equilibria along a vertical line or/and a horizontal line, which lead to interesting phenomena on the global dynamics.

2. Existence of Equilibria

The study of dynamics of difference equations often requires that equilibria be calculated first, followed by a local stability analysis of the equilibria. This is then complemented by other considerations (existence of periodic points, etc.). If the analysis is applied to a class of equations dependent on one or more parameters, the task is complicated by the fact that a formula is not always available for equilibria, and even if it is, determination of stability of parameter-dependent equilibria may be a daunting task.

First of all, we obtain the existence of equilibria of (1).

As is known, the equilibrium $(\bar{x}, \bar{y})$ of (1) satisfies

$$\begin{aligned}\bar{x} &= \frac{2\bar{x}}{q + (\bar{y})^2}, \\ \bar{y} &= \frac{2\bar{y}}{p + (\bar{x})^2}\end{aligned}\quad (4)$$

from which it follows that one of five cases holds for the equilibrium points of (1):

- (1) $E_0 = (0, 0)$ if one of the following conditions holds:
 - (i) $p > 2$ and $q > 2$.
 - (ii) $p > 2$ and $q < 2$.
 - (iii) $p < 2$ and $q > 2$.
- (2) E_0 and $E_+ = (\sqrt{2-p}, \sqrt{2-q})$ if $p < 2$ and $q < 2$.
- (3) E_0 and $E_{\bar{x}} = (\bar{x}, 0)$ with arbitrary $\bar{x} > 0$ if $p \neq 2$ and $q = 2$.
- (4) E_0 and $E_{\bar{y}} = (0, \bar{y})$ with arbitrary $\bar{y} > 0$ if $p = 2$ and $q \neq 2$.
- (5) E_0 and $E_{\bar{x}} = (\bar{x}, 0)$ and $E_{\bar{y}} = (0, \bar{y})$ with arbitrary $\bar{x} > 0$ and $\bar{y} > 0$ if $p = 2$ and $q = 2$.

3. Local Stability of Equilibria

Now, we consider the local stability of these equilibria of (1).

3.1. Local Stability of E_0

Theorem 1. Suppose that E_0 is the equilibrium of (1). Then one of the following holds:

- (1) E_0 is locally asymptotically stable for $p > 2$ and $q > 2$.

- (2) E_0 is nonhyperbolic of the stable type for one of the following three cases:
 - Case 2.1: $p = 2, q > 2$.
 - Case 2.2: $p > 2, q = 2$.
 - Case 2.3: $p = 2, q = 2$.

- (3) E_0 is nonhyperbolic of the unstable type for one of the following two cases:
 - Case 3.1: $p = 2, q < 2$.
 - Case 3.2: $p < 2, q = 2$.

- (4) E_0 is a saddle point for one of the following three cases:
 - Case 4.1: $p < 2$ and $q < 2$.
 - Case 4.2: $p < 2$ and $q > 2$.
 - Case 4.3: $p > 2$ and $q < 2$.

Proof. The linearized system of (1) about E_0 is

$$\begin{aligned}x_{n+1} &= \frac{1}{q}x_n + \frac{1}{q}x_{n-1}, \\ y_{n+1} &= \frac{1}{p}y_n + \frac{1}{p}y_{n-1}.\end{aligned}\quad (5)$$

As is shown in [1], we have its characteristic polynomial

$$\begin{aligned}f(\lambda) &= f_1(\lambda)f_2(\lambda) \\ &= \left(\lambda^2 - \frac{1}{p}\lambda - \frac{1}{p}\right)\left(\lambda^2 - \frac{1}{q}\lambda - \frac{1}{q}\right).\end{aligned}\quad (6)$$

Here, we only focus on one of these two factors. Obviously, we have

$$\begin{aligned}f_1(0) &= -\frac{1}{p} < 0, \\ f_1(1) &= 1 - \frac{2}{p}, \\ f_1(-1) &= 1 > 0.\end{aligned}\quad (7)$$

Thus, the distribution of solutions of $f_1(\lambda) = 0$ is one of the following:

- (i) Two real roots in $(-1, 1)$ for $p > 2$
- (ii) One root being 1 and the other being -0.5 for $p = 2$
- (iii) One root in $(-1, 0)$ and the other in $(1, +\infty)$ for $p < 2$

By Theorem 1.2.1 in [21], we obtain the conclusions and complete the proof. $\square$

3.2. Local Stability of E_+ . Now, we consider the local stability of the positive equilibrium $E_+ = (\sqrt{2-p}, \sqrt{2-q})$ of (1), which exists only for $p < 2$ and $q < 2$.

Theorem 2. Assume that $p < 2$ and $q < 2$ and $E_+ = (\sqrt{2-p}, \sqrt{2-q})$ is the positive equilibrium of (1); then E_+ is a saddle point.

