Research Article

Generalized Characteristic Polynomials of Join Graphs and Their Applications

Pengli Lu, Ke Gao, and Yang Yang

School of Computer and Communication, Lanzhou University of Technology, Lanzhou, Gansu 730050, China

Correspondence should be addressed to Pengli Lu; lupengli88@163.com

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The Kirchhoff index of G is the sum of resistance distances between all pairs of vertices of G in electrical networks. $LEL(G)$ is the Laplacian-Energy-Like Invariant of G in chemistry. In this paper, we define two classes of join graphs: the subdivision-vertex-vertex join $G_1 \odot G_2$ and the subdivision-edge-edge join $G_1 \ominus G_2$. We determine the generalized characteristic polynomial of them. We deduce the adjacency (Laplacian and signless Laplacian, resp.) characteristic polynomials of $G_1 \odot G_2$ and $G_1 \ominus G_2$ when G_1 is r_1 -regular graph and G_2 is r_2 -regular graph. As applications, the Laplacian spectra enable us to get the formulas of the number of spanning trees, Kirchhoff index, and LEL of $G_1 \odot G_2$ and $G_1 \ominus G_2$ in terms of the Laplacian spectra of G_1 and G_2 .

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph on n vertices and m edges. Let $d_i = d_G(v_i)$ be the degree of vertex v_i in G and $D(G)$ be the diagonal matrix with diagonal entries $d_1, d_2, \dots, d_n$. Let $A(G)$ denote the adjacency matrix of a graph G . The Laplacian matrix and the signless Laplacian matrix of G are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively. Let $\psi(A(G); x) = \det(xI_n - A(G))$, or simply $\psi(A(G))$ ($\psi(L(G))$ and $\psi(Q(G))$, resp.), be the adjacency (Laplacian and signless Laplacian, resp.) characteristic polynomial of G and its roots be the adjacency (Laplacian and signless Laplacian, resp.) eigenvalues of G , denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$ ($0 = \mu_1(G) \leq \mu_2(G) \leq \dots \leq \mu_n(G)$ and $\nu_1(G) \leq \nu_2(G) \leq \dots \leq \nu_n(G)$, resp.). The line graph of G is denoted by $\ell(G)$.

The generalized characteristic polynomial of G , introduced by Cvetkovi et al. [1], is defined to be $\phi_G(x, t) = \det(xI_n - (A(G) - tD(G)))$. The generalized characteristic polynomial covers the cases of usual characteristic polynomial $A(G)$ and Laplacian $L(G)$ and signless Laplacian $Q(G)$ polynomials of graph G , due to variation of the parameter t and, whenever necessary, replacing x by $-x$. We can get that the characteristic polynomials of $A(G)$, $L(G)$, and $Q(G)$ are equal to $\phi_G(x, 0)$, $(-1)^{|V(G)|} \phi_G(-x, 1)$, and $\phi_G(x, -1)$.

Let G be a connected graph. For two vertices u and v of G , the resistance distance between u and v is defined to be the effective resistance between them when unit resistors are placed on every edge of G . It is a distance function on graphs introduced by Klein and Randić [2]. The Kirchhoff index of G , denoted by $Kf(G)$, is the sum of resistance distances between all pairs of vertices of G . For a connected graph G of order n [3], $Kf(G) = n \sum_{i=2}^n (1/\mu_i)$. Recently, many results on Kirchhoff index were obtained [2, 4–8]. Laplacian-Energy-Like Invariant $LEL(G) = \sum_{i=2}^n \sqrt{\mu_i}$ was named in [9]. The motivation for introducing LEL was in its analogy to the earlier studied graph energy and Laplacian energy [10]. Although Kirchhoff index and LEL both depend on Laplacian eigenvalues, their comparative study started only quite recently [11, 12].

Graph operations, such as the disjoint union, the join, the corona, the edge corona, and the neighborhood corona [13–17], are techniques to construct new classes of graphs from old ones. In [18], a real molecular graph of ferrocene is a join graph obtained from graphs $G_1 = K_1$ and $G_2 = 2C_5$, where $2C_5$ is a disjoint union of two pentagons. In [4–6, 8, 14] the resistance distance and Kirchhoff index of artificial graphs are computed. Although most of the constructed graphs in the literature are contrived, they may be of use for chemical and physical applications.

FIGURE 1: An example of subdivision-vertex-vertex join and subdivision-edge-edge join.

Motivated by the work above, we define two new graph operations based on *subdivision graphs* as follows.

The subdivision graph $\mathcal{S}(G)$ of a graph G is the graph obtained by inserting a new vertex into every edge of G [19]. We denote the set of such new vertices by $I(G)$. In [15, 20], some new graph operations based on subdivision graphs were defined and the A -, L -, and Q -spectrum were computed in terms of those of the two graphs.

Let G_1 and G_2 be two vertex disjoint graphs. The subdivision-vertex-vertex join of G_1 and G_2 , denoted by $G_1 \otimes G_2$, is the graph obtained from $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ by joining every vertex in $V(G_1)$ to every vertex in $V(G_2)$. The subdivision-edge-edge join of G_1 and G_2 , denoted by $G_1 \odot G_2$, is the graph obtained from $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ by joining every vertex of $I(G_1)$ to every vertex in $I(G_2)$.

Note that if G_1 is a graph on n_1 vertices and m_1 edges and G_2 is a graph on n_2 vertices and m_2 edges, then $G_1 \otimes G_2$ has $n_1 + m_1 + n_2 + m_2$ vertices and $2m_1 + 2m_2 + n_1n_2$ edges and $G_1 \odot G_2$ has $n_1 + m_1 + n_2 + m_2$ vertices and $2m_1 + 2m_2 + m_1m_2$ edges.

Let C_n denote a cycle of order n and P_m denote a path of order m . Figure 1 depicts the *subdivision-vertex-vertex join* $C_5 \otimes P_2$ and *subdivision-edge-edge join* $C_5 \odot P_2$, respectively.

The paper is organized as follows. In Section 2, some useful lemmas are provided. In Section 3, we compute the generalized characteristic polynomial of the subdivision-vertex-vertex join and obtain the A -, L -, and Q -spectrum in terms of the corresponding spectrum of G_1 and G_2 when G_1 and G_2 are regular graphs. In Section 4, we compute the generalized characteristic polynomial of the subdivision-edge-edge join and obtain the A -, L -, and Q -spectrum in terms of the corresponding spectrum of G_1 and G_2 when G_1 and G_2 are regular graphs. In Section 5, as the applications, the number of spanning trees, Kirchhoff index, and LEL of the subdivision-vertex-vertex join and the subdivision-edge-edge join graphs are computed. In Section 6, we conclude the paper.

