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Research Article

Existence and Exponential Stability of Solutions to Stochastic Neutral Functional Differential Equations

Ling Hu, Zheng Wu, Zhangzhi Wei, and Lianglong Wang

School of Mathematical Sciences, Anhui University, Hefei 230601, China

Correspondence should be addressed to Lianglong Wang; wangll@ahu.edu.cn

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In this paper we consider the existence and stability of solutions to stochastic neutral functional differential equations with finite delays. Under suitable conditions, the existence and exponential stability of solutions were obtained by using the semigroup approach and Banach fixed point theorem.

1. Introduction

In natural world, stochastic phenomena are everywhere. Many stochastic facts exist in biology, chemistry, physics, and economical systems. In addition, delays also appear sometimes to change the results. These facts imply the necessity to study stochastic functional differential equations (SFDEs), although there are a lot of papers on the related topics for deterministic partial functional differential equations [1–4].

A large amount of basic knowledge about stochastic differential equations (SDEs) has been given in [5, 6]. Recently, many researchers studied the existence, stability, and other properties of solutions to SFDEs. Some of these topics have been solved by the semigroup approach and others have been solved by the variational one. There are many types of equations worth attentions; say [7] is focused on retarded equations with finite delays while [8] is concerned with neutral ones. In fact the stability of SDEs is most important among all qualitative properties and also determines whether the model is significant. Aside from all kinds of stability, other properties have drawn many attentions. For example, Wu et al. [9, 10] investigated the stochastic delay population dynamics under regime switching by using generalized Itô's formula, Gronwall inequality, and Young's inequality. Furthermore, some stochastic systems have added the impulse factors [11].

On the other hand, although stochastic partial FDEs with finite delays also seem very important, the corresponding properties of these systems have not been studied in great detail.

Taniguchi et al. [12] discussed the stochastic partial FDEs with finite delay as follows:

$$dX(t) = \left[-AX(t) + f(t, X_t)\right] dt + g(t, X_t)\, dW(t), \quad t \geq t_0, \tag{1}$$
$$X_{t_0} = \phi \in L^p(\Omega, C_\alpha), \quad t_0 \geq 0.$$

Under some suitable conditions, the existence and asymptotic behavior of solutions were obtained employing the semigroup approach and fixed point theorem.

In this paper, we shall discuss the existence and uniqueness of mild solutions to a class of stochastic NFDEs with finite delays,

$$d\left[X(t) - f(t, X_t)\right] = \left[-AX(t) + F(t, X_t)\right] dt + G(t, X_t)\, dW(t), \quad t \geq t_0, \tag{2}$$
$$X_{t_0} = \phi \in L^p(\Omega, C_\alpha), \quad t_0 \geq 0,$$

where ϕ is $\mathscr{F}_{t_0}$-measurable. $-A$ is a linear and densely closed operator which generates an analytic semigroup $S(t)$, $t \geq 0$, on a real separable Hilbert space H with the inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$.

Notice that (1) is of retarded type; here we put a neutral type one, (2). When $f(t, X_t) \equiv 0$, our equation is the same as that in [12]. In this paper we assume $0 < \alpha < 1$, $p > 2$ and define $\mathscr{D}(A^\alpha)$ with the norm $\|x\|_\alpha := \|A^\alpha x\|$ for $x \in \mathscr{D}(A^\alpha)$,

where $\mathscr{D}(A^\alpha)$ is the domain of the fractional power operator $A_\alpha : H \to H$. Let $H_\alpha := \mathscr{D}(A^\alpha)$ and $C_\alpha = C([-r,0], H_\alpha)$, $0 < r < \infty$. K is another real separable Hilbert space with the inner product $(\cdot,\cdot)_K$ and norm $\|\cdot\|_K$, $W(t)$ is a K-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. Assume $f, F : (-\infty, +\infty) \times C_\alpha \to H$, $G : (-\infty, +\infty) \times C_\alpha \to \mathscr{L}_2^0(K, H)$ are measurable mappings, satisfying that $f(t,0)$, $F(t,0)$, $G(t,0)$ are locally bounded in H-norm and $\mathscr{L}_2^0(K,H)$-norm, respectively.

This paper is organized as follows. In Section 2 some preliminary results are given, which are fundamental for the subsequent developments. The existence and uniqueness of solutions are investigated in Section 3. At last, we obtain the almost sure exponential stability of the solutions.

2. Preliminaries

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $X_t : \Omega \to H_\alpha$, $t \geq t_0 - r$ is a continuous $\mathscr{F}_t$-adapted, H_α-valued stochastic process. We give process $X_t : \Omega \to C_\alpha$, $t \geq t_0$, by setting $X_t(s)(\omega) = X(t+s)(\omega)$, $s \in [-r, 0]$.