Proof. The linearized equation of (1) about E_+ is

$$\begin{aligned} x_{n+1} &= \frac{1}{2}x_n + \frac{1}{2}x_{n-1} + \alpha y_n + \alpha y_{n-1}, \\ y_{n+1} &= \alpha x_n + \alpha x_{n-1} + \frac{1}{2}y_n + \frac{1}{2}y_{n-1}, \end{aligned} \quad (8)$$

where $\alpha = -\sqrt{(2-p)(2-q)}/2$. It is obvious that $0 < \alpha^2 < 1$ for $p < 2$ and $q < 2$.

As is shown in [1], its characteristic polynomial is

$$g(\lambda) = \lambda^4 - \lambda^3 - \left(\frac{3}{4} + \alpha^2\right)\lambda^2 + \left(\frac{1}{2} - 2\alpha^2\right)\lambda + \frac{1}{4} - \alpha^2. \quad (9)$$

We have $g(-1) = 1$, $g(-1/2) = -\alpha^2$, $g(1) = -4\alpha^2$, and $g(5/2) = 81/4 - 49\alpha^2/4 > 0$ for $0 < \alpha^2 < 1$. Thus, $g(\lambda) = 0$ has one solution in $(-1, -1/2)$ and one in $(1, 5/2)$.

Now, we divide it into two cases to show the distribution of the other two solutions of $g(\lambda) = 0$.

Case 1 ($\alpha^2 \leq 1/4$). In view of $g(0) = 1/4 - \alpha^2 \geq 0$, we conclude that $g(\lambda) = 0$ has three solutions in $(-1, 1)$ and one in $(1, 2.5)$ and thus E_+ is a saddle point by Theorem 1.2.1 in [21].

Case 2 ($\alpha^2 > 1/4$). In this case, we rewrite (9) as follows:

$$g(\lambda) = (\lambda + \lambda_1)(\lambda - \lambda_2)(\lambda^2 + s\lambda + t), \quad (10)$$

where $1/2 < \lambda_1 < 1$, $1 < \lambda_2 < 5/2$, s and t satisfying

$$s + \lambda_1 - \lambda_2 = -1, \quad (11)$$

$$t + (\lambda_1 - \lambda_2)s - \lambda_1\lambda_2 = -\left(\frac{3}{4} + \alpha^2\right), \quad (12)$$

$$(\lambda_1 - \lambda_2)t - \lambda_1\lambda_2s = \frac{1}{2} - 2\alpha^2, \quad (13)$$

$$-\lambda_1\lambda_2t = \frac{1}{4} - \alpha^2. \quad (14)$$

Next, we try to prove all roots of the equation $\lambda^2 + s\lambda + t = 0$ to be inside the unit disk for $\alpha^2 > 1/4$, which is necessary and sufficient to prove $|s| < 1 + t < 2$ by Theorem 1.2.2 in [21].

First, we try to show $0 < t < 1$ for $\alpha^2 > 1/4$.

From (14), it is obvious $t > 0$ and thus $t < 1$ is equivalent to $\alpha^2 < 1/4 + \lambda_1\lambda_2$. In view of $0 < \alpha^2 < 1$ for $p < 2$ and $q < 2$, we try to prove $\lambda_1\lambda_2 > 3/4$. To this end, we try to determine the exact range of λ_2 .

In view of $\alpha^2 > 1/4$, from $g(3/2) = 1/8 - 25\alpha^2/4 < 0$ and $g(2.5) > 0$, we could obtain $\lambda_2 \in (3/2, 5/2)$. Thus, we have $\lambda_1\lambda_2 > 3/4$ and $t < 1$ is proved.

Second, we show $|s| < 1 + t$ for $\alpha^2 > 1/4$. To this end, we only need to show $|s| < 1$ for $t < 1$.

From (11), we obtain $s + 1 = \lambda_2 - \lambda_1$ and thus $-1/2 < s < 1$ as desired.

In fact, more precisely, in view of $g(2) = 25/4 - 9\alpha^2$, we have that $3/2 < \lambda_2 < 2$ for $1/4 < \alpha^2 < 25/36$ and $2 \leq \lambda_2 < 5/2$ for $\alpha^2 \geq 25/36$.

Therefore, for $\alpha^2 > 1/4$, we have $-1/2 < s < 1/2$; for $1/4 < \alpha^2 < 25/36$ and $0 \leq s < 1$ for $\alpha^2 \geq 25/36$. Hence, we

obtain $|s| < 1 + t$ for $\alpha^2 > 1/4$ and thus all roots of the equation $\lambda^2 + s\lambda + t = 0$ lie inside the unit disk. By Theorem 1.2.1 in [21], E_+ of (1) is a saddle point for $\alpha^2 > 1/4$.

Thus, we conclude that E_+ of (1) is a saddle point if it exists and we complete the proof. $\square$

3.3. Local Stability of $E_{\bar{x}}$. Now, we consider the local stability of the equilibria $E_{\bar{x}} = (\bar{x}, 0)$ ($\bar{x} > 0$) of (1), which exists only for $p \neq 2$ and $q = 2$.

