Research Article

$L^p$ Estimates for Weak Solutions to Nonlinear Degenerate Parabolic Systems

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This paper is devoted to the $L^p$ estimates for weak solutions to nonlinear degenerate parabolic systems related to Hörmander’s vector fields. The reverse Hölder inequalities for degenerate parabolic system under the controllable growth conditions and natural growth conditions are established, respectively, and an important multiplicative inequality is proved; finally, we obtain the $L^p$ estimates for the weak solutions by combining the results of Gianazza and the Caccioppoli inequality.

1. Introduction

The $L^p$ estimates of the weak solutions to elliptic and parabolic systems under the controllable growth conditions and natural growth conditions in the Euclidean space $\mathbb{R}^n$ were well studied (see [1–3]). Roughly speaking, if the weak solution is $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ with $\Omega \subset \mathbb{R}^n$ being an open set, then there exists an index $p > 2$ such that $u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N)$. This result plays an important role in the study of the Hölder partial regularities. However, the index $p$ cannot be arbitrarily large. Folland established the local $L^p$ regularity and local Schauder estimates of the sub-Laplace operator structured on the vector fields of homogeneous group (satisfying Hörmander’s rank condition) in [4]. Rothschild and Stein [5] proved the local $L^p$ estimates of the square sum operators structured on the smooth Hörmander’s vector fields, through establishing the lifting and approximating theory of Hörmander’s vector fields and the local theory of the second-order invariant differential operator on the homogeneous group. Then, the estimates of weak solutions to the linear and nonlinear subelliptic equations and degenerate parabolic equations stemmed from the noncommutative vector fields that caused extensive concern; see [6–13]. In particular, Gianazza obtained in [11] the $L^p$ estimates of the derivatives of the weak solutions to nonlinear diagonal equations related to the Hörmander vector fields. Recently, the $L^p$ estimates of weak solutions to nonlinear subelliptic systems related to Hörmander’s vector fields under the superquadratic controllable growth condition and natural growth condition were studied in [14] and optimal partial regularity for subelliptic systems related to Hörmander’s vector fields was obtained in [15]. There are also some papers related on regularity for weak solutions to the quasilinear degenerate elliptic systems and degenerate elliptic systems with VMO coefficients were studied in [16, 17]. Those works motivated us to study the interesting question: can we get the $L^p$ estimates of weak solutions to the nonlinear degenerate parabolic systems structured on Hörmander’s vector fields?

Let $\Omega \subset \mathbb{R}^n$ be a bounded and open set. We consider the nonlinear degenerate parabolic systems related to Hörmander’s condition

$$\begin{align*}
\partial_t u^\alpha - \sum_{i=1}^k X^*_i A^*_i (t, \xi, u, Xu) &= B^\alpha (t, \xi, u, Xu), \\
\alpha &= 1, \ldots, N,
\end{align*}$$

(1)

estimates of the derivatives of the weak solutions to nonlinear diagonal equations related to the Hörmander vector fields.
under controllable growth conditions and natural growth conditions, where the controllable growth conditions are
\[ A^a_1(t, \xi, u, P) P^a_1 \geq \lambda |P|^2 - \Lambda |u|^r - f^2, \quad f \in L^s(\Omega), \]
\[ |A^a_1(t, \xi, u, P)| \leq \Lambda (|P| + |u|^{r/2} + f^2), \quad 0 \leq f^a_1 \in L^{s_1}(\Omega), \]
\[ |B^a(t, \xi, u, P)| \leq \Lambda (|P|^{2(1-1/r)} + |u|^{s-1} + f^2), \quad 0 \leq f^a \in L^{s_1}(\Omega), \]
with \( s_1 > 2, s_2 > 2, s_3 > r/(r-1), \) and \( r = 2^* = (2Q+2)/Q \) being a Sobolev critical exponent; and the natural growth conditions are
\[ A^a_1(t, \xi, u, P) P^a_1 \geq |P|^2 - f^2, \quad f \in L^s(\Omega), \]
\[ |A^a_1(t, \xi, u, P)| \leq \Lambda |P| + f^{a_1}, \quad 0 \leq f^a_1 \in L^{s_1}(\Omega), \]
\[ |B^a(t, \xi, u, P)| \leq |P|^2 + f^a, \quad 0 \leq f^a \in L^{s_1}(\Omega), \]
with \( s_1 > 1. \)

This paper is organized as follows: in Section 2, we introduce some notation, definitions, and basic facts. In Section 3, we state our main results. Section 4 is devoted to some important lemmas, including the Sobolev embedding theorem, the Poincaré inequalities, and reverse Hölder’s inequalities for the parabolic case. In particular, we establish an important multiplicative inequality by using the Fefferman-Phong inequality. We prove the main results in Section 5.

2. Preliminary

Let \( \Omega \subset \mathbb{R}^n \) be a bounded, open, and path-connected set and \( \{X_i\}_{i=1}^k \) a family of \( C^\infty \) real-valued vector fields defined in a neighborhood of the closure \( \overline{\Omega} \) of \( \Omega \). For a multi-index \( \alpha = (i_1, \ldots, i_k) \), denote by \( X_\alpha \) the commutator
\[ [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}]]], \]
of length \( l = |\alpha| \). Throughout this paper, we assume that the vector fields \( X = \{X_1, \ldots, X_k\} \) satisfy Hömander’s condition: there exists some positive integer \( s \) such that \( [X_\alpha]_{\alpha \leq s} \) span the tangent space of \( \mathbb{R}^n \) at each point of \( \Omega \); that is,
\[ \text{rank } \text{Lie} [X_1, \ldots, X_k] = n. \]

Assume that an admissible path \( \gamma \) is a Lipschitz curve \( \gamma : [a, b] \to \Omega \) such that there exist \( c_i(t), a \leq t \leq b \) satisfying
\[ \sum_{i=1}^k c_i(t)^2 \leq 1 \]
and \( \gamma(t) = \sum_{i=1}^k c_i(t)X_i(\gamma(t)). \) Then we define a metric associated to \( \{X_1, \ldots, X_k\} \) on \( \Omega \)
\[ q_{cc}(\xi, \eta) = \min \{b \geq 0 : \exists \gamma : [0, b] \to \Omega, \text{s.t. } \gamma(0) = \xi, \gamma(b) = \eta \}. \]
We refer to it as Carnot-Carathéodory metric, and we call it C-C metric for short. It is shown in [18] that \( q_{cc} \) is a distance.

