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Research Article

Stability and Hopf Bifurcation Analysis for a Computer Virus Propagation Model with Two Delays and Vaccination

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A further generalization of an SEIQRS-V (susceptible-exposed-infectious-quarantined-recovered-susceptible with vaccination) computer virus propagation model is the main topic of the present paper. This paper specifically analyzes effects on the asymptotic dynamics of the computer virus propagation model when two time delays are introduced. Sufficient conditions for the asymptotic stability and existence of the Hopf bifurcation are established by regarding different combination of the two delays as the bifurcation parameter. Moreover, explicit formulas that determine the stability, direction, and period of the bifurcating periodic solutions are obtained with the help of the normal form theory and center manifold theorem. Finally, numerical simulations are employed for supporting the obtained analytical results.

1. Introduction

Computer viruses, including conventional viruses and network worms, can propagate among computers with no human awareness and popularization of Internet has been the major propagation channel of viruses [1, 2]. The past few decades have witnessed the great financial losses caused by computer viruses. Therefore, it is of considerable importance to investigate the laws describing propagation of computer viruses in order to provide some help with preventing computer viruses. For that purpose and in view of the fact that propagation of computer viruses among computers resembles that of biological viruses among a population, many dynamical models describing propagation of computer viruses across the Internet have been established by the scholars at home and abroad, such as conventional models [3–8], stochastic models [9–12], and delayed models [13–18]. There are also some other computer virus models [19–21] combined with network theory to investigate the impact of the network topology, the patch forwarding, and the network eigenvalue on the viral prevalence.

As is known, vaccination is regarded as one of the most effective measures of preventing computer viruses and the awareness that there exist many infected computers would enhance the probability that the user of a susceptible computer will make his computer vaccinated [22, 23]. However, the mentioned models above neglect the influence of vaccination strategy on the propagation of computer viruses. Recently, considering the importance of vaccination, Kumar et al. [24] proposed the following SEIQRS-V computer virus propagation model:

$$\begin{aligned}
\frac{dS(t)}{dt} &= A - \beta S(t) I(t) - dS(t) - \rho S(t) + \theta R(t) \\
&\quad + \chi V(t), \\
\frac{dE(t)}{dt} &= \beta S(t) I(t) - dE(t) - \gamma E(t), \\
\frac{dI(t)}{dt} &= \gamma E(t) - dI(t) - \alpha I(t) - \delta I(t) - \eta I(t), \\
\frac{dQ(t)}{dt} &= \delta I(t) - dQ(t) - \alpha Q(t) - \varepsilon Q(t), \\
\frac{dR(t)}{dt} &= \varepsilon Q(t) - dR(t) - \theta R(t) + \eta I(t), \\
\frac{dV(t)}{dt} &= \rho S(t) - dV(t) - \chi V(t),
\end{aligned} \tag{1}$$

where $S(t)$, $E(t)$, $I(t)$, $Q(t)$, $R(t)$, and $V(t)$ denote the numbers of the uninfected computers, the exposed computers, the infected computers, the quarantined computers, recovered computers, and vaccinated computers at time t, respectively. A is the birth rate of new computers in the network; d is the death rate of the computers due to the reason other than the attack of viruses; α is the death rate of computers due to the attack of viruses; β is the contact rate of the uninfected computers; ρ, θ, χ, γ, δ, η, and ε are the transition rates between the states in system (1).

Obviously, system (1) neglects the delays in the procedure of viruses' propagation and it is investigated under the assumption that the transition between the states is instantaneous. This is not reasonable with reality. For example, it needs a period to clean the viruses in the infected and quarantined computers for antivirus software and there is usually a temporary immunity period for the recovered and the vaccinated computers because of the effect of the antivirus software. In addition, a stability switch occurs even when an ignored delay is small for a dynamical system. Based on this, we introduce two delays into system (1) and get the following delayed system:

$$\begin{aligned}
\frac{dS(t)}{dt} &= A - \beta S(t) I(t) - dS(t) - \rho S(t) + \theta R(t-\tau_2) \\
&\quad + \chi V(t-\tau_2), \\
\frac{dE(t)}{dt} &= \beta S(t) I(t) - dE(t) - \gamma E(t), \\
\frac{dI(t)}{dt} &= \gamma E(t) - dI(t) - \alpha I(t) - \delta I(t) - \eta I(t-\tau_1), \\
\frac{dQ(t)}{dt} &= \delta I(t) - dQ(t) - \alpha Q(t) - \varepsilon Q(t-\tau_1), \\
\frac{dR(t)}{dt} &= \varepsilon Q(t-\tau_1) - dR(t) - \theta R(t-\tau_2) \\
&\quad + \eta I(t-\tau_1), \\
\frac{dV(t)}{dt} &= \rho S(t) - dV(t) - \chi V(t-\tau_2),
\end{aligned} \tag{2}$$

where τ_1 is the time delay due to the period that antivirus software uses to clean the viruses in the infected and quarantined computers and τ_2 is the time delay due to the temporary immunity period of the recovered and the vaccinated computers.

To the best of our knowledge, until now, there is no good analysis on system (2). Therefore, it is meaningful to analyze the proposed system with two delays.

The rest of this paper is organized as follows. In the next section, we analyze the threshold of Hopf bifurcation of system (2) by regarding different combination of the two delays as the bifurcation parameter. In Section 3, by means of the normal form theory and center manifold theorem, direction and stability of the Hopf bifurcation for $\tau_1 > 0$ and $\tau_2 > 0$ are investigated. Simulation results of system (2) are shown in Section 4. Finally, we finish the paper with conclusions in Section 5.

2. Analysis of Hopf Bifurcation

By direct computation, we know that if $AR_0(d+\chi) > d^2 + (\rho+\chi)d$ and $\beta(d+\theta)(d+\alpha+\varepsilon) > R_0\theta\varepsilon\delta + R_0\theta\eta(d+\alpha+\varepsilon)$, then system (2) has a unique viral equilibrium $P_*(S_*, E_*, I_*, Q_*, R_*, V_*)$, where

$$\begin{aligned}
S_* &= \frac{(d+\gamma)(d+\alpha+\delta+\eta)}{\beta\gamma} = \frac{1}{R_0}, \\
E_* &= \frac{d+\alpha+\delta+\eta}{\gamma} I_*, \\
R_* &= \frac{\varepsilon\delta + \eta(d+\alpha+\varepsilon)}{(d+\theta)(d+\alpha+\varepsilon)} I_*, \\
V_* &= \frac{\rho}{(d+\chi)R_0}, \\
Q_* &= \frac{\delta}{d+\alpha+\varepsilon} I_*, \\
I_* &= \frac{(d+\theta)(d+\alpha+\varepsilon)\left[d^2 + (\rho+\chi)d - AR_0(d+\chi)\right]}{(d+\chi)\left[R_0\theta\varepsilon\delta + (d+\alpha+\varepsilon)(R_0\theta\eta - \beta d - \beta\theta)\right]}, \\
R_0 &= \frac{\beta\gamma}{(d+\gamma)(d+\alpha+\delta+\eta)}.
\end{aligned} \tag{3}$$