2. Preliminaries

The M -coronal $\Gamma_M(x)$ of an $n \times n$ matrix M is defined [16] to be the sum of entries of the matrix $(xI_n - M)^{-1}$; that is

$$\Gamma_M(x) = \mathbf{1}_n^T (xI_n - M)^{-1} \mathbf{1}_n, \quad (1)$$

where $\mathbf{1}_n$ denotes the column vector of size n when all the entries are equal to one.

Lemma 1 (see [21]). *Let A be an $n \times n$ real matrix and let $J_{s \times t}$ denote the $s \times t$ matrix with all entries equal to one. Then*

$$\det(A + \alpha J_{n \times n}) = \det(A) + \alpha \mathbf{1}_n^T \text{adj}(A) \mathbf{1}_n, \quad (2)$$

where α is a real number and $\text{adj}(A)$ is the adjugate matrix of A . Moreover,

$$\det(xI_n - A - \alpha J_{n \times n}) = (1 - \alpha \Gamma_A(x)) \det(xI_n - A). \quad (3)$$

Lemma 2 (see [13]). *If M is an $n \times n$ matrix when each row sum is equal to a constant t ; then,*

$$\Gamma_M(x) = \frac{n}{x - t}. \quad (4)$$

Lemma 3 (see [22]). *Let M_1, M_2, M_3 , and M_4 be, respectively, $p \times p$, $p \times q$, $q \times p$, and $q \times q$ matrices with M_1 and M_4 invertible. Then*

$$\begin{aligned} \det \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} &= \det(M_4) \cdot \det(M_1 - M_2 M_4^{-1} M_3) \\ &= \det(M_1) \cdot \det(M_4 - M_3 M_1^{-1} M_2), \end{aligned} \quad (5)$$

where $M_1 - M_2 M_4^{-1} M_3$ and $M_4 - M_3 M_1^{-1} M_2$ are called the Schur complements of M_4 and M_1 , respectively.

Lemma 4 (see [19]). *If G is a regular graph of degree r , with n vertices and m edges; then*

$$\psi(A(\ell(G)), x) = (x + 2)^{m-n} \psi(A(G), x - r + 2). \quad (6)$$

If $A = [a_{ij}]$ is an $m \times n$ matrix and B is an $r \times s$ matrix, then the Kronecker product $A \otimes B$ is defined as the $mr \times ns$ matrix with the block form

$$\begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}. \quad (7)$$

This is an associative operation with the property that $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = AC \otimes BD$ whenever the products AC and BD exist. The latter implies $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ for nonsingular matrices A and B . Moreover, if A and B are $n \times n$ and $p \times p$ matrices, then $\det(A \otimes B) = (\det A)^p (\det B)^n$. The reader is referred to [23] for other properties of the Kronecker product not mentioned here.

3. Spectrum of Subdivision-Vertex-Vertex Join

Let G_1 be an r_1 -regular graph on n_1 vertices and m_1 edges and G_2 an r_2 -regular graph on n_2 vertices and m_2 edges, respectively. We first label the vertices of $G_1 \otimes G_2$ as follows.

Let $V(G_1) = \{v_1, v_2, \dots, v_{n_1}\}$, $I(G_1) = \{e_1, e_2, \dots, e_{m_1}\}$, $V(G_2) = \{u_1, u_2, \dots, u_{n_2}\}$, and $I(G_2) = \{e_1, e_2, \dots, e_{m_2}\}$. The adjacency matrix of $G_1 \otimes G_2$ can be written as follows:

$$A_{G_1 \otimes G_2} = \begin{pmatrix} 0_{m_1 \times m_1} & R_1^T & 0_{m_1 \times n_2} & 0_{m_1 \times m_2} \\ R_1 & 0_{n_1 \times n_1} & \mathbf{1}_{n_1 \times n_2} & 0_{n_1 \times m_2} \\ 0_{n_2 \times m_1} & \mathbf{1}_{n_2 \times n_1} & 0_{n_2 \times n_2} & R_2 \\ 0_{m_2 \times m_1} & 0_{m_2 \times n_1} & R_2^T & 0_{m_2 \times m_2} \end{pmatrix}, \quad (8)$$

where $0_{s \times t}$ denotes the $s \times t$ matrix with all entries equal to zero, R is the incidence matrix of G , I_n is the identity matrix of order n , and $\mathbf{1}_n$ is the column vector with all entries equal to 1. It is clear that the degrees of the vertices of $G_1 \otimes G_2$ are $d_{G_1 \otimes G_2}(v_i) = r_1 + n_2$ for $i = 1, 2, \dots, n_1$, $d_{G_1 \otimes G_2}(e_i) = 2$ for $i = 1, 2, \dots, m_1$, $d_{G_1 \otimes G_2}(u_i) = r_2 + n_1$ for $i = 1, 2, \dots, n_2$,

and $d_{G_1 \otimes G_2}(e_i) = 2$ for $i = 1, 2, \dots, m_2$. Then the degree matrix of subdivision-vertex-vertex join can be written as follows:

$$D_{G_1 \otimes G_2} = \begin{pmatrix} 2I_{m_1} & 0_{m_1 \times n_1} & 0_{m_1 \times n_2} & 0_{m_1 \times m_2} \\ 0_{n_1 \times m_1} & (r_1 + n_2)I_{n_1} & 0_{n_1 \times n_2} & 0_{n_1 \times m_2} \\ 0_{n_2 \times m_1} & 0_{n_2 \times n_1} & (r_2 + n_1)I_{n_2} & 0_{n_2 \times m_2} \\ 0_{m_2 \times m_1} & 0_{m_2 \times n_1} & 0_{m_2 \times n_2} & 2I_{m_2} \end{pmatrix}. \quad (9)$$

3.1. Generalized Characteristic Polynomial of $G_1 \otimes G_2$

Theorem 5. Let G_1 be an r_1 -regular graph on n_1 vertices and m_1 edges and G_2 be an r_2 -regular graph on n_2 vertices and m_2 edges. Then the generalized characteristic polynomial of subdivision-vertex-vertex join of G_1 and G_2 is

$$\begin{aligned} \phi_{G_1 \otimes G_2}(x, t) &= (x + 2t)^{m_1 + m_2 - n_1 - n_2} \prod_{i=1}^{n_1} (x^2 + (2t + r_1t + n_2t)x + 2r_1t^2 + 2n_2t^2 - \lambda_i(G_1) - r_1) \\ &\quad \cdot \prod_{i=1}^{n_2} (x^2 + (2t + r_2t + n_1t)x + 2r_2t^2 + 2n_1t^2 - \lambda_i(G_2) - r_2) \\ &\quad \cdot \left(1 - \frac{n_1n_2(x + 2t)^2}{((x + r_1t + n_2t)(x + 2t) - 2r_1)((x + r_2t + n_1t)(x + 2t) - 2r_2)} \right). \end{aligned} \quad (10)$$