Let $MC_\alpha(t,p)$, $p > 2$, be the space of all F_t-measurable C_α-valued functions $\psi : \Omega \to C_\alpha$:

$$E\|\psi\|_{C_\alpha}^p = E\left\{\sup_{-r\leq s\leq 0}\|A^\alpha\psi(s)\|^p\right\} < \infty. \tag{3}$$

Next we give three important assumptions.

Assumption A. (a) H is a separable Hilbert space on which there is an analytic semigroup $S(t)$, $t \geq 0$, and $-A$ is the infinitesimal generator of $S(t)$.

(b) There exist $M \geq 1$ and $a > 0$ such that $\|S(t)h\| \leq Me^{-at}\|h\|$, $t \geq 0$, $h \in H$.

(c) $\|A^\alpha S(t)h\| \leq M_\alpha e^{-at}t^{-\alpha}\|h\|$, $t \geq 0$, for any $h \in H$, where $M_\alpha \geq 1$, $\|A^{-\alpha}\| \leq C$.

(d) $\|S(t)h - h\| \leq N_\alpha t^\alpha \|A^\alpha h\|$, $h \in D(A^\alpha)$, $N_\alpha \geq 1$.

The details can be seen in Pazy [13].

Under Assumption A, we will consider the stochastic integral equation next instead of (2) by carrying out a semigroup type argument mentioned above:

$$\begin{aligned} X(t) = {} & S(t-t_0)[\phi(0) - f(t_0,\phi)] + f(t, X_t) \\ & - \int_{t_0}^t AS(t-u) f(u, X_u)\, du \\ & + \int_{t_0}^t S(t-u) F(u, X_u)\, du \\ & + \int_{t_0}^t S(t-u) G(u, X_u)\, dW(u) \end{aligned} \tag{4}$$

$$X_{t_0} = \phi \in MC_\alpha(t_0, p), \quad t_0 \geq 0. \tag{5}$$

The function $X(t)$ which satisfies (4) is called the mild solution of (2).

Lemma 1 (see [12]). *Suppose $\Phi(t) : \Omega \to \mathscr{L}_2^0(K,H)$, $t \geq 0$. If $\Phi(t)k \in H_\alpha$, $t \geq 0$, $k \in K$, and $\int_0^t E\|\Phi(u)\|_{\mathscr{L}_2^0}^2 du < \infty$, $\int_0^t E\|A^\alpha\Phi(u)\|_{\mathscr{L}_2^0}^2 du < \infty$ then*

$$A^\alpha \int_0^t \Phi(s)\, dW(s) = \int_0^t A^\alpha \Phi(s)\, dW(s). \tag{6}$$

Assumption B. For any $\gamma, \xi \in C_\alpha$ and $t_0 \leq t \leq T$, there exist three positive real constants $N_0 = N_0(T)$, $N_1 = N_1(T)$, and $N_2 = N_2(T) > 0$ such that

$$\begin{aligned} \|f(t,\gamma) - f(t,\xi)\|^p &\leq N_0 \|\gamma - \xi\|_{C_\alpha}^p, \\ \|F(t,\gamma) - F(t,\xi)\|^p &\leq N_1 \|\gamma - \xi\|_{C_\alpha}^p, \\ \|G(t,\gamma) - G(t,\xi)\|_{\mathscr{L}_2^0}^p &\leq N_2 \|\gamma - \xi\|_{C_\alpha}^p. \end{aligned} \tag{7}$$

Under Assumption B, we get a real number $N_3 = N_3(T) > 0$ such that

$$\begin{aligned} & \|f(t,\xi)\|^p + \|F(t,\xi)\|^p + \|G(t,\xi)\|_{\mathscr{L}_2^0}^p \\ & \quad \leq N_3\left(1 + \|\xi\|_{C_\alpha}^p\right), \end{aligned} \tag{8}$$

for $t \leq t \leq T$, where T is any fixed time.

Assumption C. There exist $\alpha \in (0,1)$ and $L \geq 0$ such that

$$\begin{aligned} & \|A^\alpha f(t,\psi_1) - A^\alpha f(s,\psi_2)\| \\ & \quad \leq L\left(|t-s| + \|\psi_1 - \psi_2\|_{C_\alpha}\right), \\ & \|A^\alpha f(t,\psi)\| \leq L\left(1 + \|\psi\|_{C_\alpha}\right), \end{aligned} \tag{9}$$

for every $t_0 \leq s, t \leq T$, $\psi_1, \psi_2, \psi \in \Omega$, and $4^{p-1}L^p < 1$.

3. Existence of Mild Solutions

Definition 2. If function $X(t)$ satisfies (4) on $[t_0, T]$ and $T - t_0 > 0$ is small, it is called local solution. If $X(t)$ satisfied (4) on $[t_0, +\infty]$, it is called global solution.

