Research Article

Stability of a Class of Hybrid Neutral Stochastic Differential Equations with Unbounded Delay

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Received 12 April 2017; Revised 2 July 2017; Accepted 19 July 2017; Published 21 August 2017

Academic Editor: Rigoberto Medina

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This paper studies the stability of hybrid neutral stochastic differential equations with unbounded delay. Some novel exponential stability criteria and boundedness conditions are established based on the generalized Itô formula and Lyapunov functions. The factor $e^{-\varepsilon\delta(t)}$ is used to overcome the difficulties caused by the unbounded delay $\delta(t)$ effectively. In particular, our results generalize and improve some previous stability results from bounded delay to unbounded delay conditions. Finally, an example is presented to demonstrate the effectiveness of the proposed results.

1. Introduction

With the deep development of the research on complex systems in recent years, much attention has been paid to the influence of random factors on the power system. Hence, the stochastic differential equations (SDEs) come to play important roles in many fields, such as molecular physics, population genetics, and some other branches of science (see [1–3]).

Practically, there are many systems depending on both present and past states, as well as the changing rate. To model this kind of systems primely, the delay systems and neutral stochastic differential equations (NSDEs) with time-dependent delay are studied widely. Recently, there have appeared many interesting results in the literature. For example, [4–6] studied the global asymptotic stability results and applications of time-varying delay systems in neural networks (NNs), memristive neural networks (MNNs), and chaotic Lure systems (CLSs). Zhu and Cao [7] investigated the global asymptotic stability for stochastic neural networks of neutral type with both Markovian jump parameters and mixed time delays. Liu and Zhu [8] studied mean square stability of two classes of theta method for NSDEs with bounded delay. Ma and Xi [9] established a moderate deviation principle for NSDEs driven by Poisson random measure and time delay.

On the other hand, some random abrupt changes generally exist in practical systems because of the change of internal and external environment. SDEs with Markovian switching (SDEwMs) are powerful tools for describing systems that encounter abrupt changes in structure. It is inspiring that many important results have been reported in the literature. For example, Zhu et al. [10] investigated the problem of robust exponential stability for a class of stochastically nonlinear jump systems with mixed time delays. Zhu [11] investigated the p th moment exponential stability for a class of impulsive stochastic functional differential equations with Markovian switching (SFDEwMs). Furthermore, Zhu [12] studied the Razumikhin-type theorem for a class of SFDEwMs with Lévy noise, which was a breakthrough of the traditional Brownian motion. Reference [13] discussed the asymptotic stability of SDEwMs. Also, in recent years many researchers pay more effort to applying Markovian switching into the neural networks that has various applications (e.g., see [14–16]). Besides, a few stability studies on the NSDEs with Markovian switching (NSDEwMs) can be found in [17–20].

Above references just focus on bounded time delay, leaving many practical systems unanswered where the delay terms are usually required to be unbounded. For example, in the neural network model, stochastic unbounded variable

delay differential system must be considered to model transmission and transformation of the signal in a better fashion (see [19]). Some related works on unbounded delay can be found in [20–24]. References [22, 24] investigated the existence and uniqueness, as well as the pathwise stability of the global solutions to SDEs and NSDEs with unbounded delay, respectively. Reference [23] studied the stability of SFDEs with unbounded delay. In Section 4.2 of [21], the stability and boundedness of nonlinear hybrid SDEs were discussed when the delay function $\delta(t)$ was given as θt ($0 < \theta < 1$). Particularly, [20] presented the existence and uniqueness of the exact solution for a class of NSDEwMs with unbounded delay and also showed the p th moment exponential stability and almost sure exponential stability results under the bounded delay condition and established the Euler-Maruyama method under both cases. As far as we know, few works have been devoted to study the stability of neutral stochastic hybrid systems with unbounded delay.

Motivated by the above discussion, this paper aims to develop the p th moment exponential stability and almost sure exponential stability criteria of the solution of NSDEwMs with unbounded delay after showing the corresponding existence and uniqueness theorem. Our methods rely upon the generalized Itô formula and Lyapunov functions. Besides, the factor $e^{-\varepsilon\delta(t)}$ is used to overcome the key challenges emerging from the neutral term and unbounded delay features. The contributions of this paper are twofold. The first one is that we improve the delay function $\delta(t) = \theta t$ ($0 < \theta < 1$) in [21] to the general unbounded delay and add the neutral term. The other one is that we greatly generalize and advance the p th moment exponential stability and almost sure exponential stability results in [20] where the delay term was bounded. It should be pointed out that we use the Euler-Maruyama method which was investigated in [20] to simulate a numerical example in this study.

The remainder of this paper is organized as follows. In Section 2, we define some notations and give four assumptions with respect to the system of the hybrid neutral stochastic differential equations with unbounded delay. In Section 3, we obtain our results through three theorems. A numerical example is proposed to demonstrate the results in Section 4. Finally, in Section 5, we conclude this paper with some general remarks.

2. Models, Notations, and Assumptions

We use the following notations throughout this paper without specification.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions, which means that it is increasing and right continuous, and $\mathcal{F}_0$ contains all P -null sets. Let $\{w(t) = (w_1(t), w_2(t), \dots, w_m(t))^T, t \geq 0\}$ be an m -dimensional standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. For any $x \in R^d$, let $|x|$ stand for the Euclidean norm. If A is a matrix, denote its trace norm by $|A|^2 = \text{trace}(A^T A)$, where A^T is the transpose. For a given $\tau > 0$, denote by $C([- \tau, 0]; R^d)$ the family of continuous function

$\varphi : [- \tau, 0] \rightarrow R^d$ with the supremum norm $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$. Also, denote by $C_{\mathcal{F}_0}^b([- \tau, 0]; R^d)$ the family of bound, $\mathcal{F}_0$ -measurable, $C([- \tau, 0]; R^d)$ -valued random variables and $R_+ = [0, \infty)$.

Let $\{r(t), t \geq 0\}$ be a right-continuous Markovian chain on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ taking values in a finite states space $S = \{1, 2, \dots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$P\{r(t + \Delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases} \quad (1)$$

where $\Delta > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}$. Assume that the Markov chain $\{r(t)\}_{t \geq 0}$ is independent of the Brownian motion $\{w(t)\}_{t \geq 0}$.