Proof. Let R_1 be the incidence matrix of G_1 . Then, with respect to the adjacent matrix and degree matrix of $G_1 \otimes G_2$, the generalized matrix of $G_1 \otimes G_2$ is given by

$$\begin{aligned} A_{G_1 \otimes G_2} - tD_{G_1 \otimes G_2} &= \begin{pmatrix} -2tI_{m_1} & R_1^T & 0_{m_1 \times n_2} & 0_{m_1 \times m_2} \\ R_1 & -(r_1 + n_2)tI_{n_1} & \mathbf{1}_{n_1 \times n_2} & 0_{n_1 \times m_2} \\ 0_{n_2 \times m_1} & \mathbf{1}_{n_2 \times n_1} & -(r_2 + n_1)tI_{n_2} & R_2 \\ 0_{m_2 \times m_1} & 0_{m_2 \times n_1} & R_2^T & -2tI_{m_2} \end{pmatrix}. \end{aligned} \quad (11)$$

Thus the generalized characteristic polynomial of $G_1 \otimes G_2$ is

$$\begin{aligned} \phi_{G_1 \otimes G_2}(x, t) &= \det \begin{pmatrix} (x + 2t)I_{m_1} & -R_1^T & 0_{m_1 \times n_2} & 0_{m_1 \times m_2} \\ -R_1 & (x + r_1t + n_2t)I_{n_1} & -\mathbf{1}_{n_1 \times n_2} & 0_{n_1 \times m_2} \\ 0_{n_2 \times m_1} & -\mathbf{1}_{n_2 \times n_1} & (x + r_2t + n_1t)I_{n_2} & -R_2 \\ 0_{m_2 \times m_1} & 0_{m_2 \times n_1} & -R_2^T & (x + 2t)I_{m_2} \end{pmatrix} = \det((x + 2t)I_{m_1}) \det(S) \\ &= (x + 2t)^{m_1} \det(S), \end{aligned} \quad (12)$$

where

$$S = \begin{pmatrix} (x + r_1 t + n_2 t) I_{n_1} & -\mathbf{1}_{n_1 \times n_2} & 0_{n_1 \times m_2} \\ -\mathbf{1}_{n_2 \times n_1} & (x + r_2 t + n_1 t) I_{n_2} & -R_2 \\ 0_{m_2 \times n_1} & -R_2^T & (x + 2t) I_{m_2} \end{pmatrix} - \begin{pmatrix} -R_1 \\ 0_{n_2 \times n_1} \\ 0_{m_2 \times n_1} \end{pmatrix} \cdot ((x + 2t) I_{m_1})^{-1} \begin{pmatrix} -R_1^T & 0_{m_1 \times n_1} & 0_{m_1 \times m_2} \end{pmatrix} \quad (13)$$

$$= \begin{pmatrix} (x + r_1 t + n_2 t) I_{n_1} - \frac{R_1 R_1^T}{x + 2t} & -\mathbf{1}_{n_1 \times n_2} & 0_{n_1 \times m_2} \\ -\mathbf{1}_{n_2 \times n_1} & (x + r_2 t + n_1 t) I_{n_2} & -R_2 \\ 0_{m_2 \times n_1} & -R_2^T & (x + 2t) I_{m_2} \end{pmatrix}$$

is the Schur complement of $(x + 2t)I_{m_1}$. So, $\det(S) = (x + 2t)^{m_2} \det(S_1)$, where

$$S_1 = \begin{pmatrix} (x + r_1 t + n_2 t) I_{n_1} - \frac{R_1 R_1^T}{x + 2t} & -\mathbf{1}_{n_1 \times n_2} \\ -\mathbf{1}_{n_2 \times n_1} & (x + r_2 t + n_1 t) I_{n_2} \end{pmatrix} - \begin{pmatrix} 0_{n_1 \times m_2} \\ -R_2 \end{pmatrix} ((x + 2t) I_{m_2})^{-1} \begin{pmatrix} 0_{m_2 \times n_1} & -R_2^T \end{pmatrix} \quad (14)$$

$$= \begin{pmatrix} (x + r_1 t + n_2 t) I_{n_1} - \frac{R_1 R_1^T}{x + 2t} & -\mathbf{1}_{n_1 \times n_2} \\ -\mathbf{1}_{n_2 \times n_1} & (x + r_2 t + n_1 t) I_{n_2} - \frac{R_2 R_2^T}{x + 2t} \end{pmatrix}$$

is the Schur complement of $(x + 2t)I_{m_2}$. So,

$$\det(S_1) = \det \left((x + r_1 t + n_2 t) I_{n_1} - \frac{R_1 R_1^T}{x + 2t} \right) \cdot \det \left((x + r_2 t + n_1 t) I_{n_2} - \frac{R_2 R_2^T}{x + 2t} \right) \quad (15)$$

$$- \mathbf{1}_{n_2 \times n_1} \left((x + r_1 t + n_2 t) I_{n_1} - \frac{R_1 R_1^T}{x + 2t} \right)^{-1} \mathbf{1}_{n_1 \times n_2}$$

is the Schur complement of $(x + r_1 t + n_2 t)I_{n_1} - R_1 R_1^T / (x + 2t)$. So,

$$\phi_{G_1 \otimes G_2}(x, t) = (x + 2t)^{m_1 + m_2} \det \left((x + r_1 t + n_2 t) I_{n_1} - \frac{R_1 R_1^T}{x + 2t} \right) \cdot \det \left((x + r_2 t + n_1 t) I_{n_2} - \frac{R_2 R_2^T}{x + 2t} \right) \quad (16)$$

$$- \mathbf{1}_{n_2 \times n_1} \left((x + r_1 t + n_2 t) I_{n_1} - \frac{R_1 R_1^T}{x + 2t} \right)^{-1} \mathbf{1}_{n_1 \times n_2}$$

$$= (x + 2t)^{m_1 + m_2} \det \left((x + r_1 t + n_2 t) I_{n_1} - \frac{R_1 R_1^T}{x + 2t} \right) \cdot \det \left((x + r_2 t + n_1 t) I_{n_2} - \frac{R_2 R_2^T}{x + 2t} \right)$$