Theorem 3. *Let $0 < \alpha < (p-1)/2p$. Suppose that Assumptions A, B, and C hold. There exists a unique local continuous solution to (4) for any initial value (t_0, ϕ) with $t_0 \geq 0$ and $\phi \in MC_\alpha(t_0, p)$.*

In order to prove Theorem 3, we need some lemmas. Assume $T > t_0$ is a fixed time and D_T is the subspace of all continuous processes Z which belong to the space $C([t_0 - r, T], L^p(\Omega, H))$ with $\|Z\|_{D_T} < \infty$, where

$$\begin{aligned} \|Z\|_{D_T} &:= \sup_{t_0\leq t\leq T}\left(E\|Z_t\|_C^p\right)^{1/p}, \\ E\|Z_t\|_C^p &:= E\left\{\sup_{-r\leq s\leq 0}\|Z_t(s)\|^p\right\}. \end{aligned} \tag{10}$$

Introduce the following mapping Φ on D_T:

$$\begin{aligned}(\Phi Z)(t) &= A^{\alpha} S(t-t_0)[\phi(0) - f(t_0,\phi)] \\ &\quad + A^{\alpha} f(t, A^{-\alpha} Z_t) \\ &\quad - \int_{t_0}^{t} A^{\alpha+1} S(t-s) f(s, A^{-\alpha} Z_s)\, ds \\ &\quad + \int_{t_0}^{t} A^{\alpha} S(t-s) F(s, A^{-\alpha} Z_s)\, ds \\ &\quad + \int_{t_0}^{t} A^{\alpha} S(t-s) G(s, A^{-\alpha} Z_s)\, dW(s), \\ &\qquad t \geq t_0, \\ (\Phi Z)(t) &= A^{\alpha}\phi(t), \quad t_0 \geq t \geq t_0 - r.\end{aligned} \tag{11}$$

Lemma 4. *For $Z \in D_T$, $(\Phi Z)(t)$ is continuous on $[t_0, T]$ in the L_p-sense.*

Proof. Let $t_0 < t_1 < t_2 < T$. For any fixed $Z \in D_T$,

$$\begin{aligned}&E\|(\Phi Z)(t_2) - (\Phi Z)(t_1)\|^p \\ &\quad \leq 5^{p-1} E\|[S(t_2-t_0) - S(t_1-t_0)] \\ &\quad \cdot A^{\alpha}[\phi(0) - f(t_0,\phi)]\|^p \\ &\quad + 5^{p-1} E\left\|A^{\alpha} f(t_2, A^{-\alpha} Z_{t_2}) - A^{\alpha} f(t_1, A^{-\alpha} Z_{t_1})\right\|^p \\ &\quad + 5^{p-1} E\left\|\int_{t_0}^{t_2} A^{\alpha+1} S(t_2-s) f(s, A^{-\alpha} Z_s)\right. \\ &\quad \left. - \int_{t_0}^{t_1} A^{\alpha+1} S(t_1-s) f(s, A^{-\alpha} Z_s)\, ds\right\|^p \\ &\quad + 5^{p-1} E\left\|\int_{t_0}^{t_2} A^{\alpha} S(t_2-s) F(s, A^{-\alpha} Z_s)\right. \\ &\quad \left. - \int_{t_0}^{t_1} A^{\alpha} S(t_1-s) F(s, A^{-\alpha} Z_s)\, ds\right\|^p \\ &\quad + 5^{p-1} E\left\|\int_{t_0}^{t_2} A^{\alpha} S(t_2-s) G(s, A^{-\alpha} Z_s)\right. \\ &\quad \left. - \int_{t_0}^{t_1} A^{\alpha} S(t_1-s) G(s, A^{-\alpha} Z_s)\, dW(s)\right\|^p = I_1 + I_2 \\ &\quad + I_3 + I_4 + I_5.\end{aligned} \tag{12}$$

From Assumptions A, B, and C we get that

$$\begin{aligned}I_1 &\leq 5^{p-1} N_{\alpha}^p (t_2-t_1)^{p\alpha} \\ &\quad \cdot E\|A^{\alpha} S(t_1-t_0) A^{\alpha}[\phi(0) - f(t_0,\phi)]\|^p, \\ I_2 &\leq 10^{p-1} L^p (t_2-t_1)^p + 10^{p-1} E\left\|Z_{t_2} - Z_{t_1}\right\|^p,\end{aligned} \tag{13}$$

because of the continuity of Z_t, $I_2 \to 0$, when $t_2 \to t_1$.