We can define the C-C ball and the C-C sphere as
\[ B_R(\xi) = \{ \eta : q_{cc}(\xi, \eta) < R \}, \]
\[ \partial B_R(\xi) = \{ \eta : q_{cc}(\xi, \eta) = R \}, \]
respectively, from (8). In [18], the authors proved that the Lebesgue measure satisfies the doubling property related to the C-C ball: given a bounded set \( \Omega \subset \mathbb{R}^n \), there are positive constants \( C_1 \) and \( R_0 > 0 \) such that
\[ |B_{2R}(\xi)| \leq C_1 |B_R(\xi)|, \]
for \( \xi \in \Omega \) and \( 0 < R < R_0 \). Let \( Q = \log_2 C_1 \); the number \( Q \) acts as a dimension and is called the local homogeneous dimension related to \( \Omega \) and the system \( \{X_i\}_{i=1}^k \).

We use \( (\xi, t) \) to denote a point in \( \Omega \times \mathbb{R} \) and \( \xi \in \Omega \subset \mathbb{R}^n \). We define the parabolic Carnot-Carathéodory distance between the two points in \( \Omega \times \mathbb{R} \) as
\[ d_p((\xi, t), (\xi', t')) = \sqrt{q_{cc}(\xi, \xi')^2 + |t - t'|}. \]

We use the notation \( Q_{cc}(\xi, t) = B_R(\xi) \times (t - R^2, t) \) for parabolic cylinders in \( \Omega \times \mathbb{R} \), where \( B_R(\xi) \) is defined in (9).

For an open set \( \Omega \subset \mathbb{R}^n \), we set \( \Omega_p(\xi) = B_R(\xi) \cap \Omega \). For an open set \( U \subset \mathbb{R}^{n+1} \), we let \( U_p(\xi, t) = U \cap \Omega_p(\xi, t) \). We write \( U(t_0) \) for the set of all points \( (\xi, t_0) \) in \( U \) and \( I(U) \) for the set of all \( t \) such that \( U(t) \) is nonempty. We also write \( U = U^2 \) for the cylinder \( (S, T) \times \Omega \), where \( -\infty < S < T < \infty \) and \( U_T \) for the set for \( U \) when \( S = 0 \). We denote by \( W^{1,2}_0(U) \) the Hilbert space with the inner product
\[ (u, v)_{W^{1,2}_0(U)} = \int_U u \nu + \sum_{k=1}^d X_k u X_k v. \]

Let \( W^{1,2}_0(U) \) be subspace of \( W^{2,2}_0(U) \) such that \( \partial_u u \in L^2(\Omega) \) and \( V(U) \) are to be the Banach space consisting of all elements of \( W^{1,2}_0(U) \) having a finite norm \( \|u\|_{V(U)} = \|u\|_U \), with \( \|u\|_U^2 = \|Xu\|_{L^2(U)}^2 + \sup_{t \in \Omega_p(\xi, t)} \|u(\cdot, t)\|_{L^2(U)}^2 \). Set \( V_{0,2}(U) \) to be the set of all functions \( u \) in \( V(U) \) vanishes on the lateral boundary \( \partial_p U = (S, T) \times \Omega \) of \( U \).

From the well-known Sobolev-like embedding theorem (see [19], II.3), we have
\[ \|u\|_{L^q(\Omega)} \leq N(\Omega) \|u\|_U, \quad \forall u \in V_{0,2}(U), \quad r = \frac{2(Q+2)}{Q}. \]

We denote \( H_p^{-1}(U) \) to be the space consisting of all functions \( u \) satisfying
\[ \inf \left\{ \|F\|_{L^p(U)}^p + \|h\|_{L^p(U)}^p \mid u = \text{div}_X F + h \right\} < \infty, \]
with \( \text{div}_X F = \sum_{i=1}^k X_i F_i \). It is easy to see that \( H_p^{-1}(U) \) is a Banach space. For any \( u \in H_p^{-1}(U) \), we can define the norm
\[ \|u\|_{H_p^{-1}(U)} = \inf \left\{ \|F\|_{L^p(U)}^p + \|h\|_{L^p(U)}^p \mid u = \text{div}_X F + h \right\} , \]
and define $\mathcal{H}^1_p(U) = \{u: u, Xu \in L_p(U), \partial_\alpha u \in H^1_p(U)\}$. Note that $\mathcal{H}^1_2(U) \subset V_2(U)$.

### 3. Main Results

We call $u \in V_{0,2}(U_T)$ the weak solution of the systems (1) if

$$
\int_\Omega u^\alpha(\xi, t) \varphi^\alpha(\xi, t) d\xi + \int_\Omega [-u^\alpha \partial_\alpha \varphi^\alpha + A^\alpha_i(t, \xi, u, Xu) X_i \varphi^\alpha] d\xi dt
\leq C \int_\Omega [u^\alpha]^q d\xi dt + C \int_\Omega [f^\alpha]^q d\xi dt + C \int_\Omega [\int_\Omega |u|^2 d\xi]^Q d\xi dt
$$

for any test functions $\varphi \in W^{1,2}_0(U_T)$ and $t \in [0, T]$. The main results in this paper are as follows.

**Theorem 1.** (A) Assume that $A^\alpha_i$, $B^\alpha$ satisfy (2), $u \in V_{0,2}(U_T)$ is a weak solution to systems (1), and $f \in L^2(U_T)$, $f^\alpha \in L^2(U_T)$, and $f^\alpha \in L^r_{r/(r-1)}(U_T)$. Then for any $(\xi_0, t_0) \in \mathbb{R}^{n+1}$ and $0 < R \leq 1$ with $B_{R/2}(\xi_0) \subset B_R(\xi_0) \subset \Omega$ and $t_0 \geq R^2$, one has

$$
\int_{Q_{R/2}(\xi_0, t_0)} |Xu|^2 d\xi dt \leq C \int_{Q_{R/2}(\xi_0, t_0)} [f^\alpha]^q d\xi dt + C \int_{Q_{R/2}(\xi_0, t_0)} [\int_\Omega |u|^2 d\xi]^Q d\xi dt
$$

for any weak solution of (I), $\sup_{Q((\xi, t)-R^2,1)} |u| = M, 2\Lambda M < \lambda$ (where $\lambda$ is the same as in (3)), and $f \in L^2(U_T)$, $f^\alpha \in L^2(U_T)$, $f^\alpha \in L^r_{r/(r-1)}(U_T)$. Then for any $(\xi_0, t_0) \in \mathbb{R}^{n+1}$ and $0 < R \leq 1$ with $B_{R/2}(\xi_0) \subset B_R(\xi_0) \subset \Omega$ and $t_0 \geq R^2$, one has

$$
\int_{Q_{R/2}(\xi_0, t_0)} |Xu|^2 d\xi dt \leq C \int_{Q_{R/2}(\xi_0, t_0)} [f^\alpha]^q d\xi dt + C \int_{Q_{R/2}(\xi_0, t_0)} [\int_\Omega |u|^2 d\xi]^Q d\xi dt
$$