$$- \mathbf{1}_{n_2 \times n_1} \left((x + r_1 t + n_2 t) I_{n_1} - \frac{R_1 R_1^T}{x + 2t} \right)^{-1} \mathbf{1}_{n_1 \times n_2}$$

$$= (x + 2t)^{m_1 + m_2} \prod_{i=1}^{n_1} \left(x + r_1 t + n_2 t - \frac{\lambda_i(G_1) + r_1}{x + 2t} \right) \cdot \det \left((x + r_2 t + n_1 t) I_{n_2} - \frac{R_2 R_2^T}{x + 2t} \right)$$

$$- \Gamma_{R_1 R_1^T / (x + 2t)}(x + r_1 t + n_2 t) \mathbf{1}_{n_2 \times n_2} \Bigg);$$

by Lemma 1,

$$\phi_{G_1 \otimes G_2}(x, t) = (x + 2t)^{m_1 + m_2} \prod_{i=1}^{n_1} \left(x + r_1 t + n_2 t - \frac{\lambda_i(G_1) + r_1}{x + 2t} \right)$$

$$\cdot \left(1 - \Gamma_{R_1 R_1^T / (x + 2t)}(x + r_1 t + n_2 t) \Gamma_{R_2 R_2^T / (x + 2t)}(x + r_2 t + n_1 t) \right) \det \left((x + r_2 t + n_1 t) I_{n_2} - \frac{R_2 R_2^T}{x + 2t} \right)$$

$$= \prod_{i=1}^{n_1} \left(x^2 + (2t + r_1 t + n_2 t)x + 2r_1 t^2 + 2n_2 t^2 - \lambda_i(G_1) - r_1 \right)$$

$$\begin{aligned}
& \cdot \prod_{i=1}^{n_2} \left(x^2 + (2t + r_2 t + n_1 t) x + 2r_2 t^2 + 2n_1 t^2 - \lambda_i(G_2) - r_2 \right) \\
& \cdot \left(1 - \Gamma_{R_1 R_1^T / (x+2t)}(x + r_1 t + n_2 t) \Gamma_{R_2 R_2^T / (x+2t)}(x + r_2 t + n_1 t) \right) (x + 2t)^{m_1 + m_2 - n_1 - n_2} \\
& = (x + 2t)^{m_1 + m_2 - n_1 - n_2} \prod_{i=1}^{n_1} \left(x^2 + (2t + r_1 t + n_2 t) x + 2r_1 t^2 + 2n_2 t^2 - \lambda_i(G_1) - r_1 \right) \\
& \cdot \prod_{i=1}^{n_2} \left(x^2 + (2t + r_2 t + n_1 t) x + 2r_2 t^2 + 2n_1 t^2 - \lambda_i(G_2) - r_2 \right) \\
& \cdot \left(1 - \frac{n_1 n_2 (x + 2t)^2}{((x + r_1 t + n_2 t)(x + 2t) - 2r_1)((x + r_2 t + n_1 t)(x + 2t) - 2r_2)} \right). \tag{17}
\end{aligned}$$

□

Theorem 6. Let G_1 be an r_1 -regular graph on n_1 vertices and m_1 edges and G_2 an r_2 -regular graph on n_2 vertices and m_2 edges. Then

(1) the adjacency characteristic polynomial of $G_1 \odot G_2$ is

$$\begin{aligned}
\psi(A_{G_1 \odot G_2}; x) &= x^{m_1 + m_2 - n_1 - n_2} \prod_{i=1}^{n_1} (x^2 - r_1 - \lambda_i(G_1)) \\
& \cdot \prod_{i=1}^{n_2} (x^2 - r_2 - \lambda_i(G_2)) \\
& \cdot \left(1 - \frac{n_1 n_2 x^2}{x^4 - 2(r_1 + r_2)x^2 + 4r_1 r_2} \right), \tag{18}
\end{aligned}$$

(2) the Laplacian characteristic polynomial of $G_1 \odot G_2$ is

$$\begin{aligned}
\psi(L_{G_1 \odot G_2}; x) &= \prod_{i=2}^{n_1} (x^2 - (r_1 + n_2 + 2)x + 2n_2 + \mu_i(G_1)) \\
& \cdot \prod_{i=2}^{n_2} (x^2 - (r_2 + n_1 + 2)x + 2n_1 + \mu_i(G_2)) \cdot x \left(x^3 \right. \\
& \quad \left. - (n_1 + n_2 + r_1 + r_2 + 4)x^2 \right. \\
& \quad \left. + (4n_1 + 4n_2 + 2r_1 + 2r_2 + 4 + r_1 r_2 + r_1 n_1 + r_2 n_2) \right. \\
& \quad \left. \cdot x - 4n_1 - 4n_2 - 2n_1 r_1 - 2n_2 r_2 \right) (x - 2)^{m_1 + m_2 - n_1 - n_2}, \tag{19}
\end{aligned}$$

(3) the signless Laplacian characteristic polynomial of $G_1 \odot G_2$ is

$$\begin{aligned}
\psi(Q_{G_1 \odot G_2}; x) &= (x - 2)^{m_1 + m_2 - n_1 - n_2} \\
& \cdot \prod_{i=1}^{n_1} (x^2 - (r_1 + n_2 + 2)x + 2n_2 + 2r_1 - \nu_i(G_1)) \cdot \left(1 \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{n_1 n_2 (x - 2)^2}{(x^2 - (2 + r_1 + n_2)x + 2n_2)(x^2 - (2 + r_2 + n_1)x + 2n_1)} \\
& \cdot \prod_{i=1}^{n_2} (x^2 - (r_2 + n_1 + 2)x + 2n_1 + 2r_2 - \nu_i(G_2)). \tag{20}
\end{aligned}$$

(20)