The linearized section of system (2) at $P_*(S_*, E_*, I_*, Q_*, R_*, V_*)$ is as follows:

$$\begin{aligned}
\frac{dS(t)}{dt} &= a_1 S(t) + a_2 I(t) + c_1 R(t-\tau_2) + c_2 V(t-\tau_2), \\
\frac{dE(t)}{dt} &= a_3 S(t) + a_4 E(t) + a_5 I(t), \\
\frac{dI(t)}{dt} &= a_6 E(t) - a_7 I(t) + b_1 I(t-\tau_1), \\
\frac{dQ(t)}{dt} &= a_8 I(t) + a_9 Q(t) + b_2 Q(t-\tau_1), \\
\frac{dR(t)}{dt} &= a_{10} R(t) + b_3 I(t-\tau_1) + b_4 Q(t-\tau_1) \\
&\quad + c_3 R(t-\tau_2), \\
\frac{dV(t)}{dt} &= a_{11} S(t) + a_{12} V(t) + c_4 V(t-\tau_2),
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
a_1 &= -(\beta I_* + d + \rho), \\
a_2 &= -\beta S_*, \\
a_3 &= \beta I_*, \\
a_4 &= -(d+\gamma),
\end{aligned}$$

$$
\begin{aligned}
a_5 &= \beta S_*, \\
a_6 &= \gamma, \\
a_7 &= -(d+\alpha+\delta), \\
a_8 &= \delta, \\
a_9 &= -(d+\alpha), \\
a_{10} &= -d, \\
a_{11} &= \rho, \\
a_{12} &= -d, \\
b_1 &= -\eta, \\
b_2 &= -\varepsilon, \\
b_3 &= \eta, \\
b_4 &= \varepsilon, \\
c_1 &= \theta, \\
c_2 &= \chi, \\
c_3 &= -\theta, \\
c_4 &= -\chi.
\end{aligned} \tag{5}
$$

Then, the characteristic equation for system (4) can be obtained:

$$
\begin{aligned}
&\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 \\
&\quad + \left(B_5\lambda^5 + B_4\lambda^4 + B_3\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0\right)e^{-\lambda\tau_1} \\
&\quad + \left(C_5\lambda^5 + C_4\lambda^4 + C_3\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0\right)e^{-\lambda\tau_2} \\
&\quad + \left(D_4\lambda^4 + D_3\lambda^3 + D_2\lambda^2 + D_1\lambda + D_0\right)e^{-\lambda(\tau_1+\tau_2)} \\
&\quad + \left(E_3\lambda^3 + E_2\lambda^2 + E_1\lambda + E_0\right)e^{-\lambda(\tau_1+2\tau_2)} \\
&\quad + \left(F_4\lambda^4 + F_3\lambda^3 + F_2\lambda^2 + F_1\lambda + F_0\right)e^{-2\lambda\tau_1} \\
&\quad + \left(G_4\lambda^4 + G_3\lambda^3 + G_2\lambda^2 + G_1\lambda + G_0\right)e^{-2\lambda\tau_2} \\
&\quad + \left(H_3\lambda^3 + H_2\lambda^2 + H_1\lambda + H_0\right)e^{-\lambda(2\tau_1+\tau_2)} \\
&\quad + \left(I_2\lambda^2 + I_1\lambda + I_0\right)e^{-2\lambda(\tau_1+\tau_2)} = 0,
\end{aligned} \tag{6}
$$