The model considered in this paper is the following hybrid neutral stochastic differential equation with unbounded delay:

$$\begin{aligned} d[x(t) - u(x(t - \delta(t)), t, r(t))] \\ = f(x(t), x(t - \delta(t)), t, r(t)) dt \\ + g(x(t), x(t - \delta(t)), t, r(t)) dw(t), \quad t \geq 0 \end{aligned} \quad (2)$$

with initial conditions

$$\begin{aligned} x_0 = \varphi = \{\varphi(\theta) : \theta \in [-\delta(0), 0]\} \in C_{\mathcal{F}_0}^b, \\ r(0) = i_0 \in S, \end{aligned} \quad (3)$$

where the functions

$$\begin{aligned} f : R^d \times R^d \times R_+ \times S &\longrightarrow R^d, \\ g : R^d \times R^d \times R_+ \times S &\longrightarrow R^{d \times m}, \\ u : R^d \times R_+ \times S &\longrightarrow R^d \end{aligned} \quad (4)$$

are all Borel-measurable and $x(t)$ is a d -dimensional state process. The unbounded delay function $\delta : R_+ \rightarrow R_+$ is Borel-measurable. It is necessary to say that model (2) was investigated in [20] to study the existence and uniqueness of the solution under the unbounded delay, as well as the p th moment exponential stability and almost sure exponential stability results under the bounded delay condition. Next we will further establish the p th moment exponential stability and almost sure exponential stability criteria of model (2) when the delay function is unbounded. Besides, the following Assumption 4 we used is quite different from the assumption $\mathcal{A}_2$ in [20].

Denote by $C(R^d \times R_+; R_+)$ the family of all continuous functions from $R^d \times R_+$ to R_+ . Let $C^{2,1}(R^d \times R_+ \times S; R_+)$ denote the family of all continuous nonnegative functions $V(x, t, i)$ from $R^d \times R_+ \times S$ to R_+ such that, for each $i \in S$, they are continuously twice differentiable in x and once in t .

For a given $V \in C^{2,1}(R^d \times R \times S; R_+)$, we define an operator $LV: R^d \times R^d \times R_+ \times S \rightarrow R$ by

$$\begin{aligned} LV(x, y, t, i) = & V_t(x - u(y, t, i), t, i) + V_x(x \\ & - u(y, t, i), t, i) f(x, y, t, i) + \frac{1}{2} \text{trace} [g^T(x, y, t, i) \\ & \cdot V_{xx}(x - u(y, t, i), t, i) g(x, y, t, i)] \\ & + \sum_{j=1}^N \gamma_{ij} V(x - u(y, t, i), t, j), \end{aligned} \quad (5)$$

where

$$\begin{aligned} V_t(x, t, i) &= \frac{\partial V(x, t, i)}{\partial t}, \\ V_x(x, t, i) &= \left(\frac{\partial V(x, t, i)}{\partial x_1}, \frac{\partial V(x, t, i)}{\partial x_2}, \dots, \frac{\partial V(x, t, i)}{\partial x_d} \right), \\ V_{xx}(x, t, i) &= \left(\frac{\partial^2 V(x, t, i)}{\partial x_i \partial x_j} \right)_{d \times d}. \end{aligned} \quad (6)$$

By the generalized Itô formula, we have

$$\begin{aligned} V(x(t) - u(x(t - \delta(t)), t, r(t)), t, r(t)) - V(x(0) - u(x(0), 0, \\ r(0)), 0, r(0)) = \int_0^t LV(x(s), x(s - \delta(s)), s, r(s)) ds \\ + \int_0^t V_x(x(s) - u(x(t - \delta(s)), s, r(s)), s, r(s)) g(x(s), x(s \\ - \delta(s)), s, r(s)) dw(s) \\ + \int_0^t \int_R [V(x(s) - u(x(s - \delta(s)), s, r(s)), s, i_0 + h(r(s), l)) \\ - V(x(s) - u(x(s - \delta(s)), s, r(s)), s, r(s))] \mu(ds, dl), \end{aligned} \quad (7)$$

where the function $h: S \times R \rightarrow R$ is defined by

$$h(i, y) = \begin{cases} j - i, & \text{if } y \in \Delta_{ij}, \\ 0, & \text{otherwise,} \end{cases} \quad (8)$$

and $\mu(ds, dl) = \nu(ds, dl) - \mu(dl)ds$ is a martingale measure while $\nu(dt, dy)$ is a Poisson random measure with intensity $dt \times \mu(dy)$, in which μ is a Lebesgue measure on R . The definition of Δ_{ij} and more related details can be found in [25].

Now we give the following primary inequalities.

Proposition 1. For any $a, b, p \in R_+$ and $\eta \in (0, 1)$, the following inequalities hold.

$$(a + b)^p \leq a^p + b^p, \quad 0 < p \leq 1, \quad (9)$$

$$(a + b)^p \leq (1 + \eta)^{p-1} (a^p + \eta^{1-p} b^p), \quad p > 1. \quad (10)$$

The detailed descriptions and proof of Proposition 1 can be found in [25].

Nevertheless, additional assumptions will be considered to achieve the further conclusions.

Assumption 2 (the local Lipschitz condition). For each positive integer h , there exists a positive constant K_h such that for all $x_1, x_2, y_1, y_2 \in R^d$ with $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq h$ and for all $t \in R_+, i \in S$,

$$\begin{aligned} & |f(x_1, y_1, t, i) - f(x_2, y_2, t, i)|^2 \\ & \vee |g(x_1, y_1, t, i) - g(x_2, y_2, t, i)|^2 \\ & \leq K_h (|x_1 - x_2|^2 + |y_1 - y_2|^2). \end{aligned} \quad (11)$$

Assumption 3. The delay function $\delta: R_+ \rightarrow R_+$ is differentiable and there exists a constant $\bar{\delta}$ satisfying

$$\delta'(t) \leq \bar{\delta} < 1. \quad (12)$$

Let $\delta^*(t) = t - \delta(t)$, $t \in R_+$; then $\delta^{*'}(t) \geq 1 - \bar{\delta} > 0$, so $\delta^*(t)$ is an increasing function of t and $t - \delta(t) \geq -\delta(0)$, $t \in R_+$.

Assumption 4. There are a function $V \in C^{2,1}(R^d \times R_+ \times S; R_+)$, nonnegative constants $\lambda_1, \lambda_2, \lambda_3$, and positive constants p, c_1, c_2 with $\lambda_2 > \lambda_3$, such that

$$c_1 |x|^p \leq V(x, t, i) \leq c_2 |x|^p, \quad (13)$$

and there also exists $\varepsilon > 0$ such that

$$\begin{aligned} LV(x, y, t, i) \leq & \lambda_1 - \lambda_2 V(x, t, i) \\ & + \lambda_3 (1 - \bar{\delta}) e^{-\varepsilon \delta(t)} V(y, t - \delta(t), i), \end{aligned} \quad (14)$$

for $x, y \in R^d, t \in R_+$, and $i \in S$.