4. Spectrum of Subdivision-Edge-Edge Join

The adjacency matrix of $G_1 \odot G_2$ can be written as follows:

$$A_{G_1 \odot G_2} = \begin{pmatrix} 0_{n_1 \times n_1} & R_1 & 0_{n_1 \times m_2} & 0_{n_1 \times n_2} \\ R_1^T & 0_{m_1 \times m_1} & \mathbf{1}_{m_1 \times m_2} & 0_{m_1 \times n_2} \\ 0_{m_2 \times n_1} & \mathbf{1}_{m_2 \times m_1} & 0_{m_2 \times m_2} & R_2^T \\ 0_{n_2 \times n_1} & 0_{n_2 \times m_1} & R_2 & 0_{n_2 \times n_2} \end{pmatrix}, \tag{21}$$

where $0_{s \times t}$ denotes the $s \times t$ matrix with all entries equal to zero, R is the incidence matrix of G , I_n is the identity matrix of order n , and $\mathbf{1}_n$ is the column vector with all entries equal to 1. It is clear that the degrees of the vertices of $G_1 \odot G_2$ are $d_{G_1 \odot G_2}(v_i) = r_1$ for $i = 1, 2, \dots, n_1$, $d_{G_1 \odot G_2}(e_i) = m_2 + 2$ for $i = 1, 2, \dots, m_1$, $d_{G_1 \odot G_2}(u_i) = r_2$ for $i = 1, 2, \dots, n_2$, and $d_{G_1 \odot G_2}(e_i) = m_1 + 2$ for $i = 1, 2, \dots, m_2$. Then the degree matrix of subdivision-edge-edge join can be written as follows:

$$\begin{aligned}
D_{G_1 \odot G_2} &= \begin{pmatrix} r_1 I_{n_1} & 0_{n_1 \times m_1} & 0_{n_1 \times m_1} & 0_{n_1 \times n_2} \\ 0_{m_1 \times n_1} & (m_2 + 2) I_{m_1} & 0_{m_1 \times m_2} & 0_{m_1 \times n_2} \\ 0_{m_2 \times n_1} & 0_{m_2 \times m_1} & (m_1 + 2) I_{m_2} & 0_{m_2 \times n_2} \\ 0_{n_2 \times n_1} & 0_{n_2 \times m_1} & 0_{n_2 \times m_2} & r_2 I_{n_2} \end{pmatrix}. \tag{22}
\end{aligned}$$

4.1. Generalized Characteristic Polynomial of $G_1 \odot G_2$

Theorem 7. Let G_1 be an r_1 -regular graph on n_1 vertices and m_1 edges and G_2 be an r_2 -regular graph on n_2 vertices and

m_2 edges. Then the generalized characteristic polynomial of subdivision-edge-edge join of G_1 and G_2 is

$$\begin{aligned} \phi_{G_1 \odot G_2}(x; t) &= \prod_{i=1}^{n_1} \left(x^2 + (m_2 t + 2t + r_1 t) x + r_1 m_2 t^2 + 2r_1 t^2 - \lambda_i(G_1) - r_1 \right) \\ &\cdot \prod_{i=1}^{n_2} \left(x^2 + (m_1 t + 2t + r_2 t) x + r_2 m_1 t^2 + 2r_2 t^2 - \lambda_i(G_2) - r_2 \right) \\ &\cdot \left(1 - \frac{m_1 m_2 (x + r_1 t)(x + r_2 t) (x^2 + (m_1 t + r_2 t + 2t) x + r_2 m_1 t^2 + 2r_2 t^2 - 2r_2)^{-1}}{(x^2 + (m_2 t + r_1 t + 2t) x + r_1 m_2 t^2 + 2r_1 t^2 - 2r_1)} \right) \\ &\cdot (x + m_1 t + 2t)^{m_2 - n_2} (x + m_2 t + 2t)^{m_1 - n_1}. \end{aligned} \quad (23)$$

Proof. Let R_1 be the incidence matrix of G_1 . Then, with respect to the adjacent matrix and degree matrix of $G_1 \odot G_2$, the generalized matrix of $G_1 \odot G_2$ is given by

Thus the generalized characteristic polynomial of $G_1 \odot G_2$ is

$$\begin{aligned} A_{G_1 \odot G_2} - tD_{G_1 \odot G_2} \\ = \begin{pmatrix} -r_1 t I_{n_1} & R_1 & 0_{n_1 \times m_2} & 0_{n_1 \times n_2} \\ R_1^T & -(m_2 + 2)t I_{m_1} & \mathbf{1}_{m_1 \times m_2} & 0_{m_1 \times n_2} \\ 0_{m_2 \times n_1} & \mathbf{1}_{m_2 \times m_1} & -(m_1 + 2)t I_{m_2} & R_2^T \\ 0_{n_2 \times n_1} & 0_{n_2 \times m_1} & R_2 & -r_2 t I_{n_2} \end{pmatrix}. \end{aligned} \quad (24)$$

$$\begin{aligned} \phi_{G_1 \odot G_2}(x; t) &= \det \begin{pmatrix} (x + r_1 t) I_{n_1} & -R_1 & 0_{n_1 \times m_2} & 0_{n_1 \times n_2} \\ -R_1^T & (x + m_2 t + 2t) I_{m_1} & -\mathbf{1}_{m_1 \times m_2} & 0_{m_1 \times n_2} \\ 0_{m_2 \times m_1} & -\mathbf{1}_{m_2 \times m_1} & (x + m_1 t + 2t) I_{m_2} & -R_2^T \\ 0_{n_2 \times n_1} & 0_{n_2 \times m_1} & -R_2 & (x + r_2 t) I_{n_2} \end{pmatrix} \\ &= \det((x + r_1 t) I_{n_1}) \det(S) = (x + r_1 t)^{n_1} \det(S), \end{aligned} \quad (25)$$

where

$$S = \begin{pmatrix} (x + m_2 t + 2t) I_{m_1} - \frac{R_1^T R_1}{x + r_1 t} & -\mathbf{1}_{m_1 \times m_2} & 0_{m_1 \times n_2} \\ -\mathbf{1}_{m_2 \times m_1} & (x + m_1 t + 2t) I_{m_2} & -R_2^T \\ 0_{n_2 \times m_1} & -R_2 & (x + r_2 t) I_{n_2} \end{pmatrix} \quad (26)$$

is the Schur complement of $(x + r_1 t) I_{n_1}$. So, $\det(S) = (x + r_2 t)^{n_2} \det(S_1)$, where

$$S_1 = \begin{pmatrix} (x + m_2 t + 2t) I_{m_1} - \frac{R_1^T R_1}{x + r_1 t} & -\mathbf{1}_{m_1 \times m_2} \\ -\mathbf{1}_{m_2 \times m_1} & (x + m_1 t + 2t) I_{m_2} - \frac{R_2^T R_2}{x + r_2 t} \end{pmatrix} \quad (27)$$

is the Schur complement of $(x + r_2 t)I_{n_2}$. So,

$$\begin{aligned} \det(S_1) &= \det\left((x + m_2 t + 2t)I_{m_1} - \frac{R_1^T R_1}{x + r_1 t}\right) \\ &\cdot \det\left((x + m_1 t + 2t)I_{m_2} - \frac{R_2^T R_2}{x + r_2 t}\right) \\ &- \mathbf{1}_{m_2 \times m_1} \left((x + m_2 t + 2t)I_{m_1} - \frac{R_1^T R_1}{x + r_1 t} \right)^{-1} \\ &\cdot \mathbf{1}_{m_1 \times m_2} \end{aligned} \quad (28)$$