Theorem 3. Assume that $p \neq 2$ and $q = 2$ and $E_{\bar{x}} = (\bar{x}, 0)$ ($\bar{x} > 0$) are the equilibria of (1).

If $p > 2$ or $p < 2$ and $p + (\bar{x})^2 \geq 2$, then $E_{\bar{x}}$ is nonhyperbolic of the stable type.

If $p < 2$ and $p + (\bar{x})^2 < 2$, then $E_{\bar{x}}$ is nonhyperbolic of the unstable type.

Proof. The linearized equation of (1) about every $E_{\bar{x}}$ is

$$\begin{aligned} x_{n+1} &= \frac{1}{2}x_n + \frac{1}{2}x_{n-1}, \\ y_{n+1} &= \frac{1}{p + (\bar{x})^2}y_n + \frac{1}{p + (\bar{x})^2}y_{n-1}, \end{aligned} \quad (15)$$

from which we have its characteristic polynomial

$$\begin{aligned} h(\lambda) &= h_1(\lambda)h_2(\lambda) \\ &= \left(\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2}\right)\left(\lambda^2 - \frac{1}{p + (\bar{x})^2}\lambda - \frac{1}{p + (\bar{x})^2}\right). \end{aligned} \quad (16)$$

It is obvious that $h_1(\lambda) = 0$ has two solutions 1 and -0.5 .

Similarly, if $p < 2$ and $p + (\bar{x})^2 = 2$, then $h_2(\lambda) = 0$ has two solutions 1 and -0.5 .

If $p > 2$ or $p < 2$ and $p + (\bar{x})^2 > 2$, then $h_2(\lambda) = 0$ has two solutions in $(-1, 1)$.

If $p < 2$ and $p + (\bar{x})^2 < 2$, then $h_2(\lambda) = 0$ has one solution in $(-1, 0)$ and the other in $(1, +\infty)$.

By Theorem 1.2.1 in [21], we derive the conclusions and complete the proof. $\square$

3.4. Local Stability of $E_{\bar{y}}$. Similar to the proof of Theorem 3, we have the following theorem.

Theorem 4. Assume that $p = 2$ and $q \neq 2$ and $E_{\bar{y}} = (0, \bar{y})$ ($\bar{y} > 0$) are the equilibria of (1).

If $q > 2$ or $q < 2$ and $q + (\bar{y})^2 \geq 2$, then $E_{\bar{y}}$ is nonhyperbolic of the stable type.

If $q < 2$ and $q + (\bar{y})^2 < 2$, then $E_{\bar{y}}$ is nonhyperbolic of the unstable type.

3.5. Local Stability of $E_{\bar{x}}$ and $E_{\bar{y}}$. In case of $p = q = 2$, the equilibria of (1) include E_0 , $E_{\bar{x}}$, and $E_{\bar{y}}$ ($\bar{x} > 0$, $\bar{y} > 0$).

By Theorem 1, for $p = q = 2$, E_0 of (1) is nonhyperbolic of the stable type.

Similar to the proof of Theorem 3, the linearized equation of (1) about $E_{\bar{x}}$ is (15) with $p = 2$ and its characteristic polynomial is (16) with $p = 2$ which has four roots in $(-1, 1)$. It implies that every $E_{\bar{x}}$ is nonhyperbolic of the stable type.

Similarly, every $E_{\bar{y}}$ is nonhyperbolic of the stable type.

TABLE 1: Local stability of equilibria of (1).

Region	Parameters	Local stability of equilibria
R_1	$p > 2, q > 2$	E_0 – L.A.S.
R_2	$p < 2, q > 2$	E_0 – Saddle
R_3	$p < 2, q < 2$	E_0 – Saddle, E_+ – Saddle
R_4	$p > 2, q < 2$	E_0 – Saddle
R_5	$p = 2, q > 2$	E_0 – N.H.(S), $E_{\bar{y}}$ – N.H.(S)
R_6	$p < 2, q = 2$	E_0 – N.H.(U), $E_{\bar{x}}$ – N.H.
R_7	$p = 2, q < 2$	E_0 – N.H.(U), $E_{\bar{y}}$ – N.H.
R_8	$p > 2, q = 2$	E_0 – N.H.(S), $E_{\bar{x}}$ – N.H.(S)
R_9	$p = 2, q = 2$	E_0 – N.H.(S), $E_{\bar{x}}$ – N.H.(S), $E_{\bar{y}}$ – N.H.(S)

Theorem 5. Assume that $p = q = 2$, E_0 , $E_{\bar{x}}$, and $E_{\bar{y}}$ ($\bar{x} > 0$ and $\bar{y} > 0$) are the equilibria of (1); then they are nonhyperbolic of the stable type.