Assumption 5. For the same ε in Assumption 4, there is a constant $\beta \in (0, 1)$ such that

$$|u(x(t - \delta(t)), t, r(t))| \leq \beta e^{-\varepsilon \delta(t)/p} |x(t - \delta(t))|, \quad (15)$$

for $x, y \in R^d, t \geq 0$, and $i \in S$.

Moreover, inequality (15) implies that

$$|u(x(t), t, i)| \leq \beta |x(t)|. \quad (16)$$

3. Main Results

In this section, we will obtain the p th moment and almost sure exponential stability criteria of the solution of model (2) under Assumptions 2–5. In the following theorem, we will obtain the existence and uniqueness for the global solution of model (2).

Theorem 6. Let Assumptions 2–5 be satisfied; then, for initial condition (3), there exists a unique global solution $x = x(t); t \geq 0$ of model (2) with the following properties hold:

$$E|x(t) - u(x(t - \delta(t)), t, r(t))|^p < \infty, \quad t \geq 0, \quad (17)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t E|x(s)|^p ds \leq \frac{\lambda_1}{c_2(\lambda_2 - \lambda_3)}. \quad (18)$$

Furthermore, if $\lambda_1 = 0$, then the solution x obeys that

$$\int_0^\infty E |x(t)|^p dt \leq \frac{1}{c_1(\lambda_2 - \lambda_3)} \cdot E \left(|x(0) - u(x(-\delta(0)), 0, r(0))|^p \right. \quad (19)$$

$$\left. + \lambda_3 E \int_{-\delta(0)}^0 |x(s)|^p ds \right), \quad (20)$$

$$\int_0^\infty |x(t)|^p dt < \infty \quad a.s.$$

Proof. It is well-known that, for a given initial condition (3), Assumption 2 supports the existence and uniqueness of maximal local solution $x(t)$ of (2) on $t \in [0, \tau_e)$, where τ_e is the explosion time.

Since x_0 is bounded, there is a positive constant k_0 such that $|x(0)| = |x_0| < k_0$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf \{t \in [0, \tau_e) : |x(t)| \geq k\}, \quad (21)$$

$$\inf \emptyset = \infty.$$

Obviously, τ_k is increasing as $k \rightarrow \infty$. Define $\tau_\infty = \lim_{k \rightarrow \infty} \tau_k$, where $\tau_\infty \leq \tau_e$, a.s.

In order to show that the solution $x(t)$, $t \in [0, \infty)$, does not explode in finite time, we need to prove that $\tau_\infty = \infty$ a.s., which implies $\tau_e = \infty$ a.s.

By the generalized Itô formula (7) and condition (14), we obtain that, for any $t \geq 0$,

$$\begin{aligned} dV(x(t) - u(x(t - \delta(t)), t, r(t)), t, r(t)) \\ = LV(x(t), x(t - \delta(t)), t, r(t)) dt + dM_1(t) \\ \leq \lambda_1 - \lambda_2 V(x(t), t, r(t)) + \lambda_3 (1 - \bar{\delta}) \\ \cdot e^{-\varepsilon \delta(t)} V(x(t - \delta(t)), t - \delta(t), r(t - \delta(t))) \\ + dM_1(t), \end{aligned} \quad (22)$$

where

$$\begin{aligned} M_1(t) = \int_0^t V_x(x(s) - u(x(s - \delta(s)), s, r(s)), s, r(s)) g(x(s), \\ x(s - \delta(s)), s, r(s)) dw(s) \\ + \int_0^t \int_R (V(x(s) - u(x(s - \delta(s)), s, r(s)), s, i_0 + h(r(s), l)) \\ - V(x(s) - u(x(s - \delta(s)), s, r(s)), s, r(s))) \mu(ds, dl) \end{aligned} \quad (23)$$

is a local martingale with $M_1(0) = 0$.

For any $k > k_0$, integrating both sides of (22) and then taking expectation, we get that, for any $t \geq 0$,

$$\begin{aligned} EV(x(\tau_k \wedge t) - u(x(\tau_k \wedge t - \delta(\tau_k \wedge t)), \tau_k \\ \wedge t, r(\tau_k \wedge t)) \tau_k \wedge t, r(\tau_k \wedge t)) \leq EV(x(0) \\ - u(x(-\delta(0)), 0, r(0)), 0, r(0)) + \lambda_1 t \\ - \lambda_2 E \int_0^{\tau_k \wedge t} V(x(s), s, r(s)) ds + \lambda_3 (1 - \bar{\delta}) \\ \cdot E \int_0^{\tau_k \wedge t} e^{-\varepsilon \delta(s)} V(x(s - \delta(s)), s \\ - \delta(s), r(s - \delta(s))) ds. \end{aligned} \quad (24)$$

Applying (13) to the first term on the right side of (24), we have

$$\begin{aligned} EV(x(0) - u(x(-\delta(0)), 0, r(0)), 0, r(0)) \\ \leq c_2 |x(0) - u(x(-\delta(0)), 0, r(0))|^p. \end{aligned} \quad (25)$$

On the other hand, according to Assumption 3 and inequality $e^{-\varepsilon \delta(t)} < 1$, we get

$$\begin{aligned} \lambda_3 (1 - \bar{\delta}) E \int_0^{\tau_k \wedge t} e^{-\varepsilon \delta(s)} V(x(s - \delta(s)), s \\ - \delta(s), r(s - \delta(s))) ds \leq \lambda_3 (1 - \bar{\delta}) \\ \cdot E \int_0^{\tau_k \wedge t} V(x(s - \delta(s)), s \\ - \delta(s), r(s - \delta(s))) ds \\ \leq \lambda_3 E \int_{-\delta(0)}^0 V(x(s), s, r(s)) ds \\ + \lambda_3 E \int_0^{\tau_k \wedge t} V(x(s), s, r(s)) ds. \end{aligned} \quad (26)$$

Substituting (25) and (26) into (24), it follows that

$$\begin{aligned} 0 \leq c_1 E |x(t \wedge \tau_k) \\ - u(x(t \wedge \tau_k - \delta(t \wedge \tau_k)), t \wedge \tau_k, r(t \wedge \tau_k))|^p \\ \leq EV(x(t \wedge \tau_k) \\ - u(x(t \wedge \tau_k - \delta(t \wedge \tau_k)), t \wedge \tau_k, r(t \wedge \tau_k)), t \\ \wedge \tau_k, r(t \wedge \tau_k)) \leq c_2 E |x(0) \\ - u(x(-\delta(0)), 0, r(0))|^p \\ + \lambda_3 E \int_{-\delta(0)}^0 V(x(s), s, r(s)) ds + \lambda_1 t - (\lambda_2 \\ - \lambda_3) E \int_0^{\tau_k \wedge t} V(x(s), s, r(s)) ds. \end{aligned} \quad (27)$$

Bearing in mind that $\lambda_2 > \lambda_3$ in Assumption 4, we obtain

$$\begin{aligned} & c_1 E |x(\tau_k \wedge t) \\ & - u(x(\tau_k \wedge t - \delta(\tau_k \wedge t)), \tau_k \wedge t, r(\tau_k \wedge t))|^p \\ & \leq K_1 + \lambda_1 t, \end{aligned} \quad (28)$$

where $K_1 = c_2 E |x(0) - u(x(-\delta(0)), 0, r(0))|^p + \lambda_3 E \int_{-\delta(0)}^0 V(x(s), s, r(s)) ds$.