is the Schur complement of $(x + m_2 t + 2t)I_{m_1} - R_1^T R_1 / (x + r_1 t)$. So,

$$\begin{aligned} \phi_{G_1 \odot G_2}(x; t) &= (x + r_1 t)^{n_1} (x + r_2 t)^{n_2} \\ &\cdot \det\left((x + m_2 t + 2t)I_{m_1} - \frac{A(\ell(G_1)) + 2I_{m_1}}{x + r_1 t}\right) \end{aligned}$$

$$\begin{aligned} &\cdot \det\left((x + m_1 t + 2t)I_{m_2} - \frac{R_2^T R_2}{x + r_2 t}\right) \\ &- \mathbf{1}_{m_2 \times m_1} \left((x + m_2 t + 2t)I_{m_1} - \frac{R_1^T R_1}{x + r_1 t} \right)^{-1} \\ &\cdot \mathbf{1}_{m_1 \times m_2} \end{aligned} \quad (29)$$

by Lemma 4,

$$\begin{aligned} \phi_{G_1 \odot G_2}(x; t) &= (x + r_1 t)^{n_1} (x + r_2 t)^{n_2} (x + m_2 t \\ &+ 2t)^{m_1 - n_1} \prod_{i=1}^{n_1} \left((x + m_2 t + 2t) \right. \\ &- \frac{\lambda_i(G_1) + r_1 - 2 + 2}{x + r_1 t} \left. \right) \cdot \det\left((x + m_1 t + 2t)I_{m_2} \right. \\ &- \frac{R_2^T R_2}{x + r_2 t} - \Gamma_{R_1^T R_1 / (x + r_1 t)}(x + m_2 t + 2t) \mathbf{1}_{m_2 \times m_2} \left. \right) \end{aligned} \quad (30)$$

and by Lemma 1,

$$\begin{aligned} \phi_{G_1 \odot G_2}(x; t) &= (x + r_1 t)^{n_1} (x + r_2 t)^{n_2} (x + m_2 t + 2t)^{m_1 - n_1} \prod_{i=1}^{n_1} \left(x + m_2 t + 2t - \frac{\lambda_i(G_1) + r_1}{x + r_1 t} \right) \\ &\cdot \left(1 - \Gamma_{R_1^T R_1 / (x + r_1 t)}(x + m_2 t + 2t) \Gamma_{R_2^T R_2 / (x + r_2 t)}(x + m_1 t + 2t) \right) \det\left((x + m_1 t + 2t)I_{m_2} - \frac{R_2^T R_2}{x + r_2 t}\right) \\ &= \prod_{i=1}^{n_1} \left(x^2 + (m_2 t + 2t + r_1 t)x + r_1 m_2 t^2 + 2r_1 t^2 - \lambda_i(G_1) - r_1 \right) \\ &\cdot \prod_{i=1}^{n_2} \left(x^2 + (m_1 t + 2t + r_2 t)x + r_2 m_1 t^2 + 2r_2 t^2 - \lambda_i(G_2) - r_2 \right) \\ &\cdot \left(1 - \frac{m_1 m_2 (x + r_1 t)(x + r_2 t)(x^2 + (m_1 t + r_2 t + 2t)x + r_2 m_1 t^2 + 2r_2 t^2 - 2r_2)^{-1}}{(x^2 + (m_2 t + r_1 t + 2t)x + r_1 m_2 t^2 + 2r_1 t^2 - 2r_1)} \right) \\ &\cdot (x + m_1 t + 2t)^{m_2 - n_2} (x + m_2 t + 2t)^{m_1 - n_1}. \end{aligned} \quad (31)$$

□

Theorem 8. Let G_1 be an r_1 -regular graph on n_1 vertices and m_1 edges and G_2 an r_2 -regular graph on n_2 vertices and m_2 edges. Then

(1) the adjacency characteristic polynomial of $G_1 \odot G_2$ is

$$\begin{aligned} \psi(A_{G_1 \odot G_2}; x) &= x^{m_1 + m_2 - n_1 - n_2} \prod_{i=1}^{n_1} (x^2 - \lambda_i(G_1) - r_1) \\ &\cdot \prod_{i=1}^{n_2} (x^2 - \lambda_i(G_2) - r_2) \\ &\cdot \left(1 - \frac{m_1 m_2 x^2}{x^4 - 2(r_1 + r_2)x^2 + 4r_1 r_2} \right), \end{aligned} \quad (32)$$

(2) the Laplacian characteristic polynomial of $G_1 \odot G_2$ is

$$\begin{aligned} \psi(L_{G_1 \odot G_2}; x) &= \prod_{i=2}^{n_1} (x^2 - (m_2 + 2 + r_1)x + r_1 m_2 \\ &+ u_i(G_1)) \prod_{i=2}^{n_2} (x^2 - (m_1 + 2 + r_2)x + r_2 m_1 \\ &+ u_i(G_2)) \cdot (x - m_1 - 2)^{m_2 - n_2} (x - m_2 - 2)^{m_1 - n_1} \\ &\cdot x(x^3 - (m_1 + m_2 + r_1 + r_2 + 4)x^2 + (r_1 m_2 \\ &+ r_2 m_1 + r_2 m_2 + r_1 m_1 + r_1 r_2 + 2m_1 + 2m_2 + 2r_1 \\ &+ 2r_2 + 4)x - m_1 r_1 r_2 - m_2 r_1 r_2 - 2m_1 r_2 - 2m_2 r_1), \end{aligned} \quad (33)$$

(3) the signless Laplacian characteristic polynomial of $G_1 \odot G_2$ is

$$\begin{aligned} \psi(Q_{G_1 \odot G_2}; x) &= \prod_{i=1}^{n_1} (x^2 - (m_2 + 2 + r_1)x + r_1(m_2 + 2) - v_i(G_1)) (x - m_2 - 2)^{m_1 - n_1} \\ &\quad \cdot \left(1 - \frac{m_1 m_2 (x - r_1)(x - r_2)}{(x^2 - (m_2 + r_1 + 2)x + m_2 r_1)(x^2 - (m_1 + r_2 + 2)x + m_1 r_2)} \right) \\ &\quad \cdot \prod_{i=1}^{n_2} (x^2 - (m_1 + 2 + r_2)x + r_2(m_1 + 2) - v_i(G_2)) (x - m_1 - 2)^{m_2 - n_2}. \end{aligned} \quad (34)$$

5. Applications

We give the number of spanning trees, the Kirchhoff index, and LEL of the two classes of join graphs, as the applications.