There are 9 cases in parametric space $p - q$ with distinct local stability of distinct equilibria. We list the above results in Table 1. For simplicity, if E_0 of (1) is locally asymptotically stable, we denote E_0 – L.A.S. If E_0 is nonhyperbolic, we denote E_0 – N.H. If E_0 is nonhyperbolic of the stable type or the unstable type, we denote E_0 – N.H.(S) or N.H.(U). If E_0 is a saddle point, we denote E_0 – Saddle.

4. Global Dynamics

For nonnegative initial conditions (x_i, y_i) ($i = -1, 0$), we assume that $\{(x_n, y_n)\}$ is the corresponding solution of (1). For simplicity, we often need to consider the behavior of $\{x_n\}$ and $\{y_n\}$, respectively.

In the following, we try to investigate the global dynamics of (1) for these nine cases.

Case 1 (R_1). By Theorem 1 in [1], E_0 of (1) is globally asymptotically stable for $p > 2$ and $q > 2$; that is, basin of attraction of E_0 of (1) is $\mathbb{B}(E_0) = \{(x, y) \mid x > 0, y > 0\}$.

Case 2 (R_2). In this case, E_0 of (1) is a saddle point for $p < 2$ and $q > 2$.

If the initial conditions (x_i, y_i) ($i = -1, 0$) are on y -axis, we have $x_n = 0$ for all n , and system (1) is changed into a single equation

$$y_{n+1} = \frac{y_n + y_{n-1}}{p}, \quad (17)$$

from which we have $y_n = c_1(r_1)^n + c_2(r_2)^n$ with r_1 and r_2 satisfying the characteristic equation $r^2 - r/p - 1/p = 0$. For $p < 2$, one of the modulus of r_1 and r_2 is smaller than one and the other is greater than one. Therefore, we have $\lim_{n \rightarrow \infty} y_n = +\infty$ with (x_i, y_i) on y -axis for $p < 2$.

If (x_i, y_i) are on x -axis, we have $y_n = 0$ for all n and thus $\lim_{n \rightarrow \infty} x_n = 0$ for $q > 2$.

We declare that the stable manifold of E_0 is $\mathbb{W}_S(E_0) = \{(x, y) \mid x > 0, y = 0\}$.

In fact, if $(x_i, y_i) \notin \mathbb{W}_S(E_0)$ in the first quadrant, then from (1), we obtain

$$x_{n+1} \leq \frac{x_n + x_{n-1}}{q}. \quad (18)$$

We ascertain $\lim_{n \rightarrow \infty} x_n = 0$ for $q > 2$. In fact, we could deduce that by comparison and the theory of linear difference equations.

Setting $u_{-1} = x_{-1}$, $u_0 = x_0$ and

$$u_{n+1} = \frac{u_n + u_{n-1}}{q}, \quad (19)$$

we obtain $x_n \leq u_n$ for all $n \geq 1$ by induction.

From (19), we have $u_n = c_1(r_1)^n + c_2(r_2)^n$ ($n \geq 1$) with r_1 and r_2 satisfying the characteristic equation $r^2 - r/q - 1/q = 0$ from which we have $|r_1| < 1$ and $|r_2| < 1$ for $q > 2$ and thus u_n goes to zero as n tends to ∞ . Therefore, we have $\lim_{n \rightarrow \infty} x_n = 0$ for $q > 2$.

Next, we consider the behavior of the component y_n .

From the fact of $\lim_{n \rightarrow \infty} x_n = 0$, there is a positive constant M satisfying $p + M^2 < 2$ such that $|x_n| \leq M$ for $n \geq n_*$ with n_* being some positive integer. From (1), for $n \geq n_* + 1$, we obtain

$$y_{n+1} > \frac{y_n + y_{n-1}}{p + M^2}. \quad (20)$$

By comparison and the theory of linear difference equations, we get $\lim_{n \rightarrow \infty} y_n = +\infty$ for $p < 2$.

Hence, we obtain the following theorem.

Theorem 6. If $p < 2$ and $q > 2$, then the global stable manifold E_0 of (1) is $\mathbb{W}_S(E_0) = \{(x, y) \mid x > 0, y = 0\}$. Whenever $(x_i, y_i) \notin \mathbb{W}_S(E_0)$ ($i = -1, 0$), then $\lim_{n \rightarrow \infty} (x_n, y_n) = (0, +\infty)$.

Case 3 (R_3). In this case, both E_0 and E_+ of (1) are saddle points for $p < 2$ and $q < 2$.

We claim that sets of the form

$$\Delta_+ = [\sqrt{2 - p + \epsilon}, +\infty) \times [0, \sqrt{2 - q - \epsilon}] \quad (21)$$

are invariant for sufficiently small $\epsilon > 0$: that is, $(x_n, y_n) \in \Delta_+$ for all n if $(x_i, y_i) \in \Delta_+$ ($i = -1, 0$).

