Research Article

Optimal and Nonlinear Dynamic Countermeasure under a Node-Level Model with Nonlinear Infection Rate

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This paper mainly addresses the issue of how to effectively inhibit viral spread by means of dynamic countermeasure. To this end, a controlled node-level model with nonlinear infection and countermeasure rates is established. On this basis, an optimal control problem capturing the dynamic countermeasure is proposed and analyzed. Specifically, the existence of an optimal dynamic countermeasure scheme and the corresponding optimality system are shown theoretically. Finally, some numerical examples are given to illustrate the main results, from which it is found that (1) the proposed optimal strategy can achieve a low level of infections at a low cost and (2) adjusting nonlinear infection and countermeasure rates and tradeoff factor can be conductive to the containment of virus propagation with less cost.

1. Introduction

In order to study the long-term behavior of computer virus and suppress viral spread macroscopically, a large number of dynamical models have been proposed in the past few decades (for the related references, see, e.g., [1–11]). From the perspective of the division scale of computers on networks, these models can be roughly divided into two categories: compartment-level models and node-level models.

Compartment-level models are those models that regard computers having the same state as an object to study. This work can be traced back to the 1980s. The first compartment-level model is proposed by Kephart and White [1], who followed the suggestions recommended by Cohen [12] and Murray [13]. Since then, multifarious propagation models have been developed (see, e.g., [14–22]). It is worth noticing that Zhu et al. [6] proposed the original compartment-level SICS (susceptible-infected-countermeasured-susceptible) model with linear static countermeasure based on the CMC (Countermeasure Competing) strategy presented by Chen and Carley [23]. However, compartment-level models ignore the effect of network eigenvalue on viral spread. Consequently, node-level models are considered.

Node-level models are those models that regard a single computer as an object to investigate. The first node-level model (i.e., SIS (susceptible-infected-susceptible) model) is proposed by Van Mieghem et al. [7]. Since then, Sahneh and Scoglio [8] presented the node-level SAIS (susceptible-alert-infected-susceptible) model, and Yang et al. [9, 10] considered the node-level SLBS (susceptible-latent-breaking-susceptible) and SIRS (susceptible-infected-recovered-susceptible) models, respectively. Very recently, Gan [11] established the node-level SIES (susceptible-infected-external-susceptible) model. Besides, for the other related work about this topic, one can refer to [24–28] and the references cited therein.

Inspired by the above-mentioned work and based on the compartment-level SICS model, this paper considers a controlled node-level SICS model. Different from the conventional node-level models, this paper mainly addresses the issue of how to effectively distribute dynamic countermeasure by optimal control strategy (for the related references of optimal models, see, e.g., [29–33]). An optimal control
problem is proposed and the existence of an optimal control is proved. The corresponding optimality system is also derived. Finally, some numerical examples are made, from which it can be seen that the proposed optimal strategy can achieve a low level of infections at a low cost.

The subsequent materials of this paper are organized as follows. Sections 2 and 3 formulate the controlled node-level model and analyze the optimal control problem, respectively. Numerical examples are provided in Section 4. Finally, Section 5 closes this work.

2. The Controlled Node-Level Model

In this paper, the propagation network of computer virus and countermeasure is represented by a graph $G = (V, E)$ with $N$ nodes labelled $1, 2, \ldots, N$, where each node and edge stand for a computer and a network link, respectively. Thus, the graph $G$ can be described by its adjacency matrix $A = \{a_{ij}\}_{N \times N}$, where $a_{ii} = 0$.

As was treated in the traditional SICS model [6], at any time all nodes in the graph $G$ are divided into three groups: $S$-nodes (susceptible nodes are uninfected but have no immunity), $I$-nodes (infected nodes), and $C$-nodes (countermeasured nodes are uninfected and have temporary immunity due to the presence of countermeasures). Let $S_i(t), I_i(t)$, and $C_i(t)$ denote the probability of node $i$ being susceptible, infected, and countermeasured at time $t$, respectively. Then the vector

$$\left(S_1(t), \ldots, S_N(t), I_1(t), \ldots, I_N(t), C_1(t), \ldots, C_N(t)\right)^T$$

probabilistically captures the state of the network at time $t$.