We can easily compute the L -spectrum of $G_1 \odot G_2$ in terms of L -spectrum of G_1 and G_2 by (2) of Theorem 6.

Corollary 9. Let G_1 be an r_1 -regular graph on n_1 vertices and m_1 edges and G_2 an r_2 -regular graph on n_2 vertices and m_2 edges. Then the Laplacian spectrum of $G_1 \odot G_2$ consists of

- (1) 2, repeated $m_1 + m_2 - n_1 - n_2$ times;
- (2) two roots of the equation $x^2 - (r_1 + n_2 + 2)x + 2n_2 + \mu_i(G_1) = 0$, for each $i = 2, \dots, n_1$;
- (3) two roots of the equation $x^2 - (r_2 + n_1 + 2)x + 2n_1 + \mu_i(G_2) = 0$, for each $i = 2, \dots, n_2$;
- (4) four roots of the equation $x(x^3 - (n_1 + n_2 + r_1 + r_2 + 4)x^2 + (4n_1 + 4n_2 + 2r_1 + 2r_2 + 4 + r_1 r_2 + r_1 n_1 + r_2 n_2)x - 4n_1 - 4n_2 - 2n_1 r_1 - 2n_2 r_2) = 0$, and the four roots are $0, \omega_1, \omega_2$, and ω_3 , respectively.

We can easily compute the L -spectrum of $G_1 \odot G_2$ in terms of L -spectrum of G_1 and G_2 by (2) of Theorem 8.

Corollary 10. Let G_1 be an r_1 -regular graph on n_1 vertices and m_1 edges and G_2 an r_2 -regular graph on n_2 vertices and

m_2 edges. Then the Laplacian spectrum of $G_1 \odot G_2$ consists of

- (1) $m_1 + 2$, repeated $m_2 - n_2$ times;
- (2) $m_2 + 2$, repeated $m_1 - n_1$ times;
- (3) two roots of the equation $x^2 - (m_2 + 2 + r_1)x + r_1 m_2 + \mu_i(G_1) = 0$, for each $i = 2, \dots, n_1$;
- (4) two roots of the equation $x^2 - (m_1 + 2 + r_2)x + r_2 m_1 + \mu_i(G_2) = 0$, for each $i = 2, \dots, n_2$;
- (5) four roots of the equation $x(x^3 - (m_1 + m_2 + r_1 + r_2 + 4)x^2 + (r_1 m_2 + r_2 m_1 + r_2 m_2 + r_1 m_1 + r_1 r_2 + 2m_1 + 2m_2 + 2r_1 + 2r_2 + 4)x - m_1 r_1 r_2 - m_2 r_1 r_2 - 2m_1 r_2 - 2m_2 r_1) = 0$, and the four roots are $0, \gamma_1, \gamma_2$, and γ_3 , respectively.

Let $t(G)$ denote the number of spanning trees [1] of G . It is well known that if G is a connected graph on n vertices with Laplacian spectrum $0 = \mu_1(G) \leq \mu_2(G) \leq \dots \leq \mu_n(G)$, then $t(G) = (\mu_2(G) \cdots \mu_n(G))/n$.

Corollary 11. Let G_1 be an r_1 -regular graph on n_1 vertices and m_1 edges and G_2 an r_2 -regular graph on n_2 vertices and m_2 edges. The number of spanning trees of $G_1 \odot G_2$ and $G_1 \ominus G_2$ are, respectively,

$$\begin{aligned} t(G_1 \odot G_2) &= \frac{2^{m_1 + m_2 - n_1 - n_2} \prod_{i=2}^{n_1} (2n_2 + \mu_i(G_1)) \prod_{i=2}^{n_2} (2n_1 + \mu_i(G_2)) (4n_1 + 4n_2 + 2r_1 n_1 + 2r_2 n_2)}{n_1 + m_1 + n_2 + m_2}, \\ t(G_1 \ominus G_2) &= \frac{(m_1 + 2)^{m_2 - n_2} (m_2 + 2)^{m_1 - n_1} \prod_{i=2}^{n_1} (r_1 m_2 + \mu_i(G_1)) \prod_{i=2}^{n_2} (r_2 m_1 + \mu_i(G_2))}{(n_1 + m_1 + n_2 + m_2) (m_1 r_1 r_2 + m_2 r_1 r_2 + 2m_1 r_2 + 2m_2 r_1)^{-1}}. \end{aligned} \quad (35)$$

Proof. For $G_1 \odot G_2$, the number of the vertices is $n_1 + m_1 + n_2 + m_2$. By Corollary 9 and Vieta's formulas, we can obtain the following: for every $\mu_i(G_1)$, $\prod_{i=2}^{n_1} (x_1 x_2) = \prod_{i=2}^{n_1} (2n_2 + \mu_i(G_1))$; for every $\mu_i(G_2)$, $\prod_{i=2}^{n_2} (x_1 x_2) = \prod_{i=2}^{n_2} (2n_1 + \mu_i(G_2))$; for the equation $x^3 - (n_1 + n_2 + r_1 + r_2 + 4)x^2 + (4n_1 + 4n_2 + 2r_1 + 2r_2 + 4 + r_1 r_2 + r_1 n_1 + r_2 n_2)x - 4n_1 - 4n_2 - 2n_1 r_1 - 2n_2 r_2 = 0$,

$\omega_1 \omega_2 \omega_3 = 4n_1 + 4n_2 + 2r_1 n_1 + 2r_2 n_2$. Therefore, we complete the proof of $t(G_1 \odot G_2)$ and we omit the proof of $t(G_1 \ominus G_2)$. $\square$

The Kirchhoff index [24] $Kf(G)$ is an important physical and chemical indicator. Let G be a connected graph of order n ; then $Kf(G) = n \sum_{i=2}^n (1/\mu_i)$.