Define $\mu_k = \inf_{|y| \geq (1-\beta)k, t \geq 0} c_1 |y|^p$. By (16), we get

$$\begin{aligned} & |x(\tau_k) - u(x(\tau_k - \delta(\tau_k)), \tau_k, r(\tau_k))| I_{\{\tau_k \leq t\}} \\ & \geq (|x(\tau_k)| - |u(x(\tau_k - \delta(\tau_k)), \tau_k, r(\tau_k))|) I_{\{\tau_k \leq t\}} \\ & \geq (|x(\tau_k)| - \beta |x(\tau_k - \delta(\tau_k))|) I_{\{\tau_k \leq t\}} \\ & \geq (1 - \beta) k. \end{aligned} \quad (29)$$

On the basis of (28) and (29), we have

$$\begin{aligned} & K_1 + \lambda_1 t \geq c_1 E |x(\tau_k \wedge t) \\ & - u(x(\tau_k \wedge t - \delta(\tau_k \wedge t)), \tau_k \wedge t, r(\tau_k \wedge t))|^p \\ & \geq E [c_1 |x(\tau_k) - u(x(\tau_k - \delta(\tau_k)), \tau_k, r(\tau_k))|^p \\ & \cdot I_{\{\tau_k \leq t\}}] \geq E \left[\inf_{|y| \geq (1-\beta)k, t \geq 0} c_1 |y|^p I_{\{\tau_k \leq t\}} \right] = \mu_k P\{\tau_k \\ & \leq t\}. \end{aligned} \quad (30)$$

Remarking that $\lim_{|x| \rightarrow \infty} c_1 |x|^p \rightarrow \infty$, then $\lim_{k \rightarrow \infty} \mu_k = \infty$. Letting $k \rightarrow \infty$ in (30), we have $P\{\tau_\infty \leq t\} = 0$; that is, $P\{\tau_\infty > t\} = 1$. For the arbitrariness of t , we know that $P\{\tau_\infty = \infty\} = 1$, which implies that $\tau_\infty = \infty$ a.s. as required.

We have established the existence and uniqueness of the global solution, and now we further verify its properties.

Noting that inequality (27) yields

$$(\lambda_2 - \lambda_3) E \int_0^{\tau_k \wedge t} V(x(s), s, r(s)) ds \leq K_1 + \lambda_1 t. \quad (31)$$

Letting $k \rightarrow \infty$, we get

$$(\lambda_2 - \lambda_3) \int_0^t EV(x(s), s, r(s)) ds \leq K_1 + \lambda_1 t. \quad (32)$$

Dividing both sides of (32) by $(\lambda_2 - \lambda_3)t$ and letting $t \rightarrow \infty$, combining with (13) we obtain the assertion (18).

Particularly, when $\lambda_1 = 0$, from (31), we have

$$c_1 (\lambda_2 - \lambda_3) E \int_0^{\tau_k \wedge t} |x(s)|^p ds \leq K_1. \quad (33)$$

Dividing both sides of this inequality by $(\lambda_2 - \lambda_3)$, and letting $t \rightarrow \infty$, we get

$$\begin{aligned} & \int_0^\infty E |x(t)|^p dt \leq \frac{K_1}{c_1 (\lambda_2 - \lambda_3)} \\ & = \frac{1}{c_1 (\lambda_2 - \lambda_3)} \left(E |x(0) \right. \\ & \left. - u(x(-\delta(0)), 0, r(0))|^p + \lambda_3 E \int_{-\delta(0)}^0 |x(s)|^p ds \right), \end{aligned} \quad (34)$$

which gives the desired estimate (19).

By Fubini's theorem, we obtain

$$E \int_0^\infty |x(t)|^p dt \leq \frac{K_1}{c_1 (\lambda_2 - \lambda_3)}; \quad (35)$$

this implies

$$\int_0^\infty |x(t)|^p dt < \infty, \quad \text{a.s.} \quad (36)$$

which is (20). $\square$

Remark 7. The results of Theorem 6 are similar to those of Theorem 1 in [20] in form, but Assumption 4 in this paper is different from $\mathcal{A}2$ in [20].

Theorem 8. Let Assumptions 2–5 be satisfied; then, for initial condition (3), the unique solution x of (2) has the property

$$\limsup_{t \rightarrow \infty} E |x(t) - u(x(t - \delta(t)), t, r(t))|^p \leq \frac{\lambda_1}{c_1 \varepsilon}, \quad (37)$$

where $0 < \varepsilon < \varepsilon_0$, while ε_0 is the unique root to the equation

$$\lambda_2 = \lambda_3 + \frac{c_2 \varepsilon (1 + \beta)^p}{c_1 (1 - \delta)}. \quad (38)$$

If, furthermore, $\lambda_1 = 0$, then the solution x obeys that

$$\limsup_{t \rightarrow \infty} \frac{\ln E |x(t) - u(x(t - \delta(t)), t, r(t))|^p}{t} \leq -\varepsilon, \quad (39)$$

$$\limsup_{t \rightarrow \infty} \frac{\ln |x(t) - u(x(t - \delta(t)), t, r(t))|^p}{t} \leq -\varepsilon, \quad (40)$$

a.s.

Proof. By the generalized Itô formula, for any $0 < \varepsilon < \varepsilon_0$, we have

$$\begin{aligned} & d \left[e^{\varepsilon t} V(x(t) - u(x(t - \delta(t)), t, r(t)), t, r(t)) \right] \\ & = e^{\varepsilon t} \varepsilon V(x(t) - u(x(t - \delta(t)), t, r(t)), t, r(t)) dt \\ & \quad + e^{\varepsilon t} LV(x(t), x(t - \delta(t)), t, r(t)) dt \\ & \quad + e^{\varepsilon t} dM_2(t), \end{aligned} \quad (41)$$

where

$$\begin{aligned}
M_2(t) = & \int_0^t e^{\varepsilon s} \left\{ V_x(x(s) - u(x(s - \delta(s)), s, r(s)), s, r(s)) g(x(s), x(s - \delta(s)), s, r(s)) dw(s) \right. \\
& + \int_R [V(x(s) - u(x(s - \delta(s)), s, r(s)), s, i_0 + h(r(s), l)) - V(x(s) - u(x(s - \delta(s)), s, r(s)), s, r(s))] \\
& \cdot \mu(ds, dl) \Big\}
\end{aligned} \quad (42)$$

is a local martingale with $M_2(0) = 0$.