For convenience, two important functions, which will be used in the sequel, are defined as follows:

$$f_i(t) = \sum_j a_{ij} \beta_i I_j(t) \left(1 + m_1 I_j(t)\right), \quad m_1 \geq 0, \quad \beta_i > 0. \quad (2)$$

Clearly, $f_i(t) \leq \sum_j a_{ij} \beta_j I_j(t)$.

$$g_i(t) = \sum_j a_{ij} \gamma_j(t) C_j(t) \left(1 + m_2 C_j(t)\right), \quad m_2 \geq 0, \quad (3)$$

where $\gamma_j(t) \in L^2[0, T]$ is a controllable rate, $\gamma \leq \gamma_j(t) \leq \overline{\gamma}$, $0 \leq t \leq T$; $\gamma$ and $\overline{\gamma}$ are positive constants, $0 < \gamma \leq \overline{\gamma} < 1$.

Now, a set of probabilistic assumptions on the state transition of node $i$ are made (see also Figure 1).

(A1) An $S$-node $i$ becomes infected at rate $f_i(t)$.

(A2) An $S$- or $I$-node $i$ becomes countermeasured at rate $g_i(t)$.

(A3) An $I$-node $i$ becomes susceptible at a constant rate $\alpha_i > 0$.

(A4) A $C$-node $i$ loses immunity at constant rate $\theta_i > 0$.

Let $\Delta t$ be a very small time interval and $o(\Delta t)$ be a higher-order infinitesimal. Assumptions (A1)–(A4) imply that the probabilities of state transition of node $i$ satisfy the following relations:

$$\Pr(i \text{ is infected at time } t) = f_i(t) \Delta t + o(\Delta t),$$

$$\Pr(i \text{ is countermeasured at time } t) = g_i(t) \Delta t + o(\Delta t),$$

$$\Pr(i \text{ is susceptible at time } t) = \alpha_i \Delta t + o(\Delta t),$$

$$\Pr(i \text{ loses immunity at time } t) = \theta_i \Delta t + o(\Delta t).$$

![Figure 1: The transfer diagram of the controlled node-level SICS model.](image-url)
Pr \{ i \text{ is countermeasured at time } t \} \\
+ \Delta t \mid i \text{ is susceptible at time } t \} = g_i(t) \Delta t \\
+ o(\Delta t),

Pr \{ i \text{ is countermeasured at time } t \} \\
+ \Delta t \mid i \text{ is infected at time } t \} = g_i(t) \Delta t + o(\Delta t),

Pr \{ i \text{ is susceptible at time } t \} \\
+ \Delta t \mid i \text{ is infected at time } t \} = \alpha_i \Delta t + o(\Delta t),

Pr \{ i \text{ is susceptible at time } t \} \\
+ \Delta t \mid i \text{ is countermeasured at time } t \} = \theta_i \Delta t \\
+ o(\Delta t).

(4)

Invoking the total probability formulas and letting $\Delta t \to 0$, the controlled node-level model (i.e., controlled node-level SICS model) can be derived.

\[
\frac{dS_i(t)}{dt} = \alpha_i S_i(t) + \theta_i C_i(t) - f_i(t) S_i(t) - g_i(t) S_i(t),
\]

\[
\frac{dI_i(t)}{dt} = -\alpha_i I_i(t) + f_i(t) S_i(t) - g_i(t) I_i(t),
\]

\[
\frac{dC_i(t)}{dt} = -\theta_i C_i(t) + g_i(t) (S_i(t) + I_i(t)),
\]

with initial condition

\[
(S_i(0), \ldots, S_N(0), I_i(0), \ldots, I_N(0), C_i(0), \ldots, C_N(0))^T \\
\in \bar{\Omega},
\]

where

\[
\bar{\Omega} = \{(S_1, \ldots, S_N, I_1, \ldots, I_N, C_1, \ldots, C_N)^T \in \mathbb{R}^{2N}_+ \mid S_i + I_i + C_i = 1, \ 1 \leq i \leq N\}.
\]

The admissible control set is

\[
U = \{ u(\cdot) \in \left(L^2(0, T)\right)^N \mid \gamma \leq \gamma_i(\cdot) \leq \gamma_i, \ 1 \leq i \leq N\},
\]

where $u(\cdot) = (\gamma_1(\cdot), \ldots, \gamma_N(\cdot))^T$.