for any weak solution of (I), $\sup_{Q((\xi, t)-R^2,1)} |u| = M, 2\Lambda M < \lambda$, and $f, f^\alpha \in L^2(U_T)$, $f^\alpha \in L^r(U_T)$, $\sigma \in (2, \infty)$, and $f^\alpha \in L^r_{r/(r-1)}(U_T)$, $r \in (r/(r-1), \infty)$. Then there exists $p > 2$ such that $u \in \mathcal{H}^1_p(U_T)$. Moreover, the inequality

$$
\|Xu\|_{L^r((\xi, T)\times \Omega)} \leq C,
$$

is valid for any $0 < \epsilon < T$, where $C = C(n, \lambda, \Lambda, \sigma, \tau, u, \|f\|_{L^r(U_T)}, \|f^\alpha\|_{L^r(U_T)}, \|f^\alpha\|_{L^r(U_T)}, \epsilon, T, \Omega, M)$, and $p = p(n, \lambda, \Lambda, \sigma, \tau, u, \|f\|_{L^r(U_T)}, \|f^\alpha\|_{L^r(U_T)}, \|f^\alpha\|_{L^r(U_T)}, \|f^\alpha\|_{L^r(U_T)}, \|f^\alpha\|_{L^r(U_T)}, M)$.

### 4. Some Lemmas

Lu established Pioncaré’s inequality in [20]; we can similarly derive the following lemma by replacing the C-C ball $B_R$ with the parabolic cylinder $Q_R$.

**Lemma 3.** There exist $C$ and $R$ such that for any $u \in W^{0,1}_q(\Omega, \xi_0, t_0)$, $1 < q < Q$, $1 \leq p \leq qQ/(Q - q)$,

$$
\left( \int_{Q_{R/2}(\xi_0, t_0)} |u - \overline{u}(t)|^p d\xi dt \right)^{1/p} \leq CR \left( \int_{Q_{R/2}(\xi_0, t_0)} |Xu|^q d\xi dt \right)^{1/q},
$$

where $\overline{u}(t) = (1/C_1) \int_{Q_{R/2}(\xi_0, t_0)} u^n \eta^q d\xi dt$; $\eta$ is the cut-off function related to $\xi$. 

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From (22), we can get the following Sobolev-Poincaré type inequality immediately.

**Corollary 4.** Let \( u \in W^{1,q}(\Omega_\beta(t_0), 1 < q < Q, \) then \( u \in L^p(\Omega_\beta(t_0), t_0), p = Qq/(Q - q), \) and one has

\[
\left( \int_{\Omega_\beta(t_0)} |u - \pi(t)|^p \, d\xi \, dt \right)^{1/p} \leq C \left( \int_{\Omega_\beta(t_0)} |Xu|^q \, d\xi \, dt \right)^{1/q}.
\]

(23)

Next, we recall Fefferman-Phong's inequality (see [21]) to get the multiplicative inequality which plays an important role in this paper.

**Theorem 5** (Fefferman-Phong's inequality). Let \( \Omega \subset \mathbb{R}^n \) be an open set and \( Q \) a homogeneous dimension. Set \( R_0 > 0; \) suppose that \( 1 < p < Q, 1 < s \leq Q/p, \beta \geq 1, \) and \( V \in L^{p,q}(\Omega, \beta) \), where \( B = B_{R_0}(\xi_0), \xi_0 \in \Omega, 0 < R < R_0, \) and \( \beta \beta \subset \Omega. \) Then there exists \( C = C(\Omega, X) > 0 \) such that for any \( u \in L^p(\beta B) \)

\[
\int_B |u - u_B|^p V \, d\xi \leq C \|V\|_{L^{Q/p}(\beta B)} \left( \int_{\beta B} |Xu|^q \, d\xi \right)^{Q/(Q + p)}.
\]

(24)

where \( u_B = (1/|B|) \int_B u \, d\xi \) and \( u \in L^{p,q}(\Omega) (\mu > 0) \) implies \( u \in L^{p,q}_{\text{loc}}(\Omega) \) and

\[
\sup_{\xi \in \Omega, 0 < r < \dim \Omega} \left( \frac{|B_r(\xi) \cap \Omega|}{|B_r(\xi)|} \right)^{1/p} \left( \int_{B_{r}(\xi) \cap \Omega} |u|^p \, d\xi \right)^{1/p} < \infty.
\]

(25)

**Lemma 6** (multiplicative inequality for \( p \)). Let \( \Omega \subset \mathbb{R}^n \) be an open set and \( Q \) a homogeneous dimension. Set \( R_0 > 0, 1 < p < Q, \beta \geq 1, \xi_0 \in \Omega, 0 < R < R_0, \) and \( \beta \beta \subset \Omega, \) where \( B = B_{R_0}(\xi_0). \) Then there exists \( C = C(\Omega, X) > 0 \) such that for any \( u \in W^{1,p}(\beta B) \)

\[
\|u\|_{L^p(\beta B)} \leq C \|u\|_{L^{Q,p}(\beta B)} \left( \int_{\beta B} |Xu|_{L^q(\beta B)} \right)^{Q/(Q + p)}
\]

(26)

where \( r = p(Q + p)/Q. \)

**Proof.** Since \( u \in W^{1,p}(\beta B), \) we have \( u \in L^{Q,p}(\Omega, \beta) \) by Sobolev's embedding theorem; then

\[
\int_B |u - u_B|^p V \, d\xi = \int_B |u - u_B|^p |u - u_B|^p \, d\xi.
\]

(27)

Let \( |u - u_B|^p V = V; \) then \( V \in L^{(Q/p)(Q - p)/(1/p)}(\beta B). \) Let \( s = Q/p; \) one has \( V \in L^{(Q/p)(Q - p)/(Q - p)}(\beta B) \subset L^{2Q/p}(\beta B) = L^{Q/2}(\beta B). \)

By Theorem 5, we obtain

\[
\int_B |u - u_B|^p V \, d\xi \leq C \|V\|_{L^{Q/p}(\beta B)} \left( \int_{\beta B} |Xu|^q \, d\xi \right)^{Q/(Q + p)}.
\]

(28)

\[
\|u - u_B\|_{L^p(\beta B)} \leq C \left( \|u - u_B\|_{L^{Q,p}(\beta B)} \left( \int_{\beta B} |Xu|_{L^q(\beta B)} \right)^{Q/(Q + p)} \right).
\]

(29)

Therefore

\[
\|u - u_B\|_{L^p(\beta B)} \leq C \left( \|u - u_B\|_{L^{Q,p}(\beta B)} \left( \int_{\beta B} |Xu|_{L^q(\beta B)} \right)^{Q/(Q + p)} \right).
\]

(30)

When \( p = 2, \) we get the multiplicative inequality that we will use in this paper.

**Lemma 7.** For any \( R > 0 \) and \( u \in V_{Q,2}(Q_0), \) it holds that

\[
\|u\|_{L^2(\beta B)} \leq C \|d\|_{L^2(\beta B)} \left( \left( \int_{\beta B} |Xu|_{L^2(\beta B)} \right)^{Q/(Q + 2)} \right).
\]

(31)

Gianazza [11] showed a reverse Hölder inequality on homogeneous spaces. Since the ball \( B_{R_{0}}(\xi_{0}) \) which is induced...
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Lemma 8. Let \( g \) and \( h \) be nonnegative functions and satisfy the following.