with

$$
\begin{aligned}
A_0 &= a_9a_{12}\left(a_1a_4a_7a_{10} + a_2a_3a_6a_{10} - a_1a_5a_6a_{10} - a_3a_6b_3c_1\right), \\
A_1 &= a_5a_6\left(a_1a_9\left(a_{10}+a_{12}\right) + a_{10}a_{12}\left(a_1+a_9\right)\right) + a_3a_6b_3c_1\left(a_9+a_{12}\right) - a_2a_3a_6\left(a_9a_{10} + a_9a_{12} + a_{10}a_{12}\right) - a_1a_4a_7a_9\left(a_{10}+a_{12}\right) \\
&\quad - a_{10}a_{12}\left(a_1a_4\left(a_7+a_9\right) + a_7a_9\left(a_1+a_4\right)\right), \\
A_2 &= a_2a_3a_6\left(a_9+a_{10}+a_{12}\right) + a_1a_4a_7a_9 - a_3a_6b_3c_1 - a_5a_6\left(a_1a_9 + a_{10}a_{12} + \left(a_1+a_9\right)\left(a_{10}+a_{12}\right)\right) \\
&\quad + a_{10}a_{12}\left(a_1a_4 + a_7a_9 + \left(a_1+a_4\right)\left(a_7+a_9\right)\right) + \left(a_{10} + a_{12}\right)\left(a_1a_4\left(a_7+a_9\right) + a_7a_9\left(a_1+a_4\right)\right), \\
A_3 &= a_5a_6\left(a_1+a_9+a_{10}+a_{12}\right) - a_2a_3a_6 - a_1a_4\left(a_7 + a_9\right) - a_7a_9\left(a_1+a_4\right) - a_{10}a_{12}\left(a_1+a_4+a_7+a_9\right) \\
&\quad - \left(a_{10}+a_{12}\right)\left(a_1a_4 + a_7a_9 + \left(a_1+a_4\right)\left(a_7+a_9\right)\right), \\
A_4 &= a_1a_4 + a_7a_9 + a_{10}a_{12} - a_5a_6 + \left(a_1+a_4\right)\left(a_7+a_9\right) + \left(a_{10}+a_{12}\right)\left(a_1+a_4+a_7+a_9\right), \\
A_5 &= -\left(a_1+a_4+a_7+a_9+a_{10}+a_{12}\right), \\
B_0 &= a_1a_4a_{10}a_{12}\left(a_7b_2 + a_9b_1\right) - a_6a_{10}a_{12}b_2\left(a_1a_5 + a_2a_3\right), \\
B_1 &= a_5a_6b_2\left(a_1a_{10} + a_1a_{12} + a_{10}a_{12}\right) - a_2a_3a_6b_2\left(a_{10} + a_{12}\right) - \left(a_7b_2 + a_9b_1\right)\left(a_1a_4\left(a_{10}+a_{12}\right) + a_{10}a_{12}\left(a_1+a_4\right)\right) \\
&\quad - a_1a_4a_{10}a_{12}\left(b_1+b_2\right) \\
B_2 &= \left(a_7b_2 + a_9b_1\right)\left(a_1a_4 + a_{10}a_{12} + \left(a_1+a_4\right)\left(a_{10}+a_{12}\right)\right) + a_2a_3a_6b_2 + \left(b_1+b_2\right) \cdot \left(a_1a_4\left(a_{10}+a_{12}\right) + a_{10}a_{12}\left(a_1+a_4\right)\right), \\
B_3 &= a_5a_6b_2 - \left(a_7b_2 + a_9b_1\right)\left(a_1+a_4+a_{10}+a_{12}\right) - \left(b_1 + b_2\right)\left(a_1a_4 + a_{10}a_{12} + \left(a_1+a_4\right)\left(a_{10}+a_{12}\right)\right), \\
B_4 &= a_7b_2 + a_9b_1 + \left(b_1+b_2\right)\left(a_1+a_4+a_{10}+a_{12}\right), \\
B_5 &= -\left(b_1+b_2\right), \\
C_0 &= a_1a_4a_7a_9\left(a_{10}c_4 + a_{12}c_3\right) + a_9a_{10}a_{11}c_2\left(a_4a_7 - a_5a_6\right) + a_6a_9\left(a_{10}c_4 + a_{12}c_3\right)\left(a_2a_3 - a_1a_5\right), \\
C_1 &= a_9\left(c_3+c_4\right)\left(a_1a_5a_6 - a_2a_3a_6 - a_1a_4a_7\right) + a_5a_6a_{11}c_2\left(a_9+a_{10}\right) + a_6\left(a_1a_5 - a_2a_3 + a_5a_9\right) \\
&\quad \cdot \left(a_{10}c_4 + a_{12}c_3\right) - a_{11}c_2\left(a_4a_7\left(a_9+a_{10}\right) + a_9a_{10}\left(a_4+a_7\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
C_2 &= (c_3 + c_4)(a_1a_4(a_7 + a_9) + a_7a_9(a_1 + a_4)) \\
&\quad + (a_{10}c_4 + a_{12}c_3)(a_1a_4 + a_7a_9 + (a_1 + a_4)(a_7 + a_9)) \\
&\quad - a_5a_6(a_{10}c_4 + a_{12}c_3 - a_{11}c_2) + a_6(c_3 + c_4)(a_2a_3 \\
&\quad - a_1a_5 - a_5a_9) + a_{11}c_2(a_4a_7 + a_9a_{10} \\
&\quad + (a_4 + a_7)(a_9 + a_{10})), \\
C_3 &= a_5a_6(c_3 + c_4) - a_{11}c_2(a_4 + a_7 + a_9 + a_{10}) - (a_{10}c_4 \\
&\quad + a_{12}c_3)(a_1 + a_4 + a_7 + a_9) - (c_3 + c_4)(a_1a_4 + a_7a_9 \\
&\quad + (a_1 + a_4)(a_7 + a_9)), \\
C_4 &= a_{10}c_4 + a_{11}c_2 + a_{12}c_3 + (c_3 + c_4)(a_1 + a_4 + a_7 \\
&\quad + a_9), \\
C_5 &= -(c_3 + c_4), \\
D_0 &= a_4(a_1a_{10}c_4 + a_1a_{12}c_3 + a_{10}a_{11}c_2)(a_7b_2 + a_9b_1) \\
&\quad + a_6(a_3a_8a_{12}b_4c_4 - a_5a_{10}a_{11}b_2c_2) - a_6b_2(a_1a_5 \\
&\quad - a_2a_3)(a_{10}c_4 + a_{12}c_3), \\
D_1 &= (c_3 + c_4)(a_5b_2(a_1a_5 - a_2a_3) - a_1a_4(a_7b_2 + a_9b_1)) \\
&\quad + a_6(a_5a_{11}b_2c_2 - a_3a_8b_4c_4) + a_5a_6b_2(a_{10}c_4 + a_{12}c_3) \\
&\quad - a_{11}c_2(a_4a_{10}(b_1 + b_2) + (a_4 + a_{10})(a_7b_2 + a_9b_1)) \\
&\quad - (a_{10}c_4 + a_{12}c_3)(a_1a_4(b_1 + b_2) \\
&\quad + (a_1 + a_4)(a_7b_2 + a_9b_1)), \\
D_2 &= (a_{10}c_4 + a_{12}c_3)(a_7b_2 + a_9b_1 + (b_1 + b_2)(a_1 + a_4)) \\
&\quad + (c_3 + c_4)(a_1a_4(b_1 + b_2) + (a_1 + a_4)(a_7b_2 + a_9b_1)) \\
&\quad + a_{11}c_2(a_7b_2 + a_9b_1 + (a_4 + a_{10})(b_1 + b_2)) \\
&\quad - a_5a_6b_2(c_3 + c_4), \\
D_3 &= -(c_3 + c_4)(a_7b_2 + a_9b_1 + (b_1 + b_2)(a_1 + a_4)) \\
&\quad - (b_1 + b_2)(a_{10}c_4 + a_{11}c_2 + a_{12}c_3), \\
D_4 &= (b_1 + b_2)(c_3 + c_4), \\
E_0 &= a_4c_3(a_7b_2 + a_9b_1)(a_1c_4 + a_{11}c_2) + a_6b_2c_3c_4(a_2a_3 \\
&\quad - a_1a_5) - a_6(a_3a_9b_3c_1c_4 + a_5a_{11}b_2c_2c_3), \\
E_1 &= a_6c_4(a_5b_2c_3 + a_3b_3c_1) - a_{11}c_2c_3(b_1(a_4 + a_9) \\
&\quad + b_2(a_4 + a_7)) - c_3c_4(a_1a_4(b_1 + b_2) \\
&\quad + (a_1 + a_4)(a_7b_2 + a_9b_1)), \\
E_2 &= c_3c_4(a_7b_2 + a_9b_1 + (a_1 + a_4)(b_1 + b_2)) \\
&\quad + a_{11}c_2c_3(b_1 + b_2), \\
E_3 &= -c_3c_4(b_1 + b_2), \\
F_0 &= a_1a_4a_{10}a_{12}b_1b_2, \\
F_1 &= -b_1b_2(a_1a_4(a_{10} + a_{12}) + a_{10}a_{12}(a_1 + a_4)), \\
F_2 &= b_1b_2(a_1a_4 + a_{10}a_{12} + (a_1 + a_4)(a_{10} + a_{12})), \\
F_3 &= -b_1b_2(a_1 + a_4 + a_{10} + a_{12}), \\
F_4 &= b_1b_2, \\
G_0 &= a_9c_3c_4(a_1a_4a_7 - a_1a_5a_6 + a_2a_3a_6) \\
&\quad + a_4a_7a_9a_{11}c_2c_3, \\
G_1 &= a_{11}c_2c_3(a_4a_7 + a_4a_9 + a_7a_9) + a_5a_6c_3c_4(a_1 + a_9) \\
&\quad - c_3c_4(a_1a_4(a_7 + a_9) + a_7a_9(a_1 + a_4)), \\
G_2 &= c_3c_4(a_1a_4 + a_7a_9 + (a_1 + a_4)(a_7 + a_9)) \\
&\quad + a_{11}c_2c_3(a_4 + a_7 + a_9) - a_5a_6c_3c_4, \\
G_3 &= c_3c_4(a_1 + a_4 + a_7 + a_9) + a_{11}c_2c_3, \\
G_4 &= c_3c_4, \\
H_0 &= a_4b_1b_2(a_1(a_{10}c_4 + a_{12}c_3) + a_{10}a_{11}c_2) \\
&\quad - a_3a_6a_{12}b_2b_3c_1, \\
H_1 &= a_3a_6b_2b_3c_1 - b_1b_2(a_1a_4(c_3 + c_4) \\
&\quad + a_{11}c_2(a_4 + a_{10})) - b_1b_2(a_1 + a_4)(a_{10}c_4 + a_{12}c_3), \\
H_2 &= b_1b_2(a_{10}c_4 + a_{11}c_2 + a_{12}c_3 + (a_1 + a_4)(c_3 + c_4)), \\
H_3 &= -b_1b_2(c_3 + c_4), \\
I_0 &= a_4b_1b_2c_3(a_{11}c_2 + a_1c_4) - a_3a_6b_2b_3c_1c_4, \\
I_1 &= b_1b_2c_3c_4(a_1 + a_4) + a_{11}b_1b_2c_2c_3, \\
I_2 &= b_1b_2c_3c_4.
\end{aligned}
\tag{7}
$$