$$\begin{aligned}I_3 &\leq 10^p E\left(\int_{t_1}^{t_2} \left\|A^{\alpha+1} S(t_2-s) f(s, A^{-\alpha} Z_s)\right\| ds\right)^p \\ &\quad + 10^p E\left(\int_{t_0}^{t_1} \left\|A^{\alpha+1} S(t_1-s)(S(t_2-t_1) - I)\right.\right. \\ &\quad \left.\left. \cdot f(s, A^{-\alpha} Z_s)\right\| ds\right)^p \leq I_{31} + I_{32}.\end{aligned} \tag{14}$$

There exist $Q_{31}, Q_{32} > 0$ and let $q = p/(p-1)$, $\epsilon_1 = 1 - \alpha + p/q > 0$ such that

$$\begin{aligned}I_{31} &\leq 10^{p-1} M_{\alpha+1}^p N_3 E\left(\int_{t_1}^{t_2} (t_2-s)^{-\alpha-1} e^{-a(t_2-s)} \left(1\right.\right. \\ &\quad \left.\left. + \|Z_s\|_C^p\right)^{1/p} ds\right)^p \leq Q_{31}(t_2-t_1)^{\epsilon_1} \left(1\right. \\ &\quad \left. + \|Z\|_{D_T}^p\right), \\ I_{32} &\leq 10^{p-1} E\left(\int_{t_0}^{t_1} \left\|A^{\alpha+1} S\left(\frac{t_1-s}{2}\right)(S(t_2-t_1) - I)\right.\right. \\ &\quad \left.\left. \cdot S\left(\frac{t_1-s}{2}\right) f(s, A^{-\alpha} Z_s)\right\| ds\right)^p \leq Q_{32}(t_2 \\ &\quad - t_1)^{p\alpha}\left(1 + \|Z\|_{D_T}^p\right).\end{aligned} \tag{15}$$

In the similar way, there exist $Q_{41}, Q_{42} > 0$ and we have

$$\begin{aligned}I_4 &\leq Q_{41}(t_2-t_1)^{1+\epsilon_1}\left(1 + \|Z\|_{D_T}^p\right) \\ &\quad + Q_{42}(t_2-t_1)^{p\alpha}\left(1 + \|Z\|_{D_T}^p\right).\end{aligned} \tag{16}$$

At last

$$\begin{aligned}I_5 &\leq 10^p E\left(\int_{t_1}^{t_2} \|A^{\alpha} S(t_2-s) G(s, A^{-\alpha} Z_s)\|_{\mathscr{L}_2^0}^2 ds\right)^{p/2} \\ &\quad + 10^p E\left(\int_{t_0}^{t_1} \|A^{\alpha} S(t_1-s)(S(t_2-t_1) - I)\right. \\ &\quad \left. \cdot G(s, A^{-\alpha} Z_s)\|_{\mathscr{L}_2^0}^2 ds\right)^{p/2} \leq I_{51} + I_{52}.\end{aligned} \tag{17}$$

Then, there exist $Q_{51}, Q_{52} > 0$ and $\epsilon_2 = (p-1-2p^2\alpha)/2p > 0$ such that

$$\begin{aligned} I_{51} &\le 10^{p-1} M_\alpha^p N_3^p E\left(\int_{t_1}^{t_2} (t_2-s)^{-2\alpha} \cdot e^{-2a(t_2-s)}\left(1+\|Z_s\|_C^2\right) ds\right)^{p/2} \le Q_{51}\left(t_2 - t_1\right)^{1+\epsilon_2}\left(1+\|Z\|_{D_T}^p\right), \\ I_{52} &\le 10^{p-1} M_\alpha^{2p} N_\alpha^p E\left[\int_{t_0}^{t_1}\left(\frac{t_1-s}{2}\right)^{-2\alpha} \cdot e^{-2a(t_1-s)}\left(t_2-t_1\right)^\alpha \left\|G\left(s, A^{-\alpha}Z_s\right)\right\|_{\mathscr{L}_2^0}^2 ds\right]^p \\ &\le Q_{52}\left(t_2-t_1\right)^{p\alpha}\left(1+\|Z\|_{D_T}^p\right). \end{aligned} \tag{18}$$

Since $Z \in D_T$, it follows that I_1, I_2, I_3, I_4, I_5 tend to zero, as $t_2 \to t_1$. Therefore, the proof of the lemma is complete. □

Lemma 5. *Suppose the operator mapping Φ and the corresponding domain D_T are defined as above; then $\Phi(D_T) \subset D_T$.*

The proof is similar to that of Lemma 4.