1. \( g \in L_\gamma(Q), \gamma > 1; h \in L_\beta(Q), \beta > \gamma. \)

2. Assume that \( \forall (\xi, t_0), \xi \in Q, \xi_0(\xi, t_0) \subset Q, \) and \( R > 0 < R < (1/2) \text{dist} ((\xi_0, t_0), \partial Q) \), one has

\[
\int_{Q_0(\xi_0, t_0)} g^\gamma d\xi dt \leq C_0 \left[ \int_{Q_0(\xi_0, t_0)} g d\xi dt \right]^\gamma + \int_{Q_0(\xi_0, t_0)} h^\gamma d\xi dt + \theta \int_{Q_0(\xi_0, t_0)} g^\gamma d\xi dt,
\]

where the constants \( R_0, \theta, \) and \( C_0 \) are positive, \( \theta \in (0, 1) \). Then there exist \( \epsilon > 0 \) and \( C > 0 \), such that \( g \in L_\gamma^1(Q), \forall p \in [\gamma, \gamma + \epsilon] \). Moreover, for \( Q_R(\xi_0, t_0) \subset Q, R < R_0 \)

\[
\left[ \int_{Q_0(\xi_0, t_0)} g^\gamma d\xi dt \right]^{1/p} \leq C \left[ \int_{Q_0(\xi_0, t_0)} g^\gamma d\xi dt \right]^{1/p} + \int_{Q_0(\xi_0, t_0)} h^\gamma d\xi dt,
\]

is valid, where \( C \) and \( \epsilon \) depend on \( C_0, \theta, \gamma, \beta, K(\geq 1) \) and \( n \).

We will use the following cut-off function. Let \( \eta(\xi) \in C_0^\infty(B_R(\xi_0)), \eta(t) \in C_0^\infty(-1, 1) \) be cut-off functions and satisfy \( 0 \leq \eta \leq 1 \) (on \( B_R(\xi_0) \)), \( \eta \equiv 1 \) (on \( B_{R/2}(\xi_0) \)), and \( \xi \equiv 1 \) (on \( (-1/2, 1/2) \)). Note that \( \int_{B_R(\xi_0)} \eta^4 d\xi = C_1 |B_R(\xi_0)|, \) where \( C_1 \) is independent of \( R \).

Lemma 9 (Caccioppoli inequality). Let \( u \in V_{2,2}(U_T) \) be the weak solutions of systems (1) and \( f \in L_2(\partial U_T), f_{\partial T} \in L_4(U_T), \) and \( f^n \in L_{r(t-1)}(U_T) \). Then for any \( \xi_0 \in \mathbb{R}^2, 0 < R < R_0, \) and \( R^2 \leq t_0 \leq T \)

\[
\sup_{t \in (0, R^2)} \int_{\Omega_R(\xi_0)} u^2 (\xi \cdot t) \eta^4 \zeta^2 d\xi + \lambda \int_{\Omega_R(\xi_0)} |Xu|^2 \eta^4 \zeta^2 \leq CR^{-2} \int_{U_R(\xi_0, t_0)} |u|^2 \eta^4 \zeta + C \int_{U_R(\xi_0, t_0)} |u|^r + C \int_{U_R(\xi_0, t_0)} (f^2 + |f_{\eta}^1|^2 + |f^n|^{(t-1)}) ,
\]

where \( C = C(n, \lambda, \Lambda) \).

Proof. Take the test function \( \phi = u(\xi, t) \eta^4 \zeta^2 \). Substituting \( \phi \) in (17), we have

\[
\frac{1}{2} \int_{\Omega_0(\xi_0)} u^2 \eta^4 \zeta^2 d\xi + \int_{t_0-R^2}^t \int_{\Omega_0(\xi_0)} A^1 (t, \xi, u, Xu) X_\gamma^1 (u^4 \zeta^2) d\xi dt = \int_{t_0-R^2}^t \int_{\Omega_0(\xi_0)} B^n (t, \xi, u, Xu) u^n \eta^4 \zeta^2 d\xi dt + \int_{t_0-R^2}^t \int_{\Omega_0(\xi_0)} u^2 \eta^4 \zeta^2 d\xi dt.
\]

By using the controllable growth conditions (2)

\[
\frac{1}{2} \sup_{t \in (t_0-R^2)} \int_{\Omega_0(\xi_0)} u^2 \eta^4 \zeta^2 d\xi + \frac{\lambda}{2} \int_{Q_0(\xi_0, t_0)} |Xu|^2 \eta^4 \zeta^2 d\xi dt 
\leq \frac{\Lambda}{2} \int_{U_R(\xi_0, t_0)} |u|^r \eta^4 \zeta^2 d\xi dt + \frac{1}{2} \int_{U_R(\xi_0, t_0)} f^2 \eta^4 \zeta^2 d\xi dt + 4A \int_{U_R(\xi_0, t_0)} |Xu| |u|^3 |X \eta| \zeta^2 d\xi dt + 4A \int_{U_R(\xi_0, t_0)} |u|^r/2 |u|^3 |X \eta| \zeta^2 d\xi dt + 4A \int_{U_R(\xi_0, t_0)} |f^n| |u|^3 |X \eta| \zeta^2 d\xi dt + \int_{U_R(\xi_0, t_0)} B^n (t, \xi, u, Xu) |u|^n \eta^4 \zeta^2 d\xi dt + \int_{U_R(\xi_0, t_0)} u^2 \eta^4 \zeta^2 d\xi dt = G_1 + G_2 + G_3 + G_4 + G_5 + G_6 + G_7.
\]

We estimate the following terms by Young's inequality:

\[
G_3 = 4A \int_{U_R(\xi_0, t_0)} |Xu| |u|^3 |X \eta| \zeta^2 
\leq \frac{\lambda}{16} \int_{U_R(\xi_0, t_0)} |Xu|^2 \eta^4 \zeta^2 + C \int_{U_R(\xi_0, t_0)} |u|^2 |X \eta| \eta^4 \zeta^2
\]
Under the same assumptions as in Lemma 9, we have

\begin{align*}
\frac{\lambda}{16} \int_{U(x, \xi_0, t)} |Xu|^2 \eta^4 \xi^2 \\
+ CR^{-2} \int_{U(x, \xi_0, t)} |u|^2 \eta^2 \xi^2,
\end{align*}

\[G_4 = 4A \int_{U(x, \xi_0, t)} |u|^\gamma |u| \eta^3 |X\eta| \xi^2\]
\[\leq \frac{\lambda}{16} \int_{U(x, \xi_0, t)} |u|^\gamma \eta^4 \xi^2 + CR^{-2} \int_{U(x, \xi_0, t)} |u|^2 \eta^2 \xi^2,
\]