Suppose (x_{-1}, y_{-1}) and $(x_0, y_0) \in \Delta_+$, from (1); then we have

$$\begin{aligned} x_1 &= \frac{x_0 + x_{-1}}{q + y_0 y_{-1}} \geq \frac{x_0 + x_{-1}}{2 - \epsilon} \geq \frac{2}{2 - \epsilon} \sqrt{2 - p + \epsilon}, \\ y_1 &= \frac{y_0 + y_{-1}}{p + x_0 x_{-1}} \leq \frac{y_0 + y_{-1}}{2 + \epsilon} \leq \frac{2}{2 + \epsilon} \sqrt{2 - q - \epsilon} \end{aligned} \quad (22)$$

from which we have $(x_1, y_1) \in \Delta_+$. By induction, we have $(x_n, y_n) \in \Delta_+$ for all n and

$$\begin{aligned} x_{n+1} &= \frac{x_n + x_{n-1}}{q + y_n y_{n-1}} \geq \frac{x_n + x_{n-1}}{2 - \epsilon}, \\ y_{n+1} &= \frac{y_n + y_{n-1}}{p + x_n x_{n-1}} \leq \frac{y_n + y_{n-1}}{2 + \epsilon} \end{aligned} \quad (23)$$

from which it follows that $\lim_{n \rightarrow \infty} (x_n, y_n) = (+\infty, 0)$.

Similarly, sets of the form

$$\Delta_- = [0, \sqrt{2 - p - \epsilon}] \times [\sqrt{2 - q + \epsilon}, +\infty) \quad (24)$$

are invariant for sufficiently small $\epsilon > 0$. For $(x_i, y_i) \in \Delta_-$, then we have $\lim_{n \rightarrow \infty} (x_n, y_n) = (0, +\infty)$.

Theorem 7. If $p < 2$ and $q < 2$, then sets of the form Δ_+ and Δ_- (defined by (21) and (24)) are invariant of (1) for sufficiently small $\epsilon > 0$.

If $(x_i, y_i) \in \Delta_-$ ($i = -1, 0$), then $\lim_{n \rightarrow \infty} (x_n, y_n) = (0, +\infty)$.

If $(x_i, y_i) \in \Delta_+$ ($i = -1, 0$), then $\lim_{n \rightarrow \infty} (x_n, y_n) = (+\infty, 0)$.

Case 4 (R_4). In this case, E_0 of (1) is a saddle point for $p > 2$ and $q < 2$.

Similar to that of Case 2, we obtain the following theorem.

Theorem 8. If $p > 2$ and $q < 2$, then the stable manifold of E_0 of (1) is $\mathbb{W}_S(E_0) = \{(x, y) \mid x = 0, y > 0\}$. Whenever $(x_i, y_i) \notin \mathbb{W}_S(E_0)$ ($i = -1, 0$), then $\lim_{n \rightarrow \infty} (x_n, y_n) = (+\infty, 0)$.

Case 5 (R_5). In this case, E_0 and $E_{\bar{y}}$ of (1) are nonhyperbolic of the stable type for $p = 2$ and $q > 2$.

For (x_i, y_i) on y -axis, we have that $x_n = 0$ for all n and y_n satisfies (17) with $p = 2$, from which it follows that $\lim_{n \rightarrow \infty} y_n$ exists depending on y_i ($i = -1, 0$).

For (x_i, y_i) on x -axis, we have that $y_n = 0$ for all n and x_n satisfies

$$x_{n+1} = \frac{x_n + x_{n-1}}{q}, \quad (25)$$

from which we have $\lim_{n \rightarrow \infty} x_n = 0$ for $q > 2$.

For positive initial conditions, similar to Case 2, we also know $\lim_{n \rightarrow \infty} x_n = 0$ for $q > 2$. Specially, it follows that $x_n = 0$ for all $n \geq n_0$ for some positive integer n_0 . From (1), for $p = 2$, we have

$$y_{n+1} = \frac{y_n + y_{n-1}}{p + x_n x_{n-1}} = \frac{y_n + y_{n-1}}{2} \quad (26)$$

for $n \geq n_0$ and thus $\lim_{n \rightarrow \infty} y_n$ exists. Thus, basin of attraction of E_0 is $\mathbb{B}(E_0) = \{(x, y) \mid x > 0, y = 0\}$. If $(x_i, y_i) \notin \mathbb{B}(E_0)$ ($i = -1, 0$) then $\lim_{n \rightarrow \infty} x_n = 0$ and $\lim_{n \rightarrow \infty} y_n$ exists.

Theorem 9. If $p = 2$ and $q > 2$, then basin of attraction of E_0 of (1) is $\mathbb{B}(E_0) = \{(x, y) \mid x > 0, y = 0\}$. Whenever $(x_i, y_i) \notin \mathbb{B}(E_0)$ ($i = -1, 0$), then $\lim_{n \rightarrow \infty} (x_n, y_n) = E_{\bar{y}} = (0, \bar{y})$.

Case 6 (R_6). In this case, E_0 and $E_{\bar{x}}$ of (1) are nonhyperbolic for $p < 2$ and $q = 2$.