For any $k \geq k_0$, let τ_k be the stopping time defined in the proof of Theorem 6. Integrating both sides of (41), and then taking expectation, we get that, for any $t \geq 0$,

$$\begin{aligned}
& E(e^{\varepsilon(t \wedge \tau_k)} V(x(t \wedge \tau_k) \\
& - u(x(t \wedge \tau_k - \delta(t \wedge \tau_k)), t \wedge \tau_k, r(t \wedge \tau_k)), t \\
& \wedge \tau_k, r(t \wedge \tau_k)) = EV(x(0) \\
& - u(x(-\delta(0)), 0, r(0)), 0, r(0)) \\
& + E \int_0^{t \wedge \tau_k} e^{\varepsilon s} \varepsilon V(x(s) \\
& - u(x(s - \delta(s)), s, r(s)), s, r(s)) ds \\
& + E \int_0^{t \wedge \tau_k} e^{\varepsilon s} LV(x(s), x(s - \delta(s)), s, r(s)) ds.
\end{aligned} \quad (43)$$

By Assumption 3, we get

$$\begin{aligned}
& E \int_0^{t \wedge \tau_k} e^{\varepsilon s} LV(x(s), x(s - \delta(s)), s, r(s)) ds \\
& \leq E \int_0^{t \wedge \tau_k} e^{\varepsilon s} [\lambda_1 - \lambda_2 V(x(s), s, r(s)) + \lambda_3 (1 \\
& - \bar{\delta}) e^{-\varepsilon \delta(s)} V(x(s - \delta(s)), s \\
& - \delta(s), r(s - \delta(s)))] ds \leq \frac{\lambda_1}{\varepsilon} e^{\varepsilon t} \\
& - \lambda_2 E \int_0^{t \wedge \tau_k} e^{\varepsilon s} V(x(s), s, r(s)) ds + \lambda_3 (1 - \bar{\delta}) \\
& \cdot E \int_0^{t \wedge \tau_k} e^{\varepsilon s} e^{-\varepsilon \delta(s)} V(x(s - \delta(s)), s - \delta(s), r(s) \\
& - \delta(s)) ds \leq \frac{\lambda_1}{\varepsilon} e^{\varepsilon t} - \lambda_2 E \int_0^{t \wedge \tau_k} e^{\varepsilon s} V(x(s), s, \\
& r(s)) ds + \lambda_3 E \int_{-\delta(0)}^0 e^{\varepsilon s} V(x(s), s, r(s)) ds \\
& + \lambda_3 E \int_0^{t \wedge \tau_k} e^{\varepsilon s} V(x(s), s, r(s)) ds \leq \frac{\lambda_1}{\varepsilon} e^{\varepsilon t}
\end{aligned}$$

$$\begin{aligned}
& + \lambda_3 E \int_{-\delta(0)}^0 e^{\varepsilon s} V(x(s), s, r(s)) ds - (\lambda_2 - \lambda_3) \\
& \cdot E \int_0^{t \wedge \tau_k} e^{\varepsilon s} V(x(s), s, r(s)) ds \leq \frac{\lambda_1}{\varepsilon} e^{\varepsilon t} \\
& + c_2 \lambda_2 E \int_{-\delta(0)}^0 e^{\varepsilon s} |x(s)|^p ds - c_1 (\lambda_2 - \lambda_3) \\
& \cdot E \int_0^{t \wedge \tau_k} e^{\varepsilon s} |x(s)|^p ds.
\end{aligned} \quad (44)$$

Next, we will estimate the second part of (43). Applying Proposition 1, as well as (13) and (15), we have the following expression when $0 < p \leq 1$:

$$\begin{aligned}
& E \int_0^{t \wedge \tau_k} e^{\varepsilon s} \varepsilon V(x(s) \\
& - u(x(s - \delta(s)), s, r(s)), s, r(s)) ds \\
& \leq c_2 E \int_0^{t \wedge \tau_k} e^{\varepsilon s} \varepsilon |x(s) \\
& - u(x(s - \delta(s)), s, r(s))|^p ds \\
& \leq c_2 E \int_0^{t \wedge \tau_k} e^{\varepsilon s} \varepsilon (|x(s)|^p \\
& + |u(x(s - \delta(s)), s, r(s))|^p) ds \\
& \leq c_2 \varepsilon E \int_0^{t \wedge \tau_k} e^{\varepsilon s} |x(s)|^p ds \\
& + c_2 \varepsilon E \int_0^{t \wedge \tau_k} e^{\varepsilon s} |u(x(s - \delta(s)), s, r(s))|^p ds \\
& \leq c_2 \varepsilon E \int_0^{t \wedge \tau_k} e^{\varepsilon s} |x(s)|^p ds \\
& + c_2 \varepsilon \beta^p E \int_0^{t \wedge \tau_k} e^{\varepsilon(s - \delta(s))} |x(s - \delta(s))|^p ds \\
& \leq \frac{1}{1 - \bar{\delta}} c_2 \varepsilon E \int_0^{t \wedge \tau_k} e^{\varepsilon s} |x(s)|^p ds + \frac{1}{1 - \bar{\delta}} \\
& \cdot c_2 \varepsilon \beta^p E \int_{-\delta(0)}^0 e^{\varepsilon s} |x(s)|^p ds + \frac{1}{1 - \bar{\delta}} \\
& \cdot c_2 \varepsilon \beta^p E \int_0^{t \wedge \tau_k} e^{\varepsilon s} |x(s)|^p ds \leq \frac{1}{1 - \bar{\delta}} c_2 \varepsilon (1 + \beta)^p
\end{aligned}$$

$$\begin{aligned}
& \cdot E \int_{-\delta(0)}^0 e^{\varepsilon s} |x(s)|^p ds + \frac{1}{1-\delta} c_2 \varepsilon (1+\beta)^p \\
& \cdot E \int_0^{t \wedge \tau_k} e^{\varepsilon s} |x(s)|^p ds.
\end{aligned} \tag{45}$$