\[G_5 = 4A \int_{U(x, \xi_0, t)} f^\alpha |u| \eta^3 |X\eta| \xi^2\]
\[\leq \frac{\lambda}{16} \int_{U(x, \xi_0, t)} |f|^\alpha |u| \eta^4 \xi^2 + CR^{-2} \int_{U(x, \xi_0, t)} |u|^2 \eta^2 \xi^2,
\]

\[G_6 = \int_{U(x, \xi_0, t)} B^\alpha (t, \xi, u, Xu) |u| \eta^4 \xi^2\]
\[\leq C \int_{U(x, \xi_0, t)} |Xu|^{2(1-\gamma/r)} |u| \eta^4 \xi^2\]
\[+ C \int_{U(x, \xi_0, t)} |u|^{-\gamma} |u| \eta^4 \xi^2\]
\[+ C \int_{Q_{2}(x, \xi_0, t)} |f^\alpha| |u| \eta^4 \xi^2\]
\[\leq \frac{\lambda}{16} \int_{U(x, \xi_0, t)} |Xu|^2 \eta^4 \xi^2 + C \int_{U(x, \xi_0, t)} |u|^\gamma\]
\[+ C \int_{Q_{2}(x, \xi_0, t)} |f^\alpha|^{\gamma/(\gamma-1)}.
\]

Combining the above estimations and \( \xi^2 \leq \xi \), then we get (34).

\textbf{Corollary 10.} Under the same assumptions as in Lemma 9, we have

\[\sup_{t \in (t_0 - R^2/4, t_0)} \int_{U(x_0, \xi_0, t)} u^2 d\xi \to 0,\]

as \( R \to 0 \), uniformly in \( \xi_0 \) and \( t_0 \).

\textbf{Proof.} By Hölder’s inequality

\[R^{-2} \int_{U(x, \xi_0, t)} |u|^2 \leq C \left( \int_{U(x, \xi_0, t)} |u|^r \right)^{2/r}.\]

From this inequality and (34) we have

\[\sup_{t \in (t_0 - R^2/4, t_0)} \int_{U(x, \xi_0, t)} u^2 d\xi\]
\[\leq C \int_{U(x, \xi_0, t)} \left( |u|^r + f^\alpha + |f^\alpha|^{r/(r-1)} \right)^{2/r}.
\]

Then \( f, f^\alpha, f^\alpha_t, u \), the assumption of \( g \), and (14), as well as the absolute continuity of Lebesgue, integrals imply (38). \( \square \)

\section{The Proof of the Main Results}

\subsection{5.1. The Proof of Theorem 1.}

Take the test function \( \varphi = \eta(\xi)^4 \xi^4 (t - \bar{u}(t)) \), where

\[\bar{u}(t) = \left( \int_{B(x, \xi_0)} \eta^4 d\xi \right)^{-1} \int_{B(x, \xi_0)} u \eta^4 d\xi\]
\[= \frac{1}{C_1} \int_{B(x, \xi_0)} u \eta^4 d\xi,
\]

and then

\[\int_{B(x, \xi_0)} \bar{u}(t) \eta^4 d\xi = C_1 \int_{B(x, \xi_0)} \frac{1}{C_1} \int_{B(x, \xi_0)} u \eta^4 d\xi\]
\[= \int_{B(x, \xi_0)} u \eta^4 d\xi,
\]

so we have

\[\int_{B(x, \xi_0)} \partial_\xi \bar{u}(t) (u - \bar{u}(t)) \eta^4 \xi^2 d\xi\]
\[= \partial_\xi \bar{u}(t) \xi^2 \int_{B(x, \xi_0)} (u - \bar{u}(t)) \eta^4 d\xi\]
\[= \partial_\xi \bar{u}(t) \xi^2 \int_{B(x, \xi_0)} (u \eta^4 - \bar{u}(t) \eta^4) d\xi = 0,
\]

and then

\[\int_{B(x, \xi_0)} \partial_\xi u (u - \bar{u}(t)) \eta^4 \xi^2 d\xi\]
\[= \int_{B(x, \xi_0)} (\partial_\xi u - \partial_\xi \bar{u}(t)) (u - \bar{u}(t)) \eta^4 \xi^2 d\xi.
\]
Substituting $\varphi$ in (17), we have
\begin{align*}
\frac{1}{2} \int_{B_\xi(\xi_0)} (u - \Pi(t))^2 \eta^4 \zeta^2 \, d\xi \\
+ \int_{t_0 - R^2}^t \int_{B_\xi(\xi_0)} A^\alpha(t, \xi, u, Xu) \\
\cdot X_i \left( (u - \Pi(t)) \eta^4 \zeta^2 \right) \, d\xi \, dt \\
= \int_{t_0 - R^2}^t \int_{B_\xi(\xi_0)} B^\alpha(t, \xi, u, Xu) (u - \Pi(t)) \eta^4 \zeta^2 \, d\xi \, dt \\
+ \int_{t_0 - R^2}^t \int_{B_\xi(\xi_0)} (u - \Pi(t))^2 \eta^4 \zeta \partial \zeta \, d\xi \, dt.
\end{align*}
(45)

By the controllable growth conditions (2)
\begin{align*}
\frac{1}{4} \sup_{t \in (t_0 - R^2, t_0]} \int_{B_\xi(\xi_0)} (u - \Pi(t))^2 \eta^4 \zeta^2 \, d\xi \\
+ \frac{1}{2} \int_{Q_\xi(\xi_0, t_0)} |Xu|^2 \eta^4 \zeta^2 \, d\xi \, dt \\
\leq \frac{\lambda}{2} \int_{Q_\xi(\xi_0, t_0)} |u|^2 \eta^4 \zeta^2 \, d\xi \, dt \\
+ \frac{1}{2} \int_{Q_\xi(\xi_0, t_0)} f^2 \eta^4 \zeta^2 \, d\xi \, dt \\
+ 4A \int_{Q_\xi(\xi_0, t_0)} |Xu| |u - \Pi(t)| \eta \, |X| |\zeta| \, d\xi \, dt \\
+ 4A \int_{Q_\xi(\xi_0, t_0)} |u|^3/2 \eta |u - \Pi(t)| \eta^3 \, |X| \eta^2 \, d\xi \, dt \\
+ 4A \int_{Q_\xi(\xi_0, t_0)} |f| |u - \Pi(t)| \eta \, |X| \eta \, d\xi \, dt \\
+ \int_{Q_\xi(\xi_0, t_0)} B^\alpha(t, \xi, u, Xu) |u - \Pi(t)| \eta^4 \zeta^2 \, d\xi \, dt \\
+ \int_{Q_\xi(\xi_0, t_0)} (u - \Pi(t))^2 \eta^4 \zeta |\partial \zeta| \, d\xi \, dt = I_3 + I_4 \\
+ I_5 + I_6 + I_7.
\end{align*}
(46)