3. The Optimal Control Problem

As $S_i(t) + I_i(t) + C_i(t) \equiv 1, 1 \leq i \leq N$, system (5) can be reduced to the following system:

\[
\frac{dI_i(t)}{dt} = -\alpha_i I_i(t) + f_i(t) (1 - I_i(t) - C_i(t)) \\
- g_i(t) I_i(t),
\]

\[
\frac{dC_i(t)}{dt} = -\theta_i C_i(t) + g_i(t) (1 - C_i(t)),
\]

with initial condition

\[
(I_1(0), \ldots, I_N(0), C_1(0), \ldots, C_N(0))^T \in \Omega,
\]

where

\[
\Omega = \{(I_1, \ldots, I_N, C_1, \ldots, C_N)^T \in \mathbb{R}^N_+ \mid I_i + C_i \leq 1, \ i = 1, 2, \ldots, N\}.
\]

Then system (9) can be written in matrix notation as

\[
\frac{dx(t)}{dt} = f(x(t), u(t)), \ 0 \leq t \leq T,
\]

with initial condition $x(0) \in \Omega$.

Now, the objective is to find a control variable $u(\cdot) \in U$ so as to minimize both the prevalence of infected computers and the total budget for dynamic countermeasures during the time period $[0, T]$. That is, the following optimal control problem needs to be solved:

\[
\text{Minimize } J(u) = \int_0^T L(x(t), u(t)) \, dt \quad \text{(P)}
\]

subject to system (12), where

\[
L(x, u) = \sum_i \left( I_i(t) + \varepsilon_i \gamma_i^2(t) \right), \ \varepsilon_i > 0,
\]

is the Lagrangian and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)^T$ is a tradeoff factor based on the control effect and control cost of dynamic countermeasure.

3.1. Existence of an Optimal Control. First, a lemma, which plays a critical role afterwards, is introduced.

**Lemma 1** (see [34, 35]). We have an optimal control problem

\[
\text{Minimize } J(u) = \int_0^T L(x(t), u(t)) \, dt \quad \text{(14)}
\]

subject to

\[
\frac{dx(t)}{dt} = f(x(t), u(t)), \ 0 \leq t \leq T,
\]

with $x(0) \in \Omega$, where $\Omega$ is positively invariant for system (15). The problem has an optimal control if the following six conditions hold simultaneously.

\begin{enumerate}
  \item[(C1)] There is $u \in U$ such that system (15) is solvable.
  \item[(C2)] $U$ is convex.
  \item[(C3)] $U$ is closed.
  \item[(C4)] $f(x, u)$ is bounded by a linear function in $x$.
  \item[(C5)] $L(x, u)$ is convex on $U$.
  \item[(C6)] $L(x, u) \geq c_1 \|u\|^2 + c_2$ for some $c_1 > 1, c_1 > 0, \text{and } c_2$.
\end{enumerate}

In order to prove the existence of an optimal control, six lemmas, one for each condition in Lemma 1, should be proved.


 Lemma 2. There is \( u \in U \) such that system (9) or (12) is solvable.

Proof. Substituting \( u \equiv \overline{u} := (\overline{y}_1, \ldots, \overline{y}_N)^T \) into system (12), one can get the uncontrolled system:

\[
\frac{dx(t)}{dt} = f(x(t), \overline{u})
\]

with \( x(0) \in \Omega \). Then the function \( f(x, \overline{u}) \) is continuously differentiable, and \( \Omega \) is positively invariant for the system. Hence, the claimed result follows from the Continuation Theorem for differential equations [36].

Lemma 3. The admissible set \( U \) is convex.