Proof of Theorem 3. Let $X, Y \in D_T$, and then for any fixed $t \in [t_0, T]$,

$$\begin{aligned} E\left\|(\Phi X)_t - (\Phi Y)_t\right\|_C^p &\le E \sup_{-r\le\theta\le 0} \|(\Phi X)(t+\theta) - (\Phi Y) \cdot (t+\theta)\|^p \le 4^{p-1} E \sup_{-r\le\theta\le 0} \left\|A^\alpha\left(f(t+\theta), A^{-\alpha}X_{t+\theta}\right) - A^\alpha\left(f(t+\theta), A^{-\alpha}Y_{t+\theta}\right)\right\|^p + 4^{p-1}E \\ &\cdot \sup_{-r\le\theta\le 0}\left\|\int_{t_0}^{t+\theta} A^{\alpha+1} S(t+\theta-s) \cdot \left(f\left(s, A^{-\alpha}X_s\right) - f\left(s, A^{-\alpha}Y_s\right)\right) ds\right\|^p + 4^{p-1}E \\ &\cdot \sup_{-r\le\theta\le 0}\left\|\int_{t_0}^{t+\theta} A^{\alpha} S(t+\theta-s) \cdot \left(F\left(s, A^{-\alpha}X_s\right) - F\left(s, A^{-\alpha}Y_s\right)\right) ds\right\|^p + 4^{p-1}E \\ &\cdot \sup_{-r\le\theta\le 0}\left\|\int_{t_0}^{t+\theta} A^{\alpha} S(t+\theta-s) \cdot \left(G\left(s, A^{-\alpha}X_s\right) - G\left(s, A^{-\alpha}Y_s\right)\right) dW(s)\right\|^p \\ &= I_{11} + I_{12} + I_{13} + I_{14}, \end{aligned}$$

$$\begin{aligned} I_{11} &\le 4^{p-1} L^p E \sup_{-r\le\theta\le 0}\left\|A^{-\alpha}X_{t+\theta} - A^{-\alpha}Y_{t+\theta}\right\|^p \le 4^{p-1} L^p \|X-Y\|_{D_T}, \\ I_{12} &\le 4^{p-1} N_0 M_{\alpha+1}^p (T-t_0)\left(\Gamma(1-\alpha q)(aq)^{q\alpha-1}\right)^{p/q} \cdot \|X-Y\|_{D_T}^p, \\ I_{13} &\le 4^{p-1} N_1 M_\alpha^p (T-t_0)\left(\Gamma(1-\alpha q)(aq)^{q\alpha-1}\right)^{p/q} \|X-Y\|_{D_T}^p. \end{aligned} \tag{19}$$

Next, let $1/p + \alpha < \rho < 1/2$ and

$$\Delta(s) = \int_{t_0}^{s} (s-\sigma)^{-\rho} S(s-\sigma) \cdot \left(G\left(\sigma, A^{-\alpha}X_\sigma\right) - G\left(\sigma, A^{-\alpha}Y_\sigma\right)\right) dW(\sigma), \tag{20}$$

and then by the use of the stochastic Fubini Theorem again we have

$$\begin{aligned} I_{14} &\le 4^{p-1} E \sup_{-r\le\theta\le 0}\left\|A^\alpha \int_{t_0}^{t+\theta} (t+\theta-s)^{\rho-1} S(t+\theta-s) \cdot \Delta(s)\, ds\right\|^p \le 4^{p-1} N_2 M_\alpha^p (T-t_0)\left(\Gamma\left(1-(1+\alpha-\rho)q\right)(aq)^{q(1+\alpha-\rho)-1}\right)^{p/q} \cdot c_p M^p \frac{T^{-p\rho+p/2}}{(1-2\rho)^{p/2}} \|X-Y\|_{D_T}^p. \end{aligned} \tag{21}$$

So we can choose a suitable $T > t_0$ to make $T - t_0 > 0$ sufficiently small; we obtain $B(T) \in (0,1)$ such that

$$\|\Phi X - \Phi Y\|_{D_T} \le B(T)\|X-Y\|_{D_T}, \tag{22}$$

for any $X, Y \in D_T$. Thus, by Banach fixed point theorem we get a unique fixed point $U \in D_T$. Setting $X(t) = A^{-a}U(t)$, we obtain

$$\begin{aligned} X(t) &= S(t-t_0)\left[\phi(0) - f(t_0, \phi)\right] + f(t, X_t) - \int_{t_0}^{t} AS(t-s) f(s, X_s)\, ds + \int_{t_0}^{t} S(t-s) F(s, X_s)\, ds + \int_{t_0}^{t} S(t-s) G(s, X_s)\, dW(s), \quad t \ge t_0, \\ X_{t_0} &= \phi(t), \quad t_0 \ge t \ge t_0 - r. \end{aligned} \tag{23}$$

The proof is complete. □

Theorem 6. *Assume $0 < \alpha < (p-2)/2p$ and let $f, F : (-\infty, +\infty) \times C_a \to H$, $g : (-\infty, +\infty) \times C_a \to \mathscr{L}_2^0(K, H)$ satisfy Assumptions A, B, and C. If there exists a constant $B_2 > 0$ such that*

$$\begin{aligned} &\|f(t,\psi)\|^p + \|F(t,\psi)\|^p + \|G(t,\psi)\|^p_{\mathscr{L}_2^0} \\ &\quad \le B_2\left(1 + \|\psi\|^p_{C_\alpha}\right) \end{aligned} \tag{24}$$

for all $\psi \in C_a$, $t \ge 0$, then there exists a unique, global continuous solution $X(t) : \Omega \to H_\alpha$ to (4) for any initial value (t_0, ϕ) with $\phi \in MC_\alpha(t_0, p)$.