For (x_i, y_i) on y -axis, we have that $x_n = 0$ for all n and y_n satisfies (17) with $p < 2$, from which it follows that $\lim_{n \rightarrow \infty} y_n = +\infty$.

For (x_i, y_i) on x -axis, we have that $y_n = 0$ for all n and x_n satisfies (25) with $q = 2$, from which it follows that $\lim_{n \rightarrow \infty} x_n$ exists depending on x_i ($i = -1, 0$).

There is a curve $\mathcal{C}_0(x)$ such that the first quadrant is divided into two connected parts and

$$\mathcal{C}_0(x) = \{(x, y) \mid x > 0, y = x\}, \quad (27)$$

$$\mathcal{W}_+^0(x) = \{(x, y) \mid x > 0, y < x\}, \quad (28)$$

$$\mathcal{W}_-^0(x) = \{(x, y) \mid x > 0, y > x\}. \quad (29)$$

If $(x_i, y_i) \in \mathcal{W}_-^0(x)$, then we have $y_n > x_n$ for all n . Thus, from (1), we obtain

$$x_{n+1} = \frac{x_n + x_{n-1}}{q + y_n y_{n-1}} < \frac{x_n + x_{n-1}}{q + x_n x_{n-1}}. \quad (30)$$

By comparison and the results of (2), we have $\lim_{n \rightarrow \infty} x_n = 0$ for $q = 2$.

Hence, similar to Case 2, we have $\lim_{n \rightarrow \infty} y_n = +\infty$ for $p < 2$.

If $(x_i, y_i) \in \mathcal{C}_0(x)$, then we also obtain the above conclusion.

If $(x_i, y_i) \in \mathcal{W}_+^0(x)$, that is, $y_i < x_i$ ($i = -1, 0$), then we choose such a $\bar{x} > 0$ that $p + (\bar{x})^2 = 2$: that is, $\bar{x}_* = \sqrt{2 - p}$. There is a curve $\mathcal{C}_{\bar{x}_*}(x)$,

$$\mathcal{C}_{\bar{x}_*}(x) = \{(x, y) \mid y > 0, y = x - \sqrt{2 - p}\}, \quad (31)$$

which is below the curve $\mathcal{C}_0(x)$ such that $\mathcal{W}_+^0(x)$ is divided into two connected parts

$$\mathcal{W}_{+,1}^0(x) = \{(x, y) \mid y > 0, \mathcal{C}_{\bar{x}_*}(x) < y < x\}, \quad (32)$$

$$\mathcal{W}_{+,2}^0(x) = \{(x, y) \mid y > 0, y < \mathcal{C}_{\bar{x}_*}(x)\}.$$

If $(x_i, y_i) \in \mathcal{W}_{+,2}^0(x)$, that is, $y_i < \mathcal{C}_{\bar{x}_*}(x_i)$ ($i = -1, 0$), then we have $y_n < \mathcal{C}_{\bar{x}_*}(x_n)$ for all n by induction. Thus, from (1), we obtain

$$\begin{aligned} y_{n+1} &= \frac{y_n + y_{n-1}}{p + x_n x_{n-1}} \\ &< \frac{y_n + y_{n-1}}{p + (y_n + \sqrt{2 - p})(y_{n-1} + \sqrt{2 - p})} \\ &< \frac{y_n + y_{n-1}}{2 + \epsilon} \end{aligned} \quad (33)$$

for arbitrarily small $\epsilon > 0$. Thus, we conclude that $\lim_{n \rightarrow \infty} y_n = 0$. Specially, we have $y_n = 0$ for $n \geq \hat{n}$ for some positive integer $\hat{n}$. Hence, from (1) we also obtain (25) with $q = 2$ for $n \geq \hat{n} + 1$, from which it follows that $\lim_{n \rightarrow \infty} x_n$ exists depending on x_i ($i = -1, 0$).

If (x_i, y_i) on the curve $\mathcal{C}_{\bar{x}_*}(x)$, then we also derive the above conclusion.

Thus, we obtain the following theorem.

Theorem 10. If $p < 2$ and $q = 2$, then E_0 of (1) is nonhyperbolic of the unstable type and every $E_{\bar{x}}$ ($\bar{x} > 0$) of (1) is nonhyperbolic. More precisely, $E_{\bar{x}}$ is nonhyperbolic of the unstable type for $p + \bar{x}^2 < 2$ and is nonhyperbolic of the stable type for $p + \bar{x}^2 \geq 2$.

There is a curve $\mathcal{C}_0(x)$ defined by (27) such that $\lim_{n \rightarrow \infty} (x_n, y_n) = (0, +\infty)$ for (x_i, y_i) ($i = -1, 0$) on and above the curve $\mathcal{C}_0(x)$.

There is a curve $\mathcal{C}_{\bar{x}_*}(x)$ defined by (31) with $\bar{x}_* = \sqrt{2 - p}$ such that $\lim_{n \rightarrow \infty} (x_n, y_n) = E_{\bar{x}} = (\bar{x}, 0)$ with $\bar{x} > 0$ for (x_i, y_i) ($i = -1, 0$) on and below the curve $\mathcal{C}_{\bar{x}_*}(x)$.