Proof. Let

\[
\begin{align*}
\mathbf{u}(1) &= (y_1^{(1)}, \ldots, y_N^{(1)}) \in U, \\
\mathbf{u}(2) &= (y_1^{(2)}, \ldots, y_N^{(2)}) \in U,
\end{align*}
\]

(17)

\[
0 < \xi < 1.
\]

As \( (L^2[0, T])^N \) is a real vector space, one can obtain

\[
(1 - \xi) \mathbf{u}(1) + \xi \mathbf{u}(2) \in (L^2[0, T])^N.
\]

Then, the convexity of \( U \) follows by the observation that

\[
y_i \leq (1 - \xi) y_i^{(1)} + \xi y_i^{(2)} \leq \overline{y}, \quad 1 \leq i \leq N.
\]

(18)

Hence, the claimed result follows.

Lemma 4. The admissible set \( U \) is closed.

Proof. Let \( \mathbf{u} = (y_1, \ldots, y_N)^T \) be a limit point of \( U \) and

\[
\mathbf{u}^{(n)} = (y_1^{(n)}, \ldots, y_N^{(n)})^T, \quad n = 1, 2, \ldots,
\]

(20)

be a sequence of points in \( U \) such that

\[
\lim_{n \to \infty} \mathbf{u}^{(n)} = \mathbf{u} \in (L^2[0, T])^N.
\]

(21)

Hence, the closeness of \( U \) follows from the observation that

\[
y_i \leq \ell_i = \lim_{n \to \infty} y_i^{(n)} \leq \overline{y}, \quad 1 \leq i \leq N.
\]

(22)

Lemma 5. \( f(x, u) \) is bounded by a linear function in \( x \).

Proof. Note that, for system (9) and for \( i = 1, 2, \ldots, N \),

\[
-\frac{\overline{y} N^2}{4} - \alpha_i I_i \leq \frac{dI_i}{dt} \leq -\alpha_i I_i + \sum_j a_{ij} \beta_j I_j,
\]

(24)

Thus, the claimed result follows.

Lemma 6. \( L(x, u) \) is convex on \( U \).

Proof. Note that the Hessian matrix of \( L(x, u) \) with respect to \( u \in U \) is as follows:

\[
\begin{bmatrix}
\frac{\partial^2 L}{\partial y_1^2} & \cdots & \frac{\partial^2 L}{\partial y_1 \partial y_N} & \frac{\partial^2 L}{\partial y_2 \partial y_1} & \cdots & \frac{\partial^2 L}{\partial y_2 \partial y_N} \\
\frac{\partial^2 L}{\partial y_2 \partial y_1} & \cdots & \frac{\partial^2 L}{\partial y_2 \partial y_N} & \frac{\partial^2 L}{\partial y_3 \partial y_1} & \cdots & \frac{\partial^2 L}{\partial y_3 \partial y_N} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\frac{\partial^2 L}{\partial y_N \partial y_1} & \cdots & \frac{\partial^2 L}{\partial y_N \partial y_{N-1}} & \frac{\partial^2 L}{\partial y_N \partial y_N} & \cdots & \frac{\partial^2 L}{\partial y_{N+1} \partial y_N}
\end{bmatrix}
\]

(25)

For any \( t \in [0, T] \), \( H_u(L) \) is real symmetric and its eigenvalues are all positive. Hence, the convexity of \( L(x, u) \) follows by the result in [37].

Lemma 7. \( L(x, u) \geq c_1 \|u\|_2^p + c_2 \) for some \( p > 1 \), \( c_1 > 0 \), and \( c_2 \).

Proof. Let \( \rho = 2 \), \( c_1 = \min_i \{\epsilon_i\}/2 \), and \( c_2 = 0 \). Then, \( L(x, u) \geq (\min_i \{\epsilon_i\}/2) \times \|u\|_2^2 \). Thus, the proof is complete.

Now, it is time to examine the main result of this subsection.

Theorem 8. The optimal control problem \( (P) \) has a solution.

Proof. Lemmas 2–7 show that the six conditions in Lemma 1 are all met. Hence, the proof is complete.