Proof. If F and G satisfy the global Lipschitz conditions, the proof of the theorem can be given similarly as a corollary of Theorem 3. If F and G satisfy the local Lipschitz conditions, the proof can be given similarly by the truncation method in [14]. Hence, we omit the proof. □

4. Pth Moment Exponential Stability

Lemma 7 (see [12]). *Let $0 < \theta < (p-2)/2p$ and assume β is any fixed real number such that $1/p + \theta < \beta < 1/2$. Then for all $t \in (t_0 + r, T)$,*

$$\begin{aligned} &E\left(\sup_{-r\le s\le 0}\left\|\int_{t_0}^{t+s} S(t+s-u)\,\Phi(u)\,dW(u)\right\|^p_\theta\right) \\ &\quad \le c_p e^{ar} M_\theta^p M^p C(p,\theta,\beta)\,\Gamma(1-2\beta)\left(2a - \frac{2a}{p}\right)^{2\beta-1} \\ &\qquad \cdot \int_{t_0}^{t} e^{-a(t-\sigma)} E\|\Phi(\sigma)\|^p_{\mathscr{L}_2^0}\,d\sigma, \end{aligned} \tag{25}$$

where $C(p,\theta,\beta) := \{\Gamma(1-q(1+\theta-\beta))a^{q(1+\theta-\beta)}\}^{p/q}$.

Now, we are in a position to present the stability.

Theorem 8. *Assume $0 < \theta < (p-1)/2p$. Let $f, F : (-\infty, +\infty) \times C_\theta \to H$, $G : (-\infty, +\infty) \times C_\theta \to \mathscr{L}_2^0(K, H)$ satisfy the local Lipschitz condition B. Furthermore assume that Assumption A is satisfied and there exist nonnegative real numbers $Q_1, Q_2 \ge 0$ and continuous functions $\xi_1, \xi_2 : [0,\infty) \to R_+$ such that*

$$\begin{aligned} E\|F(t,X_t)\|^p &\le Q_1 E\|X_t\|^p_{C_\theta} + \xi_1(t), \quad t \ge t_0, \\ E\|G(t,X_t)\|^p_{\mathscr{L}_2^0} &\le Q_2 E\|X_t\|^p_{C_\theta} + \xi_2(t), \quad t \ge t_0, \end{aligned} \tag{26}$$

for any solution $X(t)$ to (4). Suppose $a > L_0$, where $\|S(t)\| \le Me^{-at}$, $t \ge 0$, and there exist nonnegative real numbers $P_1, P_2 \ge 0$ such that $|\xi_j(t)| \le P_j e^{-(a-L_0)t}$ $(j = 1, 2)$, and then there exist positive constants $\varepsilon > 0$ and $K(p,\varepsilon,\phi) > 0$ such that for each $t \ge t_0 + 2r$

$$E\|X_t\|^p_{C_\theta} \le K(p,\varepsilon,\phi)\cdot e^{-\varepsilon(t-t_0)}. \tag{27}$$

In other words, the solution is the pth moment exponentially stable.

Proof. Without loss of generality, we suppose $t_0 = 0$. Let $-r \le s \le 0$; then for each $t > 2r$,

$$\begin{aligned} &E\|X(t+s)\|^p_\theta \\ &\quad \le 5^{p-1}E\|S(t+s)[\phi(0) - f(0,\phi)]\|^p_\theta \\ &\qquad + 5^{p-1}E\|f(t+s, X_{t+s})\|^p_\theta \\ &\qquad + 5^{p-1}E\left\|\int_0^{t+s} AS(t+s-u)\,f(u,X_u)\,du\right\|^p_\theta \\ &\qquad + 5^{p-1}E\left\|\int_0^{t+s} S(t+s-u)\,F(u,X_u)\,du\right\|^p_\theta \\ &\qquad + 5^{p-1}E\left\|\int_0^{t+s} S(t+s-u)\,G(u,X_u)\,dW(u)\right\|^p_\theta \\ &\quad = J_1 + J_2 + J_3 + J_4 + J_5, \\ &J_1 \le 5^{p-1}M_\theta^p e^{-pa(t-r)}(t-r)^{-p\theta}E\|\phi(0) - f(0,\phi)\|^p, \\ &J_2 \le 5^{p-1}L^p E\left(1 + \|X_{t+s}\|_{C_\theta}\right)^p. \end{aligned} \tag{28}$$