We estimate each term by using Young's inequality
\begin{align*}
I_3 &= 4A \int_{Q_\xi(\xi_0, t_0)} |Xu| |u - \Pi(t)| \eta \, |X| \eta \, \zeta^2 \\
&\leq \frac{\lambda}{16} \int_{Q_\xi(\xi_0, t_0)} |Xu|^2 \eta^4 \zeta^2 \\
+ C \int_{Q_\xi(\xi_0, t_0)} |u - \Pi(t)|^2 \eta^2 \zeta^2, \\
I_4 &= 4A \int_{Q_\xi(\xi_0, t_0)} |u|^{3/2} |u - \Pi(t)| \eta^3 \, |X| \eta^2 \, \zeta^2 \\
&\leq \frac{\lambda}{16} \int_{Q_\xi(\xi_0, t_0)} |u|^2 \eta^4 \zeta^2 \\
+ C \int_{Q_\xi(\xi_0, t_0)} |u - \Pi(t)|^2 \eta^2 \zeta, \\
I_5 &= 4A \int_{Q_\xi(\xi_0, t_0)} |f| |u - \Pi(t)| \eta \, |X| \eta \, \zeta^2 \\
&\leq \frac{\lambda}{16} \int_{Q_\xi(\xi_0, t_0)} |f| \eta \, |X| \eta \, \zeta^2 \\
+ C \int_{Q_\xi(\xi_0, t_0)} |u - \Pi(t)|^2 \eta^2 \zeta, \\
I_6 &= \int_{Q_\xi(\xi_0, t_0)} B^\alpha(t, \xi, u, Xu) |u - \Pi(t)| \eta^4 \zeta^2 \\
&\leq C \int_{Q_\xi(\xi_0, t_0)} |Xu|^{1-1/\gamma} |u - \Pi(t)| \eta^4 \zeta^2 \\
+ C \int_{Q_\xi(\xi_0, t_0)} |u|^{-1} |u - \Pi(t)| \eta^4 \zeta^2 \\
+ C \int_{Q_\xi(\xi_0, t_0)} |f| \eta |u - \Pi(t)| \eta^4 \zeta^2 \\
&\leq \frac{\lambda}{16} \int_{Q_\xi(\xi_0, t_0)} |Xu|^2 \eta^4 \zeta^2 + C \int_{Q_\xi(\xi_0, t_0)} |u - \Pi(t)|^r \\
+ C \int_{Q_\xi(\xi_0, t_0)} |u|^{r/(r-1)}, \\
\int_{Q_\xi(\xi_0, t_0)} (u - \Pi(t))^2 \eta^4 \zeta^2 \, d\xi \, dt \\
&\leq \frac{\lambda}{16} \int_{Q_\xi(\xi_0, t_0)} |Xu|^2 \eta^4 \zeta^2 \\
+ \lambda \int_{Q_\xi(\xi_0, t_0)} |Xu|^2 \eta^4 \zeta^2 \, d\xi \, dt.
\end{align*}
(47)
\[ \leq CR^{-2} \int_{Q_0(\xi_0, t_0)} |u - \bar{u}(t)|^2 \eta^2 \xi^2 + C \int_{Q_0(\xi_0, t_0)} (|u - \bar{u}(t)|^2 + |u'|^2) + C \int_{Q_0(\xi_0, t_0)} \left( \frac{\xi^2}{|u - \bar{u}(t)|^2} \right) \]
\[ = C_0 (J_1 + J_2 + J_3). \]  

Note that
\[ \int_{B_x(\xi_0)} |u - \bar{u}(t)|^2 \, d\xi = \int_{B_x(\xi_0)} \left| u(\xi, t) - u(\xi', t) \right| \, d\xi' \]
\[ \leq C_2 \int_{B_x(\xi_0)} \left| u(\xi, t) - u(\xi', t) \right| \, d\xi' \, d\xi. \]  

So we can use Young's inequality and Sobolev-Poincaré inequality (23) to estimate \( J_1 \):
\[ J_1 \leq CR^{-2} \left( R^2 \int_{Q_0(\xi_0, t_0)} |u - \bar{u}(t)|^2 \eta^4 \xi^2 \right) \]
\[ + C \int_{Q_0(\xi_0, t_0)} |u - \bar{u}(t)|^2 \eta^4 \xi^2 \]
\[ \leq \frac{\lambda}{16C_0} \]
\[ \leq C \left( R^{-q} \int_{Q_0(\xi_0, t_0)} |Xu|^q \right)^{2/q} \]
\[ = \left( \int_{B_x(\xi_0)} |Xu|^q \right)^{2/q} \]
\[ \leq C \left( \int_{B_x(\xi_0)} |Xu|^2 \right)^{2/2} \]
\[ = \int_{B_x(\xi_0)} |Xu|^2 \, d\xi \]
\[ = C_2 \int_{B_x(\xi_0)} \left| u(\xi, t) - u(\xi', t) \right| \, d\xi' \, d\xi. \]  

For \( J_2 \), by the triangle inequality
\[ J_2 \leq C \int_{Q_0(\xi_0, t_0)} |\bar{u}(t)|^{2/2} \]
\[ \leq C \int_{Q_0(\xi_0, t_0)} |\bar{u}(t) - c|^2 + C \int_{Q_0(\xi_0, t_0)} c' \]
\[ \leq C_2 \int_{Q_0(\xi_0, t_0)} |\bar{u}(t) - c|^2 + CR^2 |B_R(\xi_0)| c' \]
\[ = J_{221} + J_{222}. \]  

where
\[ c = \int_{t_0}^t \bar{u}(t) \, dt. \]  

By \( rq/2 > 1 \) and Hölder inequality, we have
\[ J_{222} = CR^2 |B_R(\xi_0)| \left( \int_{t_0}^t |\bar{u}(t)|^{r/2} \right)^r \]
\[ = CR^2 |B_R(\xi_0)| \left( \int_{t_0}^t \left( \frac{1}{C_1} \int_{B_x(\xi_0)} |u'|^2 \, d\xi \right)^{r/2} \right)^r \]
\[ \leq CR^2 |B_R(\xi_0)| \left( \int_{Q_0(\xi_0, t_0)} |u'|^{r/2} \right)^r. \]
To estimate $I_{221}$, we use the Poincaré inequality in $t$

$$I_{221} \leq CR^2 |B_R(\xi_0)| \left( \int_{|\xi_t| < R} |\partial_t \eta'| dt \right)^r$$

$$\leq CR^2 |B_R(\xi_0)| \left( \int_{|\xi_t| < R} |B_R(\xi_0)|^{-1} \right)$$

$$\cdot \left( \int_{|\xi_t| < R} |\partial_t \eta'| d\xi \right)^r = CR^2 |B_R(\xi_0)| K',$$

where $K$ is the control growth condition.