3.2. The Optimality System. In this subsection, a necessary condition for an optimal control of problem \( (P) \) is drawn.

Theorem 9. Suppose \( u^*(\cdot) \) is an optimal control for problem \( (P) \) and \( (I^*(\cdot), G^*(\cdot)) \) is the solution to system (9) with
\( u(\cdot) = u^*(\cdot) \). Then, there exist functions \( \lambda^*_1(t) \) and \( \lambda^*_2(t) \), \( 0 \leq t \leq T, \ 1 \leq i \leq N \), such that

\[
\frac{d\lambda^*_1(t)}{dt} = -1 + \lambda^*_1(t) [\alpha_i + f^*_i(t) + g^*_i(t)]
- \frac{\beta_i}{\left(1 + m_i I^*_i(t)\right)^2}
+ \sum_j a_{ij} [I^*_j(t) \lambda^*_i(t) - (1 - C^*_j(t)) \lambda^*_2(t)] ,
\]

\[
\frac{d\lambda^*_2(t)}{dt} = \lambda^*_2(t) f^*_i(t) + \lambda^*_2(t) [\theta_i + g^*_i(t)]
+ \frac{\gamma^*_i(t)}{\left(1 + m_2 C^*_i(t)\right)^2}
\cdot \sum_j a_{ij} [I^*_j(t) \lambda^*_i(t) - (1 - C^*_j(t)) \lambda^*_2(t)] ,
\]

\( 0 \leq t \leq T, \ i = 1, 2, \ldots, N \), with transversality conditions

\[
\lambda^*_i(T) = \lambda^*_2(T) = 0, \ i = 1, 2, \ldots, N .
\]

Furthermore, one can get

\[
\gamma^*_i(t) = \max \left\{ \min \left\{ \frac{C^*_i(t)}{\varepsilon_i (1 + m_2 C^*_i(t))} \sum_j a_{ij} [I^*_j(t) \lambda^*_i(t) - (1 - C^*_j(t)) \lambda^*_2(t)] - \sum_j \lambda^*_j \frac{dC^*_j(t)}{dt}, 0 \leq t \leq T, \ i = 1, 2, \ldots, N \right\}, \gamma^*_i(t)
\]

Thus, system (26) follows by direct calculations. As the terminal cost is unspecified and the final state is free, the transversality conditions hold. By using the optimality condition

\[
\frac{\partial H}{\partial y^*_i}(1^*, C^*, \lambda^*, u^*) = \min_{u \in U} H(I, C, \lambda, u),
\]

one can obtain that, for \( 0 \leq t \leq T \) and for \( 1 \leq i \leq N \), either

\[
\frac{\partial H}{\partial y^*_i}(1^*, C^*, \lambda^*, u^*) = \varepsilon_i \gamma^*_i(t) - \frac{C^*_i(t)}{1 + m_2 C^*_i(t)}
- \sum_j a_{ij} [I^*_j(t) \lambda^*_i(t) - (1 - C^*_j(t)) \lambda^*_2(t)] = 0
\]

or \( \gamma^*_i(t) = \gamma \) or \( \gamma^*_i(t) = \gamma \). Hence, the proof is complete. \( \Box \)

By combining the above discussions, one can get the optimality system for problem (P) as follows:

\[
\frac{dI^*_i(t)}{dt} = -\alpha_i I^*_i(t) + f^*_i(t) (1 - I^*_i(t) - C^*_i(t)) - g_i(t) I^*_i(t),
\]

\[
\frac{dC^*_i(t)}{dt} = -\theta_i C^*_i(t) + g_i(t) (1 - C^*_i(t)),
\]

\[
\frac{d\lambda^*_1(t)}{dt} = -1 + \lambda^*_1(t) [\alpha_i + f^*_i(t) + g_i(t)] - \frac{\beta_i}{\left(1 + m_i I^*_i(t)\right)^2}
\sum_j a_{ij} \lambda^*_1(t) (1 - I^*_j(t) - C^*_j(t)),
\]

\[
\frac{d\lambda^*_2(t)}{dt} = \lambda^*_2(t) f^*_i(t) + \lambda^*_2(t) [\theta_i + g_i(t)]
+ \frac{\gamma^*_i(t)}{\left(1 + m_2 C^*_i(t)\right)^2}
\cdot \sum_j a_{ij} [I^*_j(t) \lambda^*_i(t) - (1 - C^*_j(t)) \lambda^*_2(t)] ,
\]
The average control \( \gamma^*(t) \)