On the other hand, by the Hölder inequality we can obtain that

$$\begin{aligned} J_3 &\le 5^{p-1}M_{\theta+1}^p e^{-as}\left(\Gamma(1-q\theta)\,a^{q\theta-1}\right)^{p/q} \\ &\quad \cdot\left(\int_0^{t+s} e^{-a(t-u)}\left[Q_1 E\|X_u\|^p_{C_\theta} + \xi_1\right]du\right), \\ J_4 &\le 5^{p-1}M_\theta^p e^{-as}\left(\Gamma(1-q\theta)\,a^{q\theta-1}\right)^{p/q} \\ &\quad \cdot\left(\int_0^{t+s} e^{-a(t-u)}\left[Q_1 E\|X_u\|^p_{C_\theta} + \xi_1(u)\right]du\right). \end{aligned} \tag{29}$$

By virtue of Lemma 7, we can deduce

$$\begin{aligned} J_5 &\le 5^{p-1}M_\theta^p M^p e^{-as} c_p C(p,\theta,\beta)\,\Gamma(1-2\beta) \\ &\quad \cdot\left(2a - \frac{2a}{p}\right)^{2\beta-1} \\ &\quad \cdot\int_0^{t+s} e^{-a(t-u)}\left[Q_2 E\|X_u\|^p_{C_\theta} + \xi_2(u)\right]du. \end{aligned} \tag{30}$$

Then we have

$$\begin{aligned} &E\|X(t+s)\|^p_\theta \\ &\quad \le M_0\cdot e^{-at} + L_0 e^{-a(t+s)}\int_0^{t+s} e^{au}E\|X_u\|^p_{C_\theta}\,du \\ &\qquad + e^{-at}\int_0^{t+s} e^{au}\left(L_1\xi_1(u) + L_2\xi_2(u)\right)du, \end{aligned} \tag{31}$$

for each $t \geq 2r$, where

$$\begin{aligned} M_0 &= 5^{p-1} M_\theta^p r^{-p\beta} e^{par} E \left\| \phi(0) - f(0,\phi) \right\|^p, \\ L_1 &= 5^{p-1} \left(M_{\theta+1}^p + M_\theta^p \right) \left(\Gamma(1-q\theta) a^{q\theta-1} \right)^{p/q}, \\ L_2 &= 5^{p-1} M_\theta^p M^p c_p C(p,\theta,\beta) \Gamma(1-2\beta) \\ &\quad \cdot \left(2a - \frac{2a}{p} \right)^{2\beta-1} \\ L_0 &= L_1 Q_1 + L_2 Q_2. \end{aligned} \tag{32}$$

Therefore, we have for arbitrary $\varepsilon \in R_+$, $0 < \varepsilon < a$ and $T > 0$ large enough,

$$\begin{aligned} &\int_0^T e^{\varepsilon t} E \left\| X(t+s) \right\|_\theta^p dt \\ &\leq M_0 \int_0^T e^{-at+\varepsilon t} dt \\ &\quad + L_0 \int_0^T e^{\varepsilon t - a(t+s)} \int_0^{t+s} e^{au} E \left\| X_u \right\|_{C_\theta}^p du\, dt \\ &\quad + \int_0^T e^{-at+\varepsilon t} \int_0^{t+s} e^{au} \left(L_1 \xi_1(u) + L_2 \xi_2(u) \right) du\, dt. \end{aligned} \tag{33}$$

On the other hand, for $-r \leq s \leq 0$ and $t > 2r$,

$$\begin{aligned} &\int_0^T e^{\varepsilon t - a(t+s)} \int_0^{t+s} e^{au} E \left\| X_u \right\|_{C_\theta}^p du\, dt \\ &\leq \frac{1}{a-\varepsilon} \int_s^0 e^{a(u-s)} E \left\| X_u \right\|_{C_\theta}^p du \\ &\quad + \frac{1}{a-\varepsilon} \int_0^T e^{a(u-s)} \cdot e^{-(a-\varepsilon)(u-s)} E \left\| X_u \right\|_{C_\theta}^p du \\ &= \frac{1}{a-\varepsilon} \int_s^0 e^{a(u-s)} E \left\| X_u \right\|_{C_\theta}^p du \\ &\quad + \frac{1}{a-\varepsilon} \int_0^T e^{\varepsilon(u-s)} E \left\| X_u \right\|_{C_\theta}^p du, \end{aligned} \tag{34}$$