Case 1 ($\tau_1 = \tau_2 = 0$). When $\tau_1 = \tau_2 = 0$, (6) becomes

$$
\begin{aligned}
&\lambda^6 + A_{15}\lambda^5 + A_{14}\lambda^4 + A_{13}\lambda^3 + A_{12}\lambda^2 + A_{11}\lambda + A_{10} \\
&\quad = 0,
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
A_{10} &= A_0 + B_0 + C_0 + D_0 + E_0 + F_0 + G_0 + H_0 + I_0, \\
A_{11} &= A_1 + B_1 + C_1 + D_1 + E_1 + F_1 + G_1 + H_1 + I_1, \\
A_{12} &= A_2 + B_2 + C_2 + D_2 + E_2 + F_2 + G_2 + H_2 + I_2, \\
A_{13} &= A_3 + B_3 + C_3 + D_3 + E_3 + F_3 + G_3 + H_3, \\
A_{14} &= A_4 + B_4 + C_4 + D_4 + E_4 + F_4 + G_4, A_{15} \\
&\quad = A_5 + B_5 + C_5.
\end{aligned}
\tag{9}
$$

Clearly, $D_1 = A_{15} = \beta I_* + \rho + \gamma + \delta + \eta + \varepsilon + \theta + \chi + 2\alpha + 6d > 0$. Thus, if condition (H_1) (see (10)) holds, then system (2) without delay is locally asymptotically stable:

$$\begin{aligned}
D_2 &= \det\begin{pmatrix} A_{15} & 1 \\ A_{13} & A_{14} \end{pmatrix} > 0, \\
D_3 &= \det\begin{pmatrix} A_{15} & 1 & 0 \\ A_{13} & A_{14} & A_{15} \\ A_{11} & A_{12} & A_{13} \end{pmatrix} > 0, \\
D_4 &= \det\begin{pmatrix} A_{05} & 1 & 0 & 0 \\ A_{03} & A_{14} & A_{15} & 1 \\ A_{01} & A_{12} & A_{13} & A_{14} \\ 0 & A_{10} & A_{11} & A_{12} \end{pmatrix} > 0, \\
D_5 &= \det\begin{pmatrix} A_{15} & 1 & 0 & 0 & 0 \\ A_{13} & A_{14} & A_{15} & 1 & 0 \\ A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ 0 & A_{10} & A_{11} & A_{12} & A_{13} \\ 0 & 0 & 0 & A_{10} & A_{11} \end{pmatrix} > 0, \\
D_6 &= A_{10} > 0.
\end{aligned} \tag{10}$$

Case 2 ($\tau_1 > 0$; $\tau_2 = 0$). Equation (6) equals

$$\begin{aligned}
&\lambda^6 + A_{25}\lambda^5 + A_{24}\lambda^4 + A_{23}\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20} \\
&\quad + \left(B_{25}\lambda^5 + B_{24}\lambda^4 + B_{23}\lambda^3 + B_{22}\lambda^2 + B_{21}\lambda + B_{20}\right) \\
&\quad \cdot e^{-\lambda\tau_1} + \left(F_{24}\lambda^4 + F_{23}\lambda^3 + F_{22}\lambda^2 + F_{21}\lambda + F_{20}\right) \\
&\quad \cdot e^{-2\lambda\tau_1} = 0,
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
A_{20} &= A_0 + C_0 + G_0, \\
A_{21} &= A_1 + C_1 + G_1, \\
A_{22} &= A_2 + C_2 + G_2, \\
A_{23} &= A_3 + C_3 + G_3, \\
A_{24} &= A_4 + C_4 + G_4, \\
A_{25} &= A_5 + G_5, \\
B_{20} &= B_0 + D_0 + E_0, \\
B_{21} &= B_1 + D_1 + E_1, \\
B_{22} &= B_2 + D_2 + E_2, \\
B_{23} &= B_3 + D_3 + E_3, \\
B_{24} &= B_4 + D_4 + E_4, \\
B_{25} &= B_5.
\end{aligned} \tag{12}$$

Multiplying $e^{\lambda\tau_1}$ on left and right of (11), one has

$$\begin{aligned}
&B_{25}\lambda^5 + B_{24}\lambda^4 + B_{23}\lambda^3 + B_{22}\lambda^2 + B_{21}\lambda + B_{20} + \lambda^6 \\
&\quad + \left(A_{25}\lambda^5 + A_{24}\lambda^4 + A_{23}\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda \right. \\
&\quad \left. + A_{20}\right) e^{\lambda\tau_1} + \left(F_{24}\lambda^4 + F_{23}\lambda^3 + F_{22}\lambda^2 + F_{21}\lambda \right. \\
&\quad \left. + F_{20}\right) e^{-\lambda\tau_1} = 0.
\end{aligned} \tag{13}$$

Assume that $\lambda = i\omega_1$ ($\omega_1 > 0$) is the root of (13):

$$\begin{aligned}
L_{21}(\omega_1)\cos\tau_1\omega_1 - L_{22}(\omega_1)\sin\tau_1\omega_1 &= L_{23}(\omega_1), \\
L_{24}(\omega_1)\sin\tau_1\omega_1 + L_{25}(\omega_1)\cos\tau_1\omega_1 &= L_{26}(\omega_1),
\end{aligned} \tag{14}$$

with

$$\begin{aligned}
L_{21}(\omega_1) &= (A_{24} + F_{24})\omega_1^4 - \omega_1^6 - (A_{22} + F_{22})\omega_1^2 \\
&\quad + A_{20} + F_{20}, \\
L_{22}(\omega_1) &= A_{25}\omega_1^5 - (A_{23} - F_{23})\omega_1^3 + (A_{21} - F_{21})\omega_1, \\
L_{23}(\omega_1) &= B_{22}\omega_1^2 - B_{24}\omega_1^4 - B_{20}, \\
L_{24}(\omega_1) &= (A_{24} - F_{24})\omega_1^4 - \omega_1^6 - (A_{22} - F_{22})\omega_1^2 \\
&\quad + A_{20} - F_{20}, \\
L_{25}(\omega_1) &= A_{25}\omega_1^5 - (A_{23} + F_{23})\omega_1^3 + (A_{21} + F_{21})\omega_1, \\
L_{26}(\omega_1) &= B_{23}\omega_1^3 - B_{25}\omega_1^5 - B_{21}\omega_1.
\end{aligned} \tag{15}$$