(a) \( \gamma^*(t) \)

(b) \( I^*(t) \)

Figure 2: \( \gamma^*(t) \) and \( I^*(t) \) under different control strategies with \( m_1 = m_2 = 2 \) for Example 1.

The average control \( \gamma^*(t) \)

(a) \( m_2 = 5 \)

(b) \( m_1 = 5 \)

Figure 3: \( \gamma^*(t) \) for different \( m_1 \) and \( m_2 \) for Example 1.

\[
\gamma_i(t) = \max \left\{ \min \left\{ \frac{C_i(t)}{\varepsilon_i (1 + m_2 C_i(t))} \sum_j a_{ij} \left[ I_j(t) \lambda_{1j}(t) - (1 - C_j(t)) \lambda_{2j}(t) \right], \gamma \right\}, \gamma \right\},
\]

\( 0 \leq t \leq T, \ i = 1, 2, \ldots, N, \)

(33)

with \( (I(0), C(0))^T \in \Omega \) and \( \lambda(T) = 0 \).
4. Numerical Examples

In this section, the effectiveness of the optimal dynamic countermeasure will be verified by some numerical examples.

For our purpose, three networks are considered: a synthetic small-world network (WS network [38]), a synthetic scale-free network (BA network [39]), and a partial Facebook network [40], with $N = 150$ nodes, respectively. The parameters of system (33) are set as $\alpha_i = 0.01$, $\beta_i = 0.004887$ (the value of $\beta_i$ comes from a report on some real infection probabilities in [41]), $\theta_i = 0.02$, $\epsilon_i = 1$, $\gamma = 0.01$, $\gamma = 0.1$, and $T = 50$, $1 \leq i \leq N$, and the initial conditions are set as $I_i(0) = 0.03$ and $C_i(0) = 0.01$, $1 \leq i \leq N$. The optimality system (33) is solved by invoking the backward-forward Euler
The average control \( \gamma^*(t) \) and the proportion of infected nodes \( I^*(t) \) under different control strategies. Table 1 gives the final proportion of infected nodes and the value of objective function \( J \) under different control strategies, where the value of static control \( u = 0.08895 \) is an average of several real curing probabilities reported in [42]. From Figure 2 and Table 1, one can conclude that \( u^* \) is indeed the optimal control strategy to minimize the objective function \( J \) and reduce virus prevalence to a low level simultaneously.

**Example 1.** Take a WS network with 150 nodes and 150 links as the propagation network.

Suppose \( u^*(t) \) is an optimal control for problem (P) and \( x^*(t) \) is a solution to the corresponding controlled system. Let \( \gamma^*(t) \) and \( I^*(t) \) denote the average control and the proportion of infected nodes under \( u^*(t) \), respectively, where

\[
\gamma^*(t) = \frac{1}{N} \sum_i y_i^*(t), \\
I^*(t) = \frac{1}{N} \sum_i I_i^*(t).
\]

(34)
The average control $\gamma^*(t)$

\[ \gamma^*(t) \]

Proportion of infected nodes $I^*(t)$

\[ I^*(t) \]

Figure 7: $\gamma^*(t)$ and $I^*(t)$ under different control strategies with $m_1 = m_2 = 2$ for Example 2.

Figure 8: $\gamma^*(t)$ for different $m_1$ and $m_2$ for Example 2.

Table 1: $I^*(T)$ and $J$ under different control strategies with $m_1 = m_2 = 2$ for Example 1.