While for $p > 1$, applying (10) and letting $\eta = \beta$ in Assumption 5, we get

$$\begin{aligned}
& E \int_0^{t \wedge \tau_k} e^{\varepsilon s} \varepsilon V(x(s)) \\
& - u(x(s-\delta(s)), s, r(s)), s, r(s)) ds \\
& \leq c_2 E \int_0^{t \wedge \tau_k} e^{\varepsilon s} \varepsilon |x(s)|^p ds \\
& - u(x(s-\delta(s)), s, r(s))|^p ds \\
& \leq c_2 E \int_0^{t \wedge \tau_k} e^{\varepsilon s} \varepsilon ((1+\beta)^{p-1} |x(s)|^p + (1+\beta)^{p-1} \\
& \cdot \beta^{1-p} |u(x(s-\delta(s)), s, r(s))|^p) ds \leq c_2 \varepsilon (1 \\
& + \beta)^{p-1} E \int_0^{t \wedge \tau_k} e^{\varepsilon s} |x(s)|^p ds + c_2 \varepsilon (1+\beta)^{p-1} \\
& \cdot \beta E \int_0^{t \wedge \tau_k} e^{\varepsilon(s-\delta(s))} |x(s-\delta(s))|^p ds \leq \frac{1}{1-\delta} \\
& \cdot c_2 \varepsilon (1+\beta)^{p-1} E \int_0^{t \wedge \tau_k} e^{\varepsilon s} |x(s)|^p ds + \frac{1}{1-\delta} c_2 \varepsilon (1 \\
& + \beta)^{p-1} \beta E \int_{-\delta(0)}^0 e^{\varepsilon s} |x(s)|^p ds + \frac{1}{1-\delta} c_2 \varepsilon (1 \\
& + \beta)^{p-1} \beta E \int_0^{t \wedge \tau_k} e^{\varepsilon s} |x(s)|^p ds \leq \frac{1}{1-\delta} c_2 \varepsilon (1+\beta)^p \\
& \cdot E \int_{-\delta(0)}^0 e^{\varepsilon s} |x(s)|^p ds + \frac{1}{1-\delta} c_2 \varepsilon (1+\beta)^p \\
& \cdot E \int_0^{t \wedge \tau_k} e^{\varepsilon s} |x(s)|^p ds.
\end{aligned} \tag{46}$$

From (45) and (46), we get, for any $p > 0$,

$$\begin{aligned}
& E \int_0^{t \wedge \tau_k} e^{\varepsilon s} \varepsilon V(x(s)) \\
& - u(x(s-\delta(s)), s, r(s)), s, r(s)) ds \leq \frac{1}{1-\delta} \\
& \cdot c_2 \varepsilon (1+\beta)^p E \int_{-\delta(0)}^0 e^{\varepsilon s} |x(s)|^p ds + \frac{1}{1-\delta} c_2 \varepsilon (1 \\
& + \beta)^p E \int_0^{t \wedge \tau_k} e^{\varepsilon s} |x(s)|^p ds.
\end{aligned} \tag{47}$$

Substituting (44) and (47) into (43), we obtain that

$$\begin{aligned}
& E \left(e^{\varepsilon(t \wedge \tau_k)} V(x(t \wedge \tau_k)) \right. \\
& \left. - u(x(t \wedge \tau_k - \delta(t \wedge \tau_k)), t \wedge \tau_k, r(t \wedge \tau_k)), t \right. \\
& \left. \wedge \tau_k, r(t \wedge \tau_k)) \right) \leq c_2 E |x(0)|^p \\
& - u(x(-\delta(0)), 0, r(0))|^p + \frac{c_2}{1-\delta} (\lambda_3 + \varepsilon (1 \\
& + \beta)^p) E \int_{-\delta(0)}^0 e^{\varepsilon s} |x(s)|^p ds + \frac{\lambda_1}{\varepsilon} e^{\varepsilon t} - \left[c_1 (\lambda_2 \right. \\
& \left. - \lambda_3) - \frac{c_2 \varepsilon}{1-\delta} (1+\beta)^p \right] E \int_0^{t \wedge \tau_k} e^{\varepsilon s} |x(s)|^p ds.
\end{aligned} \tag{48}$$

Bearing in mind that ε_0 is the unique root to (38), as well as (13), we see, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned}
& c_1 E \left(e^{\varepsilon(t \wedge \tau_k)} |x(t \wedge \tau_k) \right. \\
& \left. - u(x(t \wedge \tau_k - \delta(t \wedge \tau_k)), t \wedge \tau_k, r(t \wedge \tau_k))|^p \right) \\
& \leq K_2 + \frac{\lambda}{\varepsilon} e^{\varepsilon t},
\end{aligned} \tag{49}$$

where

$$\begin{aligned}
& K_2 \\
& = c_2 E |x(0) - u(x(-\delta(0)), 0, r(0))|^p \\
& + \frac{c_2}{1-\delta} (\lambda_3 + \varepsilon (1+\beta)^p) E \int_{-\delta(0)}^0 e^{\varepsilon s} |x(s)|^p ds.
\end{aligned} \tag{50}$$

Then letting $k \rightarrow \infty$, the inequality becomes

$$c_1 e^{\varepsilon t} E |x(t) - u(x(t-\delta(t)), t, r(t))|^p \leq K_2 + \frac{\lambda_1}{\varepsilon} e^{\varepsilon t}. \tag{51}$$

Dividing both sides of (51) by $e^{\varepsilon t}$, and letting $t \rightarrow \infty$, it leads to

$$\limsup_{t \rightarrow \infty} E |x(t) - u(x(t-\delta(t)), t, r(t))|^p \leq \frac{\lambda_1}{c_1 \varepsilon}, \tag{52}$$

which is (37).

In the case when $\lambda_1 = 0$, from (51), we have

$$c_1 E |x(t) - u(x(t-\delta(t)), t, r(t))|^p \leq \frac{K_2}{e^{\varepsilon t}}, \tag{53}$$

which can get (39).

Now we show (40). Rewriting the integral form of (41), we have

$$\begin{aligned}
& e^{\varepsilon t} V(x(t) - u(x(t - \delta(t)), t, r(t)), t, r(t)) \\
& - V(x(0) - u(x(-\delta(0)), 0, r(0)), 0, r(0)) \\
& = \int_0^t \left[e^{\varepsilon s} \varepsilon V(x(s) \right. \\
& \quad \left. - u(x(s - \delta(s)), s, r(s)), s, r(s)) \right. \\
& \quad \left. + LV(x(s), x(s - \delta(s)), s, r(s)) \right] ds \\
& \quad + M_2(t),
\end{aligned} \tag{54}$$

where $M_2(t)$ is the same as (42). Proceeding as in the proof before, with $\lambda_1 = 0$, we can state that

$$c_1 e^{\varepsilon t} |x(t) - u(x(t - \delta(t)), t, r(t))|^p \leq K_2 + M_2(t). \tag{55}$$

By the nonnegative semimartingale convergence theorem (see [26]), we have

$$\limsup_{t \rightarrow \infty} \left[c_1 e^{\varepsilon t} |x(t) - u(x(t - \delta(t)), t, r(t))|^p \right] < \infty, \tag{56}$$

a.s.,

which means there exists a finite positive random variable η , such that

$$\sup_{0 \leq t < \infty} \left[e^{\varepsilon t} |x(t) - u(x(t - \delta(t)), t, r(t))|^p \right] \leq \eta, \tag{57}$$

a.s.