Case 7 (R_7). In this case, E_0 and $E_{\bar{y}}$ of (1) are nonhyperbolic for $p = 2$ and $q < 2$.

Similar to that of Case 6, we obtain the following theorem.

TABLE 2: Global dynamics of (1).

Region	Theorem	Global dynamics of (1)
R_1 ($p > 2, q > 2$)	Theorem 1 [1]	$\mathbb{B}(E_0) = \{(x, y) \mid x > 0, y > 0\}$
R_2 ($p < 2, q > 2$)	Theorem 6	$\mathbb{W}_S(E_0) = \{(x, y) \mid x > 0, y = 0\}$ $\lim_{n \rightarrow \infty} (x_n, y_n) = (0, +\infty)$ for $(x_i, y_i) \notin \mathbb{W}_S(E_0)$
R_3 ($p < 2, q < 2$)	Theorem 7	$\lim_{n \rightarrow \infty} (x_n, y_n) = (0, +\infty)$ for $(x_i, y_i) \in \Delta_-$ $\lim_{n \rightarrow \infty} (x_n, y_n) = (+\infty, 0)$ for $(x_i, y_i) \in \Delta_+$
R_4 ($p > 2, q < 2$)	Theorem 8	$\mathbb{W}_S(E_0) = \{(x, y) \mid x = 0, y > 0\}$ $\lim_{n \rightarrow \infty} (x_n, y_n) = (+\infty, 0)$ for $(x_i, y_i) \notin \mathbb{W}_S(E_0)$
R_5 ($p = 2, q > 2$)	Theorem 9	$\mathbb{B}(E_0) = \{(x, y) \mid x > 0, y = 0\}$ $\lim_{n \rightarrow \infty} (x_n, y_n) = E_{\bar{y}} \ (\bar{y} > 0)$ for $(x_i, y_i) \notin \mathbb{B}(E_0)$
R_6 ($p < 2, q = 2$)	Theorem 10	$\lim_{n \rightarrow \infty} (x_n, y_n) = (0, +\infty)$ for (x_i, y_i) on and above $\mathcal{C}_0(x)$. $\lim_{n \rightarrow \infty} (x_n, y_n) = E_{\bar{x}} \ (\bar{x} > 0)$ for (x_i, y_i) on and below $\mathcal{C}_{\bar{x}_*}(x)$
R_7 ($p = 2, q < 2$)	Theorem 11	$\lim_{n \rightarrow \infty} (x_n, y_n) = (+\infty, 0)$ for (x_i, y_i) on and below $\mathcal{C}_0(x)$. $\lim_{n \rightarrow \infty} (x_n, y_n) = E_{\bar{y}} \ (\bar{y} > 0)$ for (x_i, y_i) on and above $\mathcal{C}_{\bar{y}_*}(x)$
R_8 ($p > 2, q = 2$)	Theorem 12	$\mathbb{B}(E_0) = \{(x, y) \mid x = 0, y > 0\}$ $\lim_{n \rightarrow \infty} (x_n, y_n) = E_{\bar{x}} \ (\bar{x} > 0)$ for $(x_i, y_i) \notin \mathbb{B}(E_0)$
R_9 ($p = 2, q = 2$)	Theorem 13	$\mathbb{B}(E_0) = \mathcal{C}_0(x) = \{(x, y) \mid x > 0, y = x\}$ $\lim_{n \rightarrow \infty} (x_n, y_n) = E_{\bar{x}} \ (\bar{x} > 0)$ for (x_i, y_i) below $\mathcal{C}_0(x)$ $\lim_{n \rightarrow \infty} (x_n, y_n) = E_{\bar{y}} \ (\bar{y} > 0)$ for (x_i, y_i) above $\mathcal{C}_0(x)$

Theorem 11. If $p = 2$ and $q < 2$, then E_0 of (1) is nonhyperbolic of the unstable type and every $E_{\bar{y}} \ (\bar{y} > 0)$ of (1) is nonhyperbolic. More precisely, $E_{\bar{y}}$ is nonhyperbolic of the unstable type for $p + \bar{y}^2 < 2$ and is nonhyperbolic of the stable type for $p + \bar{y}^2 \geq 2$.

There is a curve $\mathcal{C}_0(x)$ defined by (27) such that $\lim_{n \rightarrow \infty} (x_n, y_n) = (+\infty, 0)$ for $(x_i, y_i) \ (i = -1, 0)$ on and below the curve $\mathcal{C}_0(x)$.