and therefore,

$$\begin{aligned} &\int_0^T e^{\varepsilon t} E \left\| X(t+s) \right\|_\theta^p dt \\ &\leq M_0 \int_0^T e^{-at+\varepsilon t} dt + \frac{L_0 e^{ar}}{a-\varepsilon} \int_s^0 e^{au} E \left\| X_u \right\|_{C_\theta}^p du \\ &\quad + \frac{L_0 e^{\varepsilon r}}{a-\varepsilon} \int_0^T e^{\varepsilon t} E \left\| X_u \right\|_{C_\theta}^p dt \\ &\quad + \int_0^T e^{-at+\varepsilon t} \int_0^{t+s} e^{au} \left(L_1 \xi_1(u) + L_2 \xi_2(u) \right) du\, dt, \end{aligned} \tag{35}$$

which, by virtue of the continuity of $X(t)$, $t \geq 0$, immediately implies

$$\begin{aligned} &\int_0^T e^{\varepsilon t} E \left\| X_t \right\|_{C_\theta}^p dt \\ &\leq M_0 \int_0^T e^{-at+\varepsilon t} dt + \frac{L_0 e^{ar}}{a-\varepsilon} \int_s^0 e^{au} E \left\| X_u \right\|_{C_\theta}^p du \\ &\quad + \frac{L_0 e^{\varepsilon r}}{a-\varepsilon} \int_0^T e^{\varepsilon t} E \left\| X_u \right\|_{C_\theta}^p dt \\ &\quad + \int_0^T e^{-at+\varepsilon t} \int_0^{t+s} e^{au} \left(L_1 \xi_1(u) + L_2 \xi_2(u) \right) du\, dt. \end{aligned} \tag{36}$$

On the other hand, since $a < L_0$ by assumption, it is possible to choose a suitable $\varepsilon \in R_+$ with $0 < \varepsilon < a - L_0$ small enough so that $L_0 e^{\varepsilon r}/(a-\varepsilon) < 1$, which, letting $T > 0$ tend to infinity and using (36), immediately yields

$$\begin{aligned} &\int_0^\infty e^{\varepsilon t} E \left\| X_t \right\|_{C_\theta}^p dt \\ &\leq \frac{1}{1 - L_0 e^{\varepsilon r}/(a-\varepsilon)} \left(M_0 \int_0^\infty e^{-at+\varepsilon t} dt \right. \\ &\quad + \frac{L_0 e^{ar}}{a-\varepsilon} \int_{-r}^0 e^{au} E \left\| X_u \right\|_{C_\theta}^p du \\ &\quad \left. + \int_0^T e^{-at+\varepsilon t} \int_0^{t+s} e^{au} \left(L_1 \xi_1(u) + L_2 \xi_2(u) \right) du\, dt \right) \\ &:= K'(p,\varepsilon,\phi) < \infty. \end{aligned} \tag{37}$$

So we can easily deduce (note $0 < \varepsilon < a - L_0$)

$$\begin{aligned} E \left\| X_t \right\|_{C_\theta}^p &\leq M_0 \cdot e^{-\varepsilon t} + M_1(\varepsilon) \cdot e^{-\varepsilon t} + L_0 \\ &\quad \cdot e^{\varepsilon r} e^{-\varepsilon t} \int_0^\infty e^{\varepsilon u} E \left\| X_u \right\|_{C_\theta}^p du \\ &\leq \left(M_0 + M_1(\varepsilon) + L_0 \cdot e^{\varepsilon r} K'(p,\varepsilon,\phi) \right) e^{-\varepsilon t} \\ &:= K(p,\varepsilon,\phi) e^{-\varepsilon t}, \end{aligned} \tag{38}$$

where

$$M_1(\varepsilon) = \int_0^\infty e^{\varepsilon t} \left(L_1 \xi_1(u) + L_2 \xi_2(u) \right) du < \infty. \tag{39}$$

So

$$E \left\| X_t \right\|_{C_\theta}^p \leq K(p,\varepsilon,\phi) e^{-\varepsilon t}. \tag{40}$$

The proof is complete. □

Additional Points

Summary. In this paper we discuss a new kind of stochastic neutral functional differential equations in abstract spaces. The existence and exponential stability of solutions were

obtained employing the semigroup approach and Banach fixed point theorem.

The main concerns of this paper are the mild solutions and their stability. To this end, system (2) was rewritten as (4) by the properties of semigroup. We structure the mapping $(\Phi Z)(t)$, which was proved to be a contract mapping via the generalized Itô formula, semigroup properties, mathematical induction, and stochastic analysis theory under Assumptions A, B, and C. The fixed point of Φ was just the mild solution of (4).

Under the linear growth conditions the pth moment exponential stability was obtained by Hölder inequality, moment inequality, Γ function, and mathematical induction.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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