This leads to the desired estimate (40). $\square$

Remark 9. In Theorem 8, we obtain the results similar to those of Theorem 2 in [20] formally, but the delay $\delta(t)$ considered in this paper is unbounded, while the delay function $\delta(t)$ in [20] is bounded. $e^{-\varepsilon \delta(t)}$ plays an important role in dealing with problems of unboundedness. Moreover, the expressions of both $0 < p \leq 1$ and $p > 1$ are unified in the estimation (47), which is important for the proof of the results.

The following result will establish p th moment exponential stability and almost sure exponential stability of the solution x to system (2).

Theorem 10. *Let Assumptions 2–5 be satisfied, with $\lambda_1 = 0$; then for any constant $p > 0$ and initial condition (3), the unique solution x of model (2) obeys*

$$\limsup_{t \rightarrow \infty} \frac{\ln(E|x(t)|^p)}{t} \leq -\varepsilon, \tag{58}$$

$$\limsup_{t \rightarrow \infty} \frac{\ln|x(t)|}{t} \leq -\frac{\varepsilon}{p} \quad \text{a.s.}, \tag{59}$$

where $\varepsilon \in (0, \varepsilon_0)$, while ε_0 is the unique positive root to the equation $\lambda_2 = \lambda_3 + c_2 \varepsilon (1 + \beta)^p / c_1 (1 - \bar{\delta})$.

Proof. By recalling (53), we know that

$$c_1 E |x(t) - u(x(t - \delta(t)), t, r(t))|^p \leq \frac{K_2}{e^{\varepsilon t}}. \tag{60}$$

Letting $K_3 = K_2/c_1$, we have

$$e^{\varepsilon t} E |x(t) - u(x(t - \delta(t)), t, r(t))|^p \leq K_3. \tag{61}$$

Now, we need to discuss two cases when $0 < p \leq 1$ and $p > 1$.

For $0 < p \leq 1$, by (9), (15), and (61), we obtain

$$\begin{aligned}
e^{\varepsilon t} E |x(t)|^p & \leq e^{\varepsilon t} E |x(t) - u(x(t - \delta(t)), t, r(t))|^p \\
& \quad + e^{\varepsilon t} E |u(x(t - \delta(t)), t, r(t))|^p \\
& \leq K_3 + e^{\varepsilon t} E |u(x(t - \delta(t)), t, r(t))|^p \\
& \leq K_3 + \beta^p e^{\varepsilon(t - \delta(t))} E |x(t - \delta(t))|^p.
\end{aligned} \tag{62}$$

From (62), for any $T > 0$, we have

$$\begin{aligned}
& \sup_{0 \leq t \leq T} e^{\varepsilon t} E |x(t)|^p \leq K_3 \\
& \quad + \beta^p \sup_{0 \leq t \leq T} e^{\varepsilon(t - \delta(t))} E |x(t - \delta(t))|^p \\
& \leq K_3 + \beta^p \sup_{-\delta(0) \leq t \leq 0} e^{\varepsilon t} E |x(t)|^p \\
& \quad + \beta^p \sup_{0 \leq t \leq T} e^{\varepsilon t} E |x(t)|^p \\
& \leq K_3 + \beta^p k_0^p \\
& \quad + \beta^p \sup_{0 \leq t \leq T} e^{\varepsilon t} E |x(t)|^p,
\end{aligned} \tag{63}$$

where k_0 is a upper bound of initial condition (3) as defined in the proof of Theorem 6. Noting that $1 - \beta^p > 0$, we have

$$\begin{aligned}
\sup_{0 \leq t \leq T} e^{\varepsilon t} E |x(t)|^p & \leq \frac{K_3 + \beta^p k_0^p}{1 - \beta^p} \\
& \leq \frac{(1 + (1 - \beta)^{1-p}) K_3 + k_0^p}{1 - \beta^p}.
\end{aligned} \tag{64}$$

For the case when $p > 1$, letting $\eta = \beta/(1 - \beta)$ in (10), where β is defined in Assumption 5, from (61), we obtain

$$\begin{aligned}
& e^{\varepsilon t} E |x(t)|^p \\
& \leq (1 - \beta)^{1-p} e^{\varepsilon t} E |x(t) - u(x(t - \delta(t)), t, r(t))|^p \\
& \quad + \beta^{1-p} e^{\varepsilon t} E |u(x(t - \delta(t)), t, r(t))|^p \\
& \leq (1 - \beta)^{1-p} K_3 + \beta e^{\varepsilon(t - \delta(t))} E |x(t - \delta(t))|^p.
\end{aligned} \tag{65}$$

Inequality (65) follows that, for any $T > 0$,

$$\begin{aligned}
\sup_{0 \leq t \leq T} e^{\varepsilon t} E |x(t)|^p &\leq (1 - \beta)^{1-p} K_3 \\
&\quad + \beta \sup_{0 \leq t \leq T} e^{\varepsilon(t-\delta(t))} E |x(t - \delta(t))|^p \\
&\leq (1 - \beta)^{1-p} K_3 \\
&\quad + \beta \sup_{-\delta(0) \leq t \leq 0} e^{\varepsilon t} E |x(t)|^p \\
&\quad + \beta \sup_{0 \leq t \leq T} e^{\varepsilon t} E |x(t)|^p \\
&\leq (1 - \beta)^{1-p} K_3 + k_0^p \\
&\quad + \beta \sup_{0 \leq t \leq T} e^{\varepsilon t} E |x(t)|^p.
\end{aligned} \tag{66}$$

From (66), we have

$$\sup_{0 \leq t \leq T} e^{\varepsilon t} E |x(t)|^p \leq \frac{(1 + (1 - \beta)^{1-p}) K_3 + k_0^p}{1 - \beta^p}. \tag{67}$$

Letting $T \rightarrow \infty$ in (64) and (67), we get, for any $p > 0$,

$$\sup_{0 \leq t \leq \infty} e^{\varepsilon t} E |x(t)|^p \leq \frac{(1 + (1 - \beta)^{1-p}) K_3 + k_0^p}{1 - \beta^p}. \tag{68}$$

A routine computation to (68) gives rise to the desired estimate (58).