There is a curve $\mathcal{C}_{\bar{y}_*}(x)$ defined by $\mathcal{C}_{\bar{y}_*}(x) = \{(x, y) \mid x > 0, y = x + \sqrt{2 - q}\}$ such that $\lim_{n \rightarrow \infty} (x_n, y_n) = E_{\bar{y}} = (0, \bar{y})$ with $\bar{y} > 0$ for $(x_i, y_i) \ (i = -1, 0)$ on and above the curve $\mathcal{C}_{\bar{y}_*}(x)$ with $\bar{y}_* = \sqrt{2 - q}$.

Case 8 (R_8). In this case, both E_0 and $E_{\bar{x}}$ of (1) are nonhyperbolic of the stable type for $p > 2$ and $q = 2$.

Similar to that of Case 5, we obtain the following theorem.

Theorem 12. If $p > 2$ and $q = 2$, then basin of attraction of E_0 of (1) is $\mathbb{B}(E_0) = \{(x, y) \mid x = 0, y > 0\}$. Whenever $(x_i, y_i) \notin \mathbb{B}(E_0) \ (i = -1, 0)$, then $\lim_{n \rightarrow \infty} (x_n, y_n) = E_{\bar{x}} = (\bar{x}, 0)$.

Case 9 (R_9). Here, $E_0, E_{\bar{x}}$, and $E_{\bar{y}} \ (\bar{x} > 0 \text{ and } \bar{y} > 0)$ of (1) are nonhyperbolic of the stable type for $p = q = 2$.

Now, we focus on E_0 . There is a curve $\mathcal{C}_0(x)$ (defined by (27)) passing through E_0 such that the first quadrant is divided into two connected parts and

$$\mathcal{W}_+^0(x) = \{(x, y) \mid x > 0, y < x\}, \quad (34)$$

$$\mathcal{W}_-^0(x) = \{(x, y) \mid x > 0, y > x\}. \quad (35)$$

If $(x_i, y_i) \ (i = -1, 0)$ are on the curve $\mathcal{C}_0(x)$, system (1) is reduced to a single equation and every positive solution of (1) converges to E_0 .

If $(x_i, y_i) \in \mathcal{W}_+^0(x)$, we have $x_i > y_i \ (i = -1, 0)$. By induction, from (1), we have $x_n > y_n$ for all n . For $p = 2$, from (1), we have

$$y_{n+1} = \frac{y_n + y_{n-1}}{2 + x_n x_{n-1}} < \frac{y_n + y_{n-1}}{2 + y_n y_{n-1}} \quad (36)$$

and thus $\lim_{n \rightarrow \infty} y_n = 0$ by comparison and the results of (2). Specially, it follows that $y_n = 0$ for all $n \geq n_*$ for some positive integer n_* . Thus, from (1), we have (25) for $q = 2$ and hence $\lim_{n \rightarrow \infty} x_n$ exists depending on initial conditions. Therefore, we have $\lim_{n \rightarrow \infty} (x_n, y_n) = E_{\bar{x}} \ (\bar{x} > 0)$ for $(x_i, y_i) \in \mathcal{W}_+^0(x)$.

Similarly, we obtain $\lim_{n \rightarrow \infty} (x_n, y_n) = E_{\bar{y}} \ (\bar{y} > 0)$ for $(x_i, y_i) \in \mathcal{W}_-^0(x)$.

Theorem 13. If $p = 2$ and $q = 2$, then basin of attraction of E_0 of (1) is

$$\mathbb{B}(E_0) = \mathcal{C}_0(x) = \{(x, y) \mid x > 0, y = x\}. \quad (37)$$

If $(x_i, y_i) \in \mathcal{W}_+^0(x) \ (i = -1, 0)$, then $\lim_{n \rightarrow \infty} (x_n, y_n) = E_{\bar{x}}$.

If $(x_i, y_i) \in \mathcal{W}_-^0(x) \ (i = -1, 0)$, then $\lim_{n \rightarrow \infty} (x_n, y_n) = E_{\bar{y}}$.

Here, $\mathcal{W}_+^0(x)$ and $\mathcal{W}_-^0(x)$ are defined by (34) and (35), respectively.

These above theorems completely describe the global dynamics of (1) and are listed in Table 2. The solution of system (1) converged to either the equilibria or the boundary

point $(+\infty, 0)$ or $(0, +\infty)$ depending on nonnegative initial conditions and parameters.

5. Conclusion

It is known that the techniques in the investigation of the behavior of difference equations can be used in investigating equations arising in mathematical models describing real life situations in biology, economics, physics, sociology, control theory, and vice versa.

In this paper, we investigate a system of nonlinear difference equations, which has complex dynamics. We use the results of linearized stability to analyze the local stability of five kinds of equilibria of (1) with parameters (p, q) in nine regions in the first quadrant. Specially, for particular parameters p and q , dynamics of the system is very interesting with a continuum of nonhyperbolic equilibria along a line. Generally speaking, the solution of (1) converges to its equilibria if the equilibrium is locally asymptotically stable or nonhyperbolic of the stable type, depending on initial conditions. Otherwise, it may converge to the boundary point $(+\infty, 0)$ or $(0, +\infty)$ or exhibit somewhat chaos.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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