<table>
<thead>
<tr>
<th></th>
<th>$I^*(T)$</th>
<th>$J(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = u^*$</td>
<td>0.0089</td>
<td>172.55</td>
</tr>
<tr>
<td>$u = 0.01$</td>
<td>0.0283</td>
<td>218.97</td>
</tr>
<tr>
<td>$u = 0.04$</td>
<td>0.0231</td>
<td>211.61</td>
</tr>
<tr>
<td>$u = 0.08995$</td>
<td>0.0072</td>
<td>181.14</td>
</tr>
<tr>
<td>$u = 0.1$</td>
<td>0.0053</td>
<td>177.27</td>
</tr>
</tbody>
</table>

Figure 3 demonstrates the average control $\gamma^*(t)$ for different $m_1$ and $m_2$. From this figure, one can see that (a) enhancing $m_1$ and $m_2$ roughly reduces $\gamma^*(t)$, (b) the smaller $m_2$ is, the longer $\gamma^*(t)$ stays at $\bar{\gamma}$, and (c) $m_2$ has a more significant impact on $\gamma^*(t)$ than $m_1$ does.

Figure 4 displays $I^*(t)$ for different $m_1$ and $m_2$. From this figure, it can be seen that (a) lower $m_1$ favors virus spreading, whereas lower $m_2$ is conducive to the containment of virus prevalence, (b) $m_2$ affects $I^*(t)$ more significantly than $m_1$ does, which implies that dynamic countermeasure plays a
Figure 9: $I^*(t)$ for different $m_1$ and $m_2$ for Example 2.

Figure 10: $I^*(T)$ and $J(u^*)$ for different $m_1$ and $m_2$ for Example 2.

dominant role in the suppression of virus diffusion, and (c) linear infection rate overestimates virus prevalence, which is in accordance with the result in [7].

Figure 5 depicts the final proportion of infected nodes $I^*(T)$ and the objective function $J(u^*)$ for varied $m_1$ and $m_2$. From this figure, it can be seen that $J$ is decreasing and increasing with respect to $m_1$ and $m_2$, respectively, which makes a suggestion that enhancing $m_1$ and diminishing $m_2$ are beneficial to the containment of viral spread and reduce $J$ to a low level simultaneously.

Figure 6 shows $\gamma^*(t)$, $I^*(t)$, $I^*(T)$, and $J(u^*)$ for different $\epsilon$. From this figure, it is found that decreasing $\epsilon$ is effective on the suppression of virus propagation and attains a lower $J(u^*)$ simultaneously, although it creates more control cost. This is in good agreement with the fact that when the control effect (i.e., to obtain a low level of infections) is given priority
(i.e., with lower $\epsilon$), often the decision is made to spend enough control cost. Hence, the tradeoff factor $\epsilon$ plays a critical role in the balance between control effect and control cost.

Example 2. Take a BA network with 150 nodes and 150 links as the propagation network.

Figure 7 displays $y^*(t)$ and $I^*(t)$ under different control strategies. Table 2 shows the values of $I^*(T)$ and $J(u)$ under different control strategies. Figures 8 and 9 depict $y^*(t)$ and $I^*(t)$ for different $m_1$ and $m_2$, respectively. Figure 10 demonstrates $I^*(T)$ and $J(u^*)$ for different $m_1$ and $m_2$. Figure 11 exhibits $y^*(t)$, $I^*(t)$, $I^*(T)$, and $J(u^*)$ for different $\epsilon$. From them, one can get the same results in Example 1. So they are omitted here for brevity.

Example 3. Take a partial Facebook network with 150 nodes and 603 links as the propagation network.

Figure 12 shows $y^*(t)$ and $I^*(t)$ under different control strategies. Table 3 gives the values of $I^*(T)$ and $J(u)$ under different control strategies. Figures 13 and 14 display $y^*(t)$
The average control $\gamma^*(t)$

![Graph](image1.png)

(a) $\gamma^*(t)$

The proportion of infected nodes $I^*(t)$

![Graph](image2.png)

(b) $I^*(t)$

Figure 12: $\gamma^*(t)$ and $I^*(t)$ under different control strategies with $m_1 = m_2 = 2$ for Example 3.