Integrating by parts for $K$, combining with the controllable growth conditions

$$K \leq CR^2 \left( \int_{Q_{\xi,\xi_0}} |A_i^\alpha | \eta^3 |X\eta| \right)$$

$$+ CR^2 \left( \int_{Q_{\xi,\xi_0}} |B^\alpha | \eta^4 \right)$$

$$\leq CR \left( \int_{Q_{\xi,\xi_0}} (|X\eta| + |u|^{r/2} + |\partial_t \eta'|) \right)$$

$$+ CR^2 \left( \int_{Q_{\xi,\xi_0}} (|X\eta|^{2(1-1/r)} + |u|^{-1} + |\partial_t \eta'|) \right)$$

$$= C(K_1 + K_2 + K_3 + K_4 + K_5 + K_6),$$

It remains to estimate each term by the Hölder inequality

$$K_1 = R \int_{Q_{\xi,\xi_0}} |X\eta| \leq R \left( \int_{Q_{\xi,\xi_0}} |X\eta|^{(2/r)(q/2)} \right)^{2/r}$$

$$\cdot \left( \int_{Q_{\xi,\xi_0}} |X\eta|^{(r-2)/2r} |\partial_t \eta'|^{2r/r-2} \right)^{r-2/2r}$$

$$= \left( \int_{Q_{\xi,\xi_0}} |X\eta|^q \right)^{2/r} \left( \int_{Q_{\xi,\xi_0}} |X\eta|^{2(r-2)/r} \right)^{r-2/2r},$$

$$K_2 = R \int_{Q_{\xi,\xi_0}} |u|^{r/2} \leq R \left( \int_{Q_{\xi,\xi_0}} |u|^{q/2} \right)^{2/r}$$

By $f, f^\alpha, f^\alpha_t, u$, the estimation of $g$, and (14), we get

$$\int_{Q_{\xi,\xi_0}} |X\eta|^2,$$

$$\int_{Q_{\xi,\xi_0}} |f|^2,$$
that are uniformly bounded. Combining all the above inequalities

\[
\lambda \int_{Q_0(\xi_0,t_0)} |Xu|^q \eta^2 \, d\xi \leq C \left( \int_{Q_0(\xi_0,t_0)} |Xu|^q + |u|^{q/2} \right)^{2/q} + C \left( \int_{Q_0(\xi_0,t_0)} \left( |f| \right)^2 + |f|^2 + |\nabla f|^2 \right) \right) + C \sup_{t \in (t_0 - R, t_0)} \left( \int_{B_R(\xi_0)} |u|^2 \, d\xi \right)^{2/3},
\]

where \( C = C(n, \lambda, \Lambda, \|u\|_{L^q}, \|f\|_{L^1}, \|f^a\|_{L^{(r-1)/r}}, \|f^a\|_{L^r} \). \( \text{(A)} \) is obtained by this estimation and the estimate of \( J_k \).

Next we will prove (B): firstly, we extend \( u \) to \( Q_T := (0, T) \times \mathbb{R}^n \) such that if \( \xi \in \mathbb{R}^n \setminus \Omega \), then \( u(\xi, t) = 0 \). It is obvious that \( u \in V_2(Q) \). Similarly, we extend \( f, f^a \), and \( f^a \) and let \( R < R_0/4 \).

Set \( (\xi_0, t_0) \) such that \( Q_{4R}((\xi_0, t_0)) \subset Q_T \). We consider the case \( B_{4R}((\xi_0, t_0)) \subset \Omega \). Following the results in (A)

\[
C \sup_{t \in (t_0 - R, t_0)} \left( \int_{B_R(\xi_0)} |u|^2 \, d\xi \right)^{2/3} < 1.
\]

For \( p \in (2, \min \{\sigma(2(r-1)/r) \} \) and \( Q_{4R}((\xi_0, t_0)) \subset Q_T \), we take \( \gamma = 2/q > 1 \) in Lemma 8, \( g = |Xu|^q + |u|^q/2, \) \( h = |f|^q + |f^a|^q + |F|^q \); then

\[
\int_{Q_{4R}((\xi_0, t_0))} \left( |Xu|^p + |u|^p/2 \right) \, d\xi dt \leq C \left[ \int_{Q_{4R}((\xi_0, t_0))} \left( |Xu|^q + |u|^q/2 \right) \, d\xi dt \right]^{p/2} + C \int_{Q_{4R}((\xi_0, t_0))} \left( |f|^p + |F|^p \right) \, d\xi dt,
\]

where \( p \) and \( C \) depend on \( n, \lambda, \Lambda, \|u\|_{L^q}, \|f\|_{L^1}, \|f^a\|_{L^{(r-1)/r}}, \|f^a\|_{L^r} \). Take the suitable parabolic cylinder \( Q_{4R}((\xi_0, t_0)) \) to cover \( (\xi, t) \times \Omega \) such that \( Q_{4R}((\xi_0, t_0)) \subset Q_T \); the result is obtained.

5.2. The Proof of Theorem 2. In the similar way to the proof of Theorem 1, we get (45). By the natural growth conditions (3)–(5), we have

\[
\frac{1}{4} \sup_{t \in (t_0 - R, t_0)} \int_{B_R(\xi_0)} (u - \bar{u}(t))^2 \eta^2 \, d\xi \\
+ \frac{1}{2} \int_{Q_{4R}((\xi_0, t_0))} \left( f^a \right)^2 \eta^2 \, d\xi dt \\
+ \frac{1}{2} \int_{Q_{4R}((\xi_0, t_0))} \left( |f| \right)^2 \eta^2 \, d\xi dt \\
+ \frac{1}{2} \int_{Q_{4R}((\xi_0, t_0))} \left( |F| \right)^2 \eta^2 \, d\xi dt \\
+ \frac{1}{2} \int_{Q_{4R}((\xi_0, t_0))} \left( |f^a| \right)^2 \eta^2 \, d\xi dt \\
+ \frac{1}{2} \int_{Q_{4R}((\xi_0, t_0))} \left( |\nabla u| \right)^2 \eta^2 \, d\xi dt \\
+ \frac{1}{2} \int_{Q_{4R}((\xi_0, t_0))} \left( |\nabla u| \right)^2 \eta^2 \, d\xi dt \\
= R_1 + R_2 + R_3 + R_4 + R_5 + R_6.
\]
We estimate each term by Young inequality and Sobolev-Poincaré inequality

\[
R_2 \leq \frac{εΛ}{2} \int_{Q_{δ}(ξ, t₀)} |Xu|^2 η^4 ζ^2 \\
+ \frac{CΛ}{ε} \int_{Q_{δ}(ξ, t₀)} |u - \overline{u}(t)|^2 |Xη|^2 η^4 ζ^2 \\
\leq \frac{εΛ}{2} \int_{Q_{δ}(ξ, t₀)} |Xu|^2 η^4 ζ^2 \\
+ \frac{CΛ}{ε} R^{-2} \int_{Q_{δ}(ξ, t₀)} |u - \overline{u}(t)|^2 η^4 ζ^2 \\
\leq \frac{εΛ}{2} \int_{Q_{δ}(ξ, t₀)} |Xu|^2 η^4 ζ^2 \\
+ \frac{CΛ}{ε} R^{-2} \left( \int_{Q_{δ}(ξ, t₀)} |Xu|^{4/3} \right)^{3/4},
\]