Thus, one can obtain the expressions of $\cos\tau_1\omega_1$ and $\sin\tau_1\omega_1$ as follows:

$$\begin{aligned}
&\cos\tau_1\omega_1 \\
&\quad = \frac{L_{22}(\omega_1) \times L_{26}(\omega_1) + L_{23}(\omega_1) \times L_{24}(\omega_1)}{L_{21}(\omega_1) \times L_{24}(\omega_1) + L_{22}(\omega_1) \times L_{25}(\omega_1)}, \\
&\sin\tau_1\omega_1 \\
&\quad = \frac{L_{21}(\omega_1) \times L_{26}(\omega_1) - L_{23}(\omega_1) \times L_{25}(\omega_1)}{L_{21}(\omega_1) \times L_{24}(\omega_1) + L_{22}(\omega_1) \times L_{25}(\omega_1)}.
\end{aligned} \tag{16}$$

Then, we can get

$$\cos^2\tau_1\omega_1 + \sin^2\tau_1\omega_1 = 1. \tag{17}$$

Suppose that (H_{21}) (see (17)) has at least one positive root.

If condition (H_{21}) holds, then there exists $\omega_{10} > 0$ such that (13) has a pair of purely imaginary roots $\pm i\omega_{10}$. For ω_{10},

$$\begin{aligned}
\tau_{10} &= \frac{1}{\omega_{10}} \\
&\quad \times \arccos\left\{\frac{L_{22}(\omega_{10}) \times L_{26}(\omega_{10}) + L_{23}(\omega_{10}) \times L_{24}(\omega_{10})}{L_{21}(\omega_{10}) \times L_{24}(\omega_{10}) + L_{22}(\omega_{10}) \times L_{25}(\omega_{10})}\right\}.
\end{aligned} \tag{18}$$

Differentiating (13) with respect to τ_1, one has

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = \frac{F_{21}(\lambda)}{F_{22}(\lambda)} - \frac{\tau_1}{\lambda}, \tag{19}$$

where

$$\begin{aligned}F_{21}(\lambda) &= 5B_{25}\lambda^4 + 4B_{24}\lambda^4 + 3B_{23}\lambda^2 + 2B_{22}\lambda + B_{21} \\ &\quad + \left(6\lambda^5 + 5A_{25}\lambda^4 + 4A_{24}\lambda^3 + 3A_{23}\lambda^2 + 2A_{22}\lambda \right. \\ &\quad \left. + A_{21}\right) e^{\lambda\tau_1} + \left(4F_{24}\lambda^3 + 3F_{23}\lambda^2 + 2F_{22}\lambda + F_{21}\right) \\ &\quad \cdot e^{-\lambda\tau_1}, \\ F_{22}(\lambda) &= \left(4F_{24}\lambda^4 + 3F_{23}\lambda^3 + 2F_{22}\lambda^2 + F_{21}\lambda\right) e^{-\lambda\tau_1} \\ &\quad - \left(\lambda^7 + A_{25}\lambda^6 + A_{24}\lambda^5 + A_{23}\lambda^4 + A_{22}\lambda^3 + A_{21}\lambda^2 \right. \\ &\quad \left. + A_{20}\lambda\right) e^{\lambda\tau_1}.\end{aligned} \tag{20}$$

Thus,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\lambda=i\omega_{10}} = \frac{G_{2R}\times H_{2R} + G_{2I}\times H_{2I}}{H_{2R}^2 + H_{2I}^2}, \tag{21}$$

with

$$\begin{aligned}G_{2R} &= 5B_{25}\omega_{10}^4 - 3B_{23}\omega_{10}^2 + B_{21} + \left(5A_{25}\omega_{10}^4 \right. \\ &\quad \left. - 3\left(A_{23} + F_{23}\right)\omega_{10}^2 + A_{21} + F_{21}\right)\cos\tau_{10}\omega_{10} \\ &\quad - \left(6\omega_{10}^5 - 4\left(A_{24} - F_{24}\right)\omega_{10}^3 + 2\left(A_{22} - F_{22}\right)\omega_{10}\right) \\ &\quad \cdot \sin\tau_{10}\omega_{10}, \\ G_{2I} &= 2B_{22}\omega_{10} - 4B_{24}\omega_{10}^3 + \left(5A_{25}\omega_{10}^4 \right. \\ &\quad \left. - 3\left(A_{23} - F_{23}\right)\omega_{10}^2 + A_{21} - F_{21}\right)\sin\tau_{10}\omega_{10} \\ &\quad + \left(6\omega_{10}^5 - 4\left(A_{24} + F_{24}\right)\omega_{10}^3 + 2\left(A_{22} + F_{22}\right)\omega_{10}\right) \\ &\quad \cdot \cos\tau_{10}\omega_{10}, \\ H_{2R} &= \left(\left(F_{21} + A_{20}\right)\omega_{10} + A_{24}\omega_{10}^5 - \omega_{10}^7 \right. \\ &\quad \left. - \left(3F_{23} + A_{22}\right)\omega_{10}^3\right)\sin\tau_{10}\omega_{10} + \left(\left(4F_{24} - A_{23}\right)\omega_{10}^4 \right. \\ &\quad \left. - \left(2F_{22} - A_{21}\right)\omega_{10}^2 - A_{25}\omega_{10}^6\right)\cos\tau_{10}\omega_{10}, \\ H_{2I} &= \left(\left(F_{21} - A_{20}\right)\omega_{10} - A_{24}\omega_{10}^5 + \omega_{10}^7 \right. \\ &\quad \left. - \left(3F_{23} - A_{22}\right)\omega_{10}^3\right)\cos\tau_{10}\omega_{10} \\ &\quad - \left(\left(4F_{24} + A_{23}\right)\omega_{10}^4 - \left(2F_{22} - A_{21}\right)\omega_{10}^2 \right. \\ &\quad \left. - A_{25}\omega_{10}^6\right)\sin\tau_{10}\omega_{10}.\end{aligned} \tag{22}$$

Thus, if condition (H_{22}) $G_{2R}\times H_{2R} + G_{2I}\times H_{2I} \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_1]_{\lambda=i\omega_{10}} \neq 0$. Based on the Hopf bifurcation theorem in [25], we have the following results.