By (57), proceeding as in the proof of (58), we obtain that, for $0 < p \leq 1$,

$$\sup_{0 \leq t \leq \infty} e^{\varepsilon t} |x(t)|^p \leq \frac{\eta/c_1 + k_0^p}{1 - \beta^p}, \quad \text{a.s.}, \tag{69}$$

and for $p > 1$,

$$\sup_{0 \leq t \leq \infty} e^{\varepsilon t} |x(t)|^p \leq \frac{(1 - \beta)^{1-p} \eta/c_1 + k_0^p}{1 - \beta}, \quad \text{a.s.} \tag{70}$$

By (69) and (70), we have

$$\limsup_{t \rightarrow \infty} \frac{\ln |x(t)|}{t} \leq -\frac{\varepsilon}{p} \quad \text{a.s.} \tag{71}$$

which is (59). $\square$

Remark 11. In Theorem 10, we improve the results of Theorem 3 in [20] with the condition of delay functions from being bounded to unbounded. The proof of our results is different from that in [20] due to the unboundedness of delay functions.

4. Example

In this section, a numerical example is given to illustrate the effectiveness of the obtained results.

To avoid the tedious computation, we consider the example when $p = 2$, it is appropriate to illustrate the results. As a premise, in model (2) with initial condition (3), provide that $w(t)$ is a scalar Brownian motion; $r(t)$ is a right-continuous Markov chain on the state space $S = \{1, 2\}$ with the generator $\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$.

Letting $\delta(t) = 0.5t + 1$, it is clear that Assumption 3 holds. Define

$$\begin{aligned}
f(x, y, t, 1) &= -9x - 0.2125e^{-0.5t}y, \\
f(x, y, t, 2) &= -8x - 0.475e^{-0.5t}y, \\
g(x, y, t, 1) &= 0.4e^{-0.5t}y, \\
g(x, y, t, 2) &= 0.3e^{-0.5t}y, \\
u(x, t, 1) &= 0.025e^{-0.5t}x, \\
u(x, t, 2) &= 0.05e^{-0.5t}x.
\end{aligned} \tag{72}$$

Then the functions f and g satisfy Assumption 2, while the neutral term $u : (R \times R_+ \times S) \rightarrow R_+$ obeys Assumption 5 with $\beta = 0.2$.

For the sake of showing our problem well and truly, we define $V(x, t, 1) = x^2$, $V(x, t, 2) = 2x^2$. That is in support of conditions (13) with $c_1 = 1$, $c_2 = 2$ in Assumption 4.

Under the expressions aforesaid, as well as formula (5), we compute

$$\begin{aligned}
LV(x, y, t, 1) &\leq -17x^2 + 2.6 \times 0.5e^{-2(0.5t+1)}y^2, \\
LV(x, y, t, 2) &\leq -17 \times (2x^2) + 2.6 \\
&\quad \times 0.5e^{-2(0.5t+1)}(2y^2).
\end{aligned} \tag{73}$$

Consequently, we infer the fact that, for all $x, y \in R$, $t \in R_+$, and $i \in S$, condition (14) holds with parameters $\bar{\delta} = 0.5$, $\lambda_1 = 0$, $\lambda_2 = 17$, and $\lambda_3 = 2.6$. Furthermore, we get $\varepsilon_0 = 2.5$ from (38). So it could be a consideration of the case when $\varepsilon = 2$.

Based on the above argument, we can show that the example is advisable, and model (2) comes into following scalar equation:

$$\begin{aligned}
&d[x(t) - 0.025e^{-0.5t}x(0.5t - 1)] \\
&= [-9x(t) - 0.2125e^{-0.5t}x(0.5t - 1)]dt \\
&\quad - 0.4e^{-0.5t}x(0.5t - 1)dw(t), \\
&d[x(t) - 0.05e^{-0.5t}x(0.5t - 1)] \\
&= [-8x(t) - 0.475e^{-0.5t}x(0.5t - 1)]dt \\
&\quad - 0.3e^{-0.5t}x(0.5t - 1)dw(t).
\end{aligned} \tag{74}$$

FIGURE 1: Computer simulation of $\ln |x(t) - u(x(t - \delta(t)), t, r(t))|/t$ of the solution $x(t)$ of (74).

FIGURE 2: Computer simulation of $\ln |x(t)|/t$ of the solution $x(t)$ of (74).

Applying Theorems 8 and 10, we can conclude that the solution of (74) obeys

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln |x(t) - u(x(t - \delta(t)), t, r(t))|^p}{t} &\leq -\varepsilon \\ &= -2, \quad \text{a.s.,} \\ \limsup_{t \rightarrow \infty} \frac{\ln |x(t)|}{t} &\leq -\frac{\varepsilon}{p} = -1 \quad \text{a.s.,} \end{aligned} \quad (75)$$

where $\varepsilon = 2$ and $p = 2$. Figures 1 and 2 give computer simulations of $\ln(|x(t)|)/t$ and $\ln(|x(t) - u(x(t - \delta(t)), t, r(t))|)/t$ for the solution of (74) using the Euler-Maruyama method (see [13]) with step size 0.01 and initial data $\phi(s) = 2 + s$, $s \in [-\delta(0), 0]$, $r(0) = 2$ for 200 samples. Figures 1 and 2 illustrate the results accurately.

Remark 12. We use the Euler-Maruyama method which was investigated in [20] to simulate the numerical example considered above. In [20], the Euler-Maruyama method was established under both bounded delay and unbounded delay conditions. The above example we considered satisfies not only the conditions of Assumptions 2–5 in this paper, but also the conditions of $\mathcal{A}1$ – $\mathcal{A}7$ in [20]; hence it is applicable to stimulate the example with the Euler-Maruyama method.

5. Conclusion

In this paper, we have improved the stability result of a class of hybrid stochastic neutral differential equations with unbounded delay. The method of Lyapunov functions and $e^{-\varepsilon \delta(t)}$ function are fully used to solve our problems. Our main contribution is generalizing the delay functions from bounded case to the unbounded in NSDE with time-dependent delay and Markovian switching. The concurrent consideration of neutral term, unbounded delay, and Markovian switching in this paper makes our results more general and adoptable to many practical systems. Compared with the existing literatures, one contribution of this paper is that our results advance the delay terms ($\delta(t) = \theta t, 0 < \theta < 1$) in [21] to general unbounded functions and add the neutral term. The other contribution is that we develop the p th and almost sure exponential stability criteria under the unbounded delay conditions, which generalizes and improves the results greatly in [20] where the delay function is limited by $\sup_{t \geq 0} \delta(t) = h < \infty$.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors would like to thank the Postdoctoral Research Funding Program of Jiangsu Province (1601147B), the National Natural Science Foundation of China (11201221), the Natural Science Foundation of Jiangsu Province (BK2012468, BK20161552), and the Postgraduate Innovation Research Project of Nanjing University of Finance and Economics (1) for their financial support.

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