(67)

\[
R_3 \leq C \int_{Q_{δ}(ξ, t₀)} |f_i^a|^{2} η^4 ζ^2 \\
+ C \int_{Q_{δ}(ξ, t₀)} |u - \overline{u}(t)|^2 |Xη|^2 η^4 ζ^2 \\
\leq C \int_{Q_{δ}(ξ, t₀)} |f_i^a|^{2} η^4 ζ^2 \\
+ CR^{-2} \int_{Q_{δ}(ξ, t₀)} |u - \overline{u}(t)|^2 η^4 ζ^2 \\
\leq C \int_{Q_{δ}(ξ, t₀)} |f_i^a|^{2} η^4 ζ^2 \\
+ CR^{-2} \left( \int_{Q_{δ}(ξ, t₀)} |Xu|^{4/r} \right)^{r/4},
\]

\[
R_4 = \frac{Λ}{2} \int_{Q_{δ}(ξ, t₀)} |Xu|^2 (u - \overline{u}(t)) η^4 ζ^2 \\
\leq ΛM \int_{Q_{δ}(ξ, t₀)} |Xu|^2 η^4 ζ^2,
\]

\[
R_5 \leq C \int_{Q_{δ}(ξ, t₀)} |f_i^a|^{r/(r-1)} η^4 ζ^2 \\
+ C \int_{Q_{δ}(ξ, t₀)} |u - \overline{u}(t)|^r.
\]

Therefore

\[
\sup_{t ∈ (t₀ - R^{-2}, t₀)} \left( \int_{B_R(ξ)} (u - \overline{u}(t))^2 η^4 ζ^2 dξ \right)
+ (λ - 2ΛM - εΛ) \int_{Q_{δ}(ξ, t₀)} |Xu|^2 η^4 ζ^2 dξ dt
\leq \frac{CΛR^{-2}}{ε} \int_{Q_{δ}(ξ, t₀)} |u - \overline{u}(t)|^2 η^4 ζ^2 \\
+ C \int_{Q_{δ}(ξ, t₀)} (|u - \overline{u}(t)|^2) \\
+ C \int_{Q_{δ}(ξ, t₀)} \left( f^2 + |f_i^a|^2 + |f_i^a|^{r/(r-1)} \right).
\]

(68)

Since λ - 2ΛM > 0, we take ε = (λ - 2ΛM)/3Λ; then

\[
\sup_{t ∈ (t₀ - R^{-2}, t₀)} \int_{B_R(ξ)} (u - \overline{u}(t))^2 η^4 ζ^2 dξ \\
+ \int_{Q_{δ}(ξ, t₀)} |Xu|^2 η^4 ζ^2 dξ dt
\leq CR^{-2} \int_{Q_{δ}(ξ, t₀)} |u - \overline{u}(t)|^2 η^4 ζ^2 \\
+ C \int_{Q_{δ}(ξ, t₀)} (|u - \overline{u}(t)|^r) \\
+ C \int_{Q_{δ}(ξ, t₀)} \left( f^2 + |f_i^a|^2 + |f_i^a|^{r/(r-1)} \right)
\]

\[
= C_0 (S_1 + S_2 + S_3).
\]

Following the estimate (50) in Theorem 1, one has

\[
S_1 \leq \frac{λ}{16C_0} \sup_{t ∈ (t₀ - R^{-2}, t₀)} \int_{B_R(ξ)} |u - \overline{u}(t)|^2 η^4 ζ^2 \\
+ C \left( R^{-2} \int_{Q_{δ}(ξ, t₀)} |Xu|^2 \right)^{2/q}.
\]

(70)

Similar to the estimate of \( I_{21} \) in Theorem 1

\[
S_2 \leq C \sup_{t ∈ (t₀ - R^{-2}, t₀)} \left( \int_{B_R(ξ)} |u|^2 dξ \right)^{2/Q} \int_{Q_{δ}(ξ, t₀)} |Xu|^2.
\]

(71)

By \( f, f_i^a, f_i^a, u \), the assumption of \( g \), and (14), we know that

\[
\int_{Q_{δ}(ξ, t₀)} |Xu|^2, \\
\int_{Q_{δ}(ξ, t₀)} |f|^2, \\
\int_{Q_{δ}(ξ, t₀)} \left| f_i^a \right|^2, \\
\int_{Q_{δ}(ξ, t₀)} \left| f_i^a \right|^{r/(r-1)}, \\
\int_{Q_{δ}(ξ, t₀)} |u|^r.
\]

(72)
They are uniformly bounded. Combining all the estimations above, we have
\[
\int_{Q_0(\xi_0,\tau_0)} |Xu|^2 \eta^4 \xi^2 \leq C \left( \int_{Q_0(\xi_0,\tau_0)} |Xu|^q \right)^{2/q} + C \int_{Q_0(\xi_0,\tau_0)} \left( |f|^p + |f^a|^p + |f^b|^p \right) d \xi d t,
\]
where \( C = C(\eta, \zeta, \lambda, \Lambda, \|u\|_{L^2}, \|f\|_{L^2}, \|f^a\|_{L^2(\Omega)}, \|f^b\|_{L^2(\Omega)}). \) Then (C) is derived. Now we prove (D). Similar to (B), we just need to take \( \gamma = 2/\beta > 1, \ g = |Xu|^\beta, \ h = |f|^p + |f^a|^p + |f^b|^p \) in Lemma 8; then
\[
\int_{Q_0(\xi_0,\tau_0)} |Xu|^p \eta^4 \xi^2 \leq C \left( \int_{Q_0(\xi_0,\tau_0)} |Xu|^2 \eta^4 \xi^2 \right)^{p/2} + C \int_{Q_0(\xi_0,\tau_0)} \left( |f|^p + |f^a|^p \right) d \xi d t,
\]
where \( p \) and \( C \) depend on \( n, \lambda, \Lambda, \|u\|_{L^2}, \|f\|_{L^2}, \|f^a\|_{L^2(\Omega)}, \|f^b\|_{L^2(\Omega)}, M. \) The proof is complete by using suitable parabolic cylinder \( Q_{R}(\xi_0,\tau_0) \) to cover \( (\varepsilon, T) \times \Omega \) such that \( Q_{4R}(\xi_0,\tau_0) \subset Q_T. \)

**Competing Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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