Theorem 1. *Suppose that conditions (H_1), (H_{21}), and (H_{22}) hold for system (2). The viral equilibrium $P_*(S_*, E_*, I_*, Q_*, R_*, V_*)$ is locally asymptotically stable when $\tau_1 \in [0, \tau_{10})$ and a Hopf bifurcation occurs at the viral equilibrium $P_*(S_*, E_*, I_*, Q_*, R_*, V_*)$ when $\tau_1 = \tau_{10}$.*

Case 3 ($\tau_1 = 0$; $\tau_2 > 0$). Equation (6) becomes

$$\begin{aligned}&\lambda^6 + A_{35}\lambda^5 + A_{34}\lambda^4 + A_{33}\lambda^3 + A_{32}\lambda^2 + A_{31}\lambda + A_{30} \\ &\quad + \left(C_{35}\lambda^5 + C_{34}\lambda^4 + C_{33}\lambda^3 + C_{32}\lambda^2 + C_{31}\lambda + C_{30}\right) \\ &\quad \cdot e^{-\lambda\tau_2} + \left(G_{34}\lambda^4 + G_{33}\lambda^3 + G_{32}\lambda^2 + G_{31}\lambda + G_{30}\right) \\ &\quad \cdot e^{-2\lambda\tau_2} = 0,\end{aligned} \tag{23}$$

where

$$\begin{aligned}A_{30} &= A_0 + B_0 + F_0, \\ A_{31} &= A_1 + B_1 + F_1, \\ A_{32} &= A_2 + B_2 + F_2, \\ A_{33} &= A_3 + B_3 + F_3, \\ A_{34} &= A_4 + B_4 + F_4, \\ A_{35} &= A_5 + B_5, \\ C_{30} &= C_0 + D_0 + H_0, \\ C_{31} &= C_1 + D_1 + H_1, \\ C_{32} &= C_2 + D_2 + H_2, \\ C_{33} &= C_3 + D_3 + H_3, \\ C_{34} &= C_4 + D_4, \\ C_{35} &= C_5, \\ G_{30} &= E_0 + G_0 + I_0, \\ G_{31} &= E_1 + G_1 + I_1, \\ G_{32} &= E_2 + G_2 + I_2, \\ G_{33} &= E_3 + G_3, \\ G_{34} &= G_4.\end{aligned} \tag{24}$$

Multiplying $e^{\lambda\tau_2}$ on left and right of (23), one has

$$\begin{aligned}&C_{35}\lambda^5 + C_{34}\lambda^4 + C_{33}\lambda^3 + C_{32}\lambda^2 + C_{31}\lambda + C_{30} + \left(\lambda^6 \right. \\ &\quad \left. + A_{35}\lambda^5 + A_{34}\lambda^4 + A_{33}\lambda^3 + A_{32}\lambda^2 + A_{31}\lambda \right. \\ &\quad \left. + A_{30}\right) e^{\lambda\tau_2} + \left(G_{34}\lambda^4 + G_{33}\lambda^3 + G_{32}\lambda^2 + G_{31}\lambda \right. \\ &\quad \left. + G_{30}\right) e^{-\lambda\tau_2} = 0.\end{aligned} \tag{25}$$

Let $\lambda = i\omega_2$ $(\omega_2 > 0)$ be the root of (25):

$$\begin{aligned}L_{31}(\omega_2)\cos\tau_2\omega_2 - L_{32}(\omega_2)\sin\tau_2\omega_2 &= L_{33}(\omega_2), \\ L_{34}(\omega_2)\sin\tau_2\omega_2 + L_{35}(\omega_2)\cos\tau_2\omega_2 &= L_{36}(\omega_2),\end{aligned} \tag{26}$$

with

$$\begin{aligned}
L_{31}(\omega_2) &= (A_{34}+G_{34})\omega_2^4 - \omega_2^6 - (A_{32}+G_{32})\omega_2^2 \\
&\quad + A_{30} + G_{30}, \\
L_{32}(\omega_2) &= A_{35}\omega_2^5 - (A_{33}-G_{33})\omega_2^3 \\
&\quad + (A_{31}-G_{31})\omega_2, \\
L_{33}(\omega_2) &= C_{32}\omega_2^2 - C_{34}\omega_2^4 - C_{30}, \\
L_{34}(\omega_2) &= (A_{34}-G_{34})\omega_2^4 - \omega_2^6 - (A_{32}-G_{32})\omega_2^2 \\
&\quad + A_{30} - G_{30}, \\
L_{35}(\omega_2) &= A_{35}\omega_2^5 - (A_{33}+G_{33})\omega_2^3 \\
&\quad + (A_{31}+G_{31})\omega_2, \\
L_{36}(\omega_2) &= C_{33}\omega_2^3 - C_{35}\omega_2^5 - C_{31}\omega_2.
\end{aligned} \tag{27}$$

Then,

$$\begin{aligned}
&\cos\tau_2\omega_2 \\
&\quad = \frac{L_{32}(\omega_2)\times L_{36}(\omega_2) + L_{33}(\omega_2)\times L_{34}(\omega_2)}{L_{31}(\omega_2)\times L_{34}(\omega_2) + L_{32}(\omega_2)\times L_{35}(\omega_2)}, \\
&\sin\tau_2\omega_2 \\
&\quad = \frac{L_{31}(\omega_2)\times L_{36}(\omega_2) - L_{33}(\omega_2)\times L_{35}(\omega_2)}{L_{31}(\omega_2)\times L_{34}(\omega_2) + L_{32}(\omega_2)\times L_{35}(\omega_2)}.
\end{aligned} \tag{28}$$

And the equation following equation regarding τ_2 can be obtained:

$$\cos^2\tau_2\omega_2 + \sin^2\tau_2\omega_2 = 1. \tag{29}$$

Suppose that (H_{31}) (see (29)) has at least one positive root.

If condition (H_{31}) holds, then there exists $\omega_{20} > 0$ such that (25) has a pair of purely imaginary roots $\pm i\omega_{20}$. For ω_{20},

$$\begin{aligned}
\tau_{20} &= \frac{1}{\omega_{20}} \\
&\quad \times \arccos\left\{\frac{L_{32}(\omega_{20})\times L_{36}(\omega_{20}) + L_{33}(\omega_{20})\times L_{34}(\omega_{20})}{L_{31}(\omega_{20})\times L_{34}(\omega_{20}) + L_{32}(\omega_{20})\times L_{35}(\omega_{20})}\right\}.
\end{aligned} \tag{30}$$

Differentiate both sides of (25) with respect to τ_2. Then,

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = \frac{F_{31}(\lambda)}{F_{32}(\lambda)} - \frac{\tau_2}{\lambda}, \tag{31}$$

where

$$\begin{aligned}
F_{31}(\lambda) &= 5C_{35}\lambda^4 + 4C_{34}\lambda^4 + 3C_{33}\lambda^2 + 2C_{32}\lambda + C_{31} \\
&\quad + \left(6\lambda^5 + 5A_{35}\lambda^4 + 4A_{34}\lambda^3 + 3A_{33}\lambda^2 + 2A_{32}\lambda\right. \\
&\quad \left. + A_{31}\right)e^{\lambda\tau_2} + \left(4G_{34}\lambda^3 + 3G_{33}\lambda^2 + 2G_{32}\lambda + G_{31}\right) \\
&\quad \cdot e^{-\lambda\tau_2}, \\
F_{32}(\lambda) &= \left(4G_{34}\lambda^4 + 3G_{33}\lambda^3 + 2G_{32}\lambda^2 + G_{31}\lambda\right)e^{-\lambda\tau_2} \\
&\quad - \left(\lambda^7 + A_{35}\lambda^6 + A_{34}\lambda^5 + A_{33}\lambda^4 + A_{32}\lambda^3 + A_{31}\lambda^2\right. \\
&\quad \left. + A_{30}\lambda\right)e^{\lambda\tau_2}.
\end{aligned} \tag{32}$$