Corollary 12. Let G_1 be an r_1 -regular graph on n_1 vertices and m_1 edges and G_2 an r_2 -regular graph on n_2 vertices and m_2

edges; then the Kirchhoff index of $G_1 \odot G_2$ and $G_1 \ominus G_2$ is, respectively,

$$\begin{aligned} \text{Kf}(G_1 \odot G_2) &= (m_1 + n_1 + m_2 + n_2) \left(\frac{m_1 + m_2 - n_1 - n_2}{2} + \sum_{i=2}^{n_1} \frac{r_1 + n_2 + 2}{2n_2 + \mu_i(G_1)} + \sum_{i=2}^{n_2} \frac{r_2 + n_1 + 2}{2n_1 + \mu_i(G_2)} \right. \\ &\quad \left. + \frac{4n_1 + 4n_2 + 2r_1 + 2r_2 + 4 + r_1r_2 + r_1n_1 + r_2n_2}{4n_1 + 4n_2 + 2r_1n_1 + 2r_2n_2} \right), \\ \text{Kf}(G_1 \ominus G_2) &= (m_1 + n_1 + m_2 + n_2) \left(\frac{m_2 - n_2}{2 + m_1} + \frac{m_1 - n_1}{2 + m_2} + \sum_{i=2}^{n_1} \frac{m_2 + 2 + r_1}{r_1m_2 + \mu_i(G_1)} + \sum_{i=2}^{n_2} \frac{m_1 + 2 + r_2}{r_2m_1 + \mu_i(G_2)} \right. \\ &\quad \left. + \frac{r_1m_2 + r_2m_1 + r_2m_2 + r_1m_1 + r_1r_2 + 2m_1 + 2m_2 + 2r_1 + 2r_2 + 4}{m_1r_1r_2 + m_2r_1r_2 + 2m_1r_2 + 2m_2r_1} \right). \end{aligned} \quad (36)$$

Proof. For $G_1 \odot G_2$, by Corollary 9 and Vieta's formulas, we can obtain the following: for every $\mu_i(G_1)$, $\sum_{i=2}^{n_1} (1/x_1 + 1/x_2) = \sum_{i=2}^{n_1} ((x_1 + x_2)/x_1x_2) = \sum_{i=2}^{n_1} ((r_1 + n_2 + 2)/(2n_2 + \mu_i(G_1)))$; for every $\mu_i(G_2)$, $\sum_{i=2}^{n_2} (1/x_1 + 1/x_2) = \sum_{i=2}^{n_2} ((r_2 + n_1 + 2)/(2n_1 + \mu_i(G_2)))$; for the equation $x^3 - (n_1 + n_2 + r_1 + r_2 + 4)x^2 + (4n_1 + 4n_2 + 2r_1 + 2r_2 + 4 + r_1r_2 + r_1n_1 + r_2n_2)x - 4n_1 - 4n_2 - 2n_1r_1 - 2n_2r_2 = 0$, $1/\omega_1 + 1/\omega_2 + 1/\omega_3 = (\omega_1\omega_2 + \omega_2\omega_3 + \omega_1\omega_3)/\omega_1\omega_2\omega_3 = (4n_1 + 4n_2 + 2r_1 + 2r_2 + 4 + r_1r_2 + r_1n_1 + r_2n_2)/(4n_1 + 4n_2 + 2r_1n_1 + 2r_2n_2)$. Therefore, we complete the proof of $\text{Kf}(G_1 \odot G_2)$ and we omit the proof of $\text{Kf}(G_1 \ominus G_2)$. $\square$

$\text{LEL}(G)$ denotes Laplacian-Energy-Like Invariant [24]. Let G be a connected graph of order n with m edges; then $\text{LEL} = \text{LEL}(G) = \sum_{i=2}^n \sqrt{\mu_i}$.

Corollary 13. Let G_1 be an r_1 -regular graph on n_1 vertices and m_1 edges and G_2 an r_2 -regular graph on n_2 vertices and m_2 edges; then

$$\begin{aligned} \text{LEL}(G_1 \odot G_2) &= \sqrt{2} (m_1 + m_2 - n_1 - n_2) + \sum_{i=1}^3 \sqrt{\omega_i} \\ &\quad + \sum_{i=2}^{n_1} \sqrt{\frac{r_1 + n_2 + 2 \pm \sqrt{\Delta_1}}{2}} \\ &\quad + \sum_{i=2}^{n_2} \sqrt{\frac{r_2 + n_1 + 2 \pm \sqrt{\Delta_2}}{2}}, \end{aligned} \quad (37)$$

where $\Delta_1 = (r_1 + n_2 + 2)^2 - 4(2n_2 + \mu_i(G_1))$, $\Delta_2 = (r_2 + n_1 + 2)^2 - 4(2n_1 + \mu_i(G_2))$, and ω_1, ω_2 , and ω_3 are three roots of the equation $x^3 - (n_1 + n_2 + r_1 + r_2 + 4)x^2 + (4n_1 + 4n_2 + 2r_1 + 2r_2 + 4 + r_1r_2 + r_1n_1 + r_2n_2)x - 4n_1 - 4n_2 - 2n_1r_1 - 2n_2r_2 = 0$.

$$\begin{aligned} \text{LEL}(G_1 \ominus G_2) &= (m_2 - n_2) \sqrt{m_1 + 2} \\ &\quad + (m_1 - n_1) \sqrt{m_2 + 2} + \sum_{i=1}^3 \sqrt{\gamma_i} \end{aligned}$$

$$\begin{aligned} &+ \sum_{i=2}^{n_1} \sqrt{\frac{m_2 + 2 + r_1 \pm \sqrt{\Delta_3}}{2}} \\ &+ \sum_{i=2}^{n_2} \sqrt{\frac{m_1 + 2 + r_2 \pm \sqrt{\Delta_4}}{2}}, \end{aligned} \quad (38)$$

where $\Delta_3 = (m_2 + 2 + r_1)^2 - 4(r_1m_2 + \mu_i(G_1))$, $\Delta_4 = (m_1 + 2 + r_2)^2 - 4(r_2m_1 + \mu_i(G_2))$, and γ_1, γ_2 , and γ_3 are three roots of the equation $x^3 - (m_1 + m_2 + r_1 + r_2 + 4)x^2 + (r_1m_2 + r_2m_1 + r_2m_2 + r_1m_1 + r_1r_2 + 2m_1 + 2m_2 + 2r_1 + 2r_2 + 4)x - m_1r_1r_2 - m_2r_1r_2 - 2m_1r_2 - 2m_2r_1 = 0$.

6. Conclusions

In this paper, two classes of join graphs, $G_1 \odot G_2$ and $G_1 \ominus G_2$, are constructed by graph operations. The formulas of generalized characteristic polynomial of the two graphs are obtained by using block matrix and Schur complement. For regular graphs G_1 and G_2 , we characterize the A -, L -, and Q -spectrum of $G_1 \odot G_2$ and $G_1 \ominus G_2$ in terms of the corresponding spectrum of G_1 and G_2 , and by the L -spectrum, we get the number of spanning trees, Kirchhoff index, and LEL . But for nonregular graphs G_1 and G_2 , we cannot determine the spectrum of $G_1 \odot G_2$ and $G_1 \ominus G_2$. The most difficult problem is that we cannot find the rule of the block matrices to deduce a simple result. Therefore, we should try to find new method to solve the problem.

Competing Interests

The authors declare that they have no competing interests.

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