Figure 13: $\gamma^*(t)$ for different $m_1$ and $m_2$ for Example 3.

Table 2: $I^*(T)$ and $J$ under different control strategies with $m_1 = m_2 = 2$ for Example 2.

<table>
<thead>
<tr>
<th></th>
<th>$u = u^*$</th>
<th>$u = 0.01$</th>
<th>$u = 0.04$</th>
<th>$u = 0.08895$</th>
<th>$u = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I^*(T)$</td>
<td>0.0132</td>
<td>0.0288</td>
<td>0.0208</td>
<td>0.0111</td>
<td>0.0097</td>
</tr>
<tr>
<td>$J(u)$</td>
<td>169.84</td>
<td>219.58</td>
<td>202.91</td>
<td>185.27</td>
<td>185.65</td>
</tr>
</tbody>
</table>

Table 3: $I^*(T)$ and $J$ under different control strategies with $m_1 = m_2 = 2$ for Example 3.

<table>
<thead>
<tr>
<th></th>
<th>$u = u^*$</th>
<th>$u = 0.01$</th>
<th>$u = 0.04$</th>
<th>$u = 0.08895$</th>
<th>$u = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I^*(T)$</td>
<td>0.0013</td>
<td>0.0504</td>
<td>0.0035</td>
<td>0.0008</td>
<td>0.0006</td>
</tr>
<tr>
<td>$J(u)$</td>
<td>61.91</td>
<td>373.77</td>
<td>112.31</td>
<td>83.53</td>
<td>86.17</td>
</tr>
</tbody>
</table>
and \( I^*(t) \) for different \( m_1 \) and \( m_2 \), respectively. Figure 15 demonstrates \( I^*(T) \) and \( J(u^*) \) for varied \( m_1 \) and \( m_2 \). Figure 16 depicts \( \gamma^*(t) \), \( I^*(t) \), \( I^*(T) \), and \( J(u^*) \) for different \( \epsilon \).

Most of the results concluded from this example are the same as those in Example 1 except the two phenomena listed as follows: (a) higher \( m_2 \) increases \( \gamma^*(t) \), which is contrary to the results in Figures 3(b) and 8(b), and (b) \( m_1 \) has a negligible impact on \( \gamma^*(t) \) and \( I^*(t) \). This indicates that the network structure, to some extent, determines the control cost and virus diffusion.

Combining the above numerical examples, the main results are listed below.

(a) \( u^* \) is indeed the optimal control strategy to minimize the objective function \( J \) and reduce the infections to a low level simultaneously.
(b) Linear infection rate overestimates the prevalence of virus.

(c) Enhancing \( m_1 \) and diminishing \( m_2 \) are conductive to the containment of viral propagation and reduce \( J \) to a low level simultaneously.

(d) \( m_2 \) has more significant influences on \( \gamma^*(t) \), \( I^*(t) \), and \( J(u^*) \) than \( m_1 \) does.

(e) Decreasing the tradeoff factor \( \varepsilon \) is beneficial to the suppression of virus spread and obtains a lower \( J(u^*) \) simultaneously, although it brings more control cost.

Additionally, the structure of network, to some extent, determines the virus prevalence and the control cost. Thus, we shall investigate how the network topology affects virus spreading and control cost in the next work.

5. Concluding Remarks

This paper has studied the issue of how to work out an optimal dynamic countermeasure for achieving a low level of infections with a low cost. In this regard, a controlled node-level SICS model with nonlinear infection rate has
been established. Furthermore, an optimal control problem has been proposed. The existence of an optimal control and the corresponding optimality system have also been derived. Additionally, some numerical examples have been given to illustrate the main results. Specifically, it has been found that the proposed optimal countermeasure scheme can achieve a low level of infections at a low cost.

In our opinions, the next work could be made as follows. First, the quadratic cost functions may be generalized to some generic functions. Second, delays [43–45], pulses [46, 47], and random fluctuations [15] may be incorporated to controlled node-level models. Last, but not least, it is worthy to carry on research on the impact of the network topology [9, 25, 48, 49] on the dynamic countermeasure strategy.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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