Thus,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]^{-1}_{\lambda=i\omega_{20}} = \frac{G_{3R}\times H_{3R} + G_{3I}\times H_{3I}}{H_{3R}^2 + H_{3I}^2}, \tag{33}$$

with

$$\begin{aligned}
G_{3R} &= 5C_{35}\omega_{20}^4 - C_{33}\omega_{20}^2 + C_{31} + \left(5A_{35}\omega_{20}^4\right. \\
&\quad \left. - 3(A_{33}+G_{33})\omega_{20}^2 + A_{31} + G_{31}\right)\cos\tau_{20}\omega_{20} \\
&\quad - \left(6\omega_{20}^5 - 4(A_{34}+G_{34})\omega_{20}^3 + 2(A_{32}+G_{32})\omega_{20}\right) \\
&\quad \cdot \sin\tau_{20}\omega_{20}, \\
G_{3I} &= 2C_{22}\omega_{20} - 4C_{24}\omega_{20}^3 + \left(5A_{35}\omega_{20}^4\right. \\
&\quad \left. - 3(A_{33}+G_{33})\omega_{20}^2 + A_{31} - G_{31}\right)\sin\tau_{20}\omega_{20} \\
&\quad + \left(6\omega_{20}^5 - 4(A_{34}+G_{34})\omega_{20}^3 + 2(A_{32}+G_{32})\omega_{20}\right) \\
&\quad \cdot \cos\tau_{20}\omega_{20}, \\
H_{3R} &= \left((G_{31}+A_{30})\omega_{20} + A_{34}\omega_{20}^5 - \omega_{20}^7\right. \\
&\quad \left. - (3G_{33}+A_{32})\omega_{20}^3\right)\sin\tau_{20}\omega_{20} \\
&\quad + \left((4G_{34}-A_{33})\omega_{20}^4 - (2G_{32}-A_{31})\omega_{20}^2\right. \\
&\quad \left. - A_{35}\omega_{20}^6\right)\cos\tau_{20}\omega_{20}, \\
H_{3I} &= \left((G_{31}-A_{30})\omega_{20} - A_{34}\omega_{20}^5 + \omega_{20}^7\right. \\
&\quad \left. - (3G_{33}-A_{32})\omega_{20}^3\right)\cos\tau_{20}\omega_{20} \\
&\quad + \left((4G_{34}+A_{33})\omega_{20}^4 - (2G_{32}+A_{31})\omega_{20}^2\right. \\
&\quad \left. - A_{35}\omega_{20}^6\right)\sin\tau_{20}\omega_{20}.
\end{aligned} \tag{34}$$

Similar to Case 2, we know that if condition (H_{32}) $G_{3R}\times H_{3R} + G_{3I}\times H_{3I} \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_2]_{\lambda=i\omega_{20}} \neq 0$. In conclusion, we have the following results.

Theorem 2. *Suppose that conditions (H_1), (H_{31}), and (H_{32}) hold for system (2). The viral equilibrium $P_*(S_*, E_*, I_*,$*

Q_*, R_*, V_*) is locally asymptotically stable when $\tau_2 \in [0, \tau_{20})$ and a Hopf bifurcation occurs at the viral equilibrium $P_*(S_*, E_*, I_*, Q_*, R_*, V_*)$ when $\tau_2 = \tau_{20}$.

Case 4 ($\tau_1 > 0; \tau_2 \in (0, \tau_{20})$). Regarding τ_1 as the bifurcation parameter when $\tau_2 \in (0, \tau_{20})$, multiplying by $e^{\lambda\tau_1}$, (6) becomes

$$\begin{aligned}
&B_5\lambda^5 + B_4\lambda^4 + B_3\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0 + \left(D_4\lambda^4\right.\\
&\quad + D_3\lambda^3 + D_2\lambda^2 + D_1\lambda + D_0\left)e^{-\lambda\tau_2} + \left(E_3\lambda^3 + E_2\lambda^2\right.\\
&\quad + E_1\lambda + E_0\left)e^{-2\lambda\tau_2} + \left(\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3\right.\\
&\quad + A_2\lambda^2 + A_1\lambda + A_0\left)e^{\lambda\tau_1} + \left(C_5\lambda^5 + C_4\lambda^4 + C_3\lambda^3\right.\\
&\quad + C_2\lambda^2 + C_1\lambda + C_0\left)e^{\lambda(\tau_1-\tau_2)} + \left(F_4\lambda^4 + F_3\lambda^3\right.\\
&\quad + F_2\lambda^2 + F_1\lambda + F_0\left)e^{-\lambda\tau_1} + \left(G_4\lambda^4 + G_3\lambda^3 + G_2\lambda^2\right.\\
&\quad + G_1\lambda + G_0\left)e^{\lambda(\tau_1-2\tau_2)} + \left(H_3\lambda^3 + H_2\lambda^2 + H_1\lambda\right.\\
&\quad + H_0\left)e^{-\lambda(\tau_1+\tau_2)} + \left(I_2\lambda^2 + I_1\lambda + I_0\right)e^{-\lambda(\tau_1+2\tau_2)}.
\end{aligned} \tag{35}$$

Let $\lambda = i\omega_1^*$ ($\omega_1^* > 0$) be the root of (35), and for the convenience we still denote ω_1^* as ω_1; then,

$$\begin{aligned}
L_{41}(\omega_1)\cos\tau_1\omega_1 - L_{42}(\omega_1)\sin\tau_1\omega_1 &= L_{43}(\omega_1),\\
L_{44}(\omega_1)\sin\tau_1\omega_1 + L_{45}(\omega_1)\cos\tau_1\omega_1 &= L_{46}(\omega_1),
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
L_{41}(\omega_1) &= (A_4 + F_4)\omega_1^4 - \omega_1^6 - (A_2 + F_2)\omega_1^2 + A_0\\
&\quad + F_0 + \left(C_4\omega_1^4 - (C_2 + H_2)\omega_1^2 + C_0 + H_0\right)\cos\tau_2\omega_1\\
&\quad + \left(C_5\omega_1^5 - (C_3 + H_3)\omega_1^3 + (C_1 + H_1)\omega_1\right)\sin\tau_2\omega_1\\
&\quad + \left(G_4\omega_1^4 - (G_2 + I_2)\omega_1^2 + G_0 + I_0\right)\cos 2\tau_2\omega_1\\
&\quad + \left((G_1 + I_1)\omega_1 - G_3\omega_1^3\right)\sin 2\tau_2\omega_1,\\
L_{42}(\omega_1) &= A_5\omega_1^5 - (A_3 - F_3)\omega_1^3 + (A_1 - F_1)\omega_1\\
&\quad + \left(C_5\omega_1^5 - (C_3 - H_3)\omega_1^3 + (C_1 - H_1)\omega_1\right)\cos\tau_2\omega_1\\
&\quad - \left(C_4\omega_1^4 - (C_2 - H_2)\omega_1^2 + C_0 - H_0\right)\sin\tau_2\omega_1\\
&\quad + \left((G_1 - I_1)\omega_1 - G_3\omega_1^3\right)\cos 2\tau_2\omega_1\\
&\quad - \left(G_4\omega_1^4 - (G_2 - I_2)\omega_1^2 + G_0 - I_0\right)\sin 2\tau_2\omega_1,\\
L_{43}(\omega_1) &= B_2\omega_1^2 - B_4\omega_1^4 - B_0\\
&\quad + \left(D_3\omega_1^3 - D_1\omega_1\right)\sin\tau_2\omega_1\\
&\quad + \left(D_2\omega_1^2 - D_4\omega_1^4 - D_0\right)\cos\tau_2\omega_1\\
&\quad + \left(E_3\omega_1^3 - E_1\omega_1\right)\sin\tau_2\omega_1 + \left(E_2\omega_1^2 - E_0\right)\cos\tau_2\omega_1,\\
L_{44}(\omega_1) &= (A_4 - F_4)\omega_1^4 - \omega_1^6 - (A_2 - F_2)\omega_1^2 + A_0\\
&\quad - F_0 + \left(C_4\omega_1^4 - (C_2 - H_2)\omega_1^2 + C_0 - H_0\right)\cos\tau_2\omega_1\\
&\quad + \left(C_5\omega_1^5 - (C_3 - H_3)\omega_1^3 + (C_1 - H_1)\omega_1\right)\sin\tau_2\omega_1\\
&\quad + \left(G_4\omega_1^4 - (G_2 - I_2)\omega_1^2 + G_0 - I_0\right)\cos 2\tau_2\omega_1\\
&\quad + \left((G_1 - I_1)\omega_1 - G_3\omega_1^3\right)\sin 2\tau_2\omega_1,\\
L_{45}(\omega_1) &= A_5\omega_1^5 - (A_3 + F_3)\omega_1^3 + (A_1 + F_1)\omega_1\\
&\quad + \left(C_5\omega_1^5 - (C_3 + H_3)\omega_1^3 + (C_1 + H_1)\omega_1\right)\cos\tau_2\omega_1\\
&\quad - \left(C_4\omega_1^4 - (C_2 + H_2)\omega_1^2 + C_0 + H_0\right)\sin\tau_2\omega_1\\
&\quad + \left((G_1 + I_1)\omega_1 - G_3\omega_1^3\right)\cos 2\tau_2\omega_1\\
&\quad - \left(G_4\omega_1^4 - (G_2 + I_2)\omega_1^2 + G_0 + I_0\right)\sin 2\tau_2\omega_1,\\
L_{46}(\omega_1) &= B_3\omega_1^3 - B_3\omega_1^3 - B_1\omega_1\\
&\quad + \left(D_3\omega_1^3 - D_1\omega_1\right)\cos\tau_2\omega_1\\
&\quad - \left(D_2\omega_1^2 - D_4\omega_1^4 - D_0\right)\sin\tau_2\omega_1\\
&\quad + \left(E_3\omega_1^3 - E_1\omega_1\right)\cos\tau_2\omega_1 - \left(E_2\omega_1^2 - E_0\right)\sin\tau_2\omega_1.
\end{aligned} \tag{37}$$

Thus,

$$\begin{aligned}
&\cos\tau_1\omega_1\\
&\quad = \frac{L_{42}(\omega_1)\times L_{46}(\omega_1) + L_{43}(\omega_1)\times L_{44}(\omega_1)}{L_{41}(\omega_1)\times L_{44}(\omega_1) + L_{42}(\omega_1)\times L_{45}(\omega_1)},\\
&\sin\tau_1\omega_1\\
&\quad = \frac{L_{41}(\omega_1)\times L_{46}(\omega_1) - L_{43}(\omega_1)\times L_{45}(\omega_1)}{L_{41}(\omega_1)\times L_{44}(\omega_1) + L_{42}(\omega_1)\times L_{45}(\omega_1)}.
\end{aligned} \tag{38}$$

Then, we get

$$\cos^2\tau_1\omega_1 + \sin^2\tau_1\omega_1 = 1. \tag{39}$$

Suppose that (H_{41}) (see (39)) has at least one positive root.

If (H_{41}) holds, then there exists $\omega_{10} > 0$ such that (35) has a pair of purely imaginary roots $\pm i\omega_{10}^*$. For ω_{10}^*,

$$\begin{aligned}
\tau_{10}^* &= \frac{1}{\omega_{10}^*}\\
&\quad \times \arccos\left\{\frac{L_{42}(\omega_{10}^*)\times L_{46}(\omega_{10}^*) + L_{43}(\omega_{10}^*)\times L_{44}(\omega_{10}^*)}{L_{41}(\omega_{10}^*)\times L_{44}(\omega_{10}^*) + L_{42}(\omega_{10}^*)\times L_{45}(\omega_{10}^*)}\right\}.
\end{aligned} \tag{40}$$

Differentiating both sides of (25) with respect to τ_2,

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = \frac{F_{41}(\lambda)}{F_{42}(\lambda)} - \frac{\tau_1}{\lambda}, \tag{41}$$

where

$$\begin{aligned}
F_{41}(\lambda) &= 5B_5\lambda^4 + 4B_4\lambda^3 + 3B_3\lambda^2 + 2B_2\lambda + B_1 \\
&\quad + \left(\left(4D_4 - \tau_2 D_3\right)\lambda^3 - \tau_2 D_4\lambda^4 + \left(3D_3 - \tau_2 D_2\right)\lambda^2\right. \\
&\quad \left. + \left(2D_2 - D_1\right)\lambda + D_1 - \tau_2 D_0\right) e^{-\lambda\tau_2} + \left(6\lambda^5\right. \\
&\quad \left. + 5A_5\lambda^4 + 4A_4\lambda^3 + 3A_3\lambda^2 + 2A_2\lambda + A_1\right) e^{\lambda\tau_1} \\
&\quad + \left(\left(3E_3 - 2\tau_2 E_2\right)\lambda^2 - 2\tau_2 E_3\lambda^3 + 2\left(E_2 - \tau_2 E_1\right)\lambda\right. \\
&\quad \left. + E_1 - 2\tau_2 E_0\right) e^{-2\lambda\tau_2} + \left(\left(5C_5 - \tau_2 C_4\right)\lambda^4 - \tau_2 C_5\lambda^5\right. \\
&\quad + \left(4C_4 - \tau_2 C_3\right)\lambda^3 + \left(3C_3 - \tau_2 C_2\right)\lambda^2 \\
&\quad \left. + \left(2C_2 - \tau_2 C_1\right)\lambda + C_1 - \tau_2 C_0\right) e^{\lambda(\tau_1 - \tau_2)} + \left(4F_4\lambda^3\right. \\
&\quad \left. + 3F_3\lambda^2 + 2F_2\lambda + F_1\right) e^{-\lambda\tau_1} + \left(\left(3H_3 - \tau_2 H_2\right)\lambda^2\right. \\
&\quad \left. - \tau_2 H_3\lambda^3 + \left(2H_2 - \tau_2 H_1\right)\lambda + H_1 - \tau_2 H_0\right) \\
&\quad \cdot e^{-\lambda(\tau_1 + \tau_2)} + \left(\left(4G_4 - 2\tau_2 G_3\right)\lambda^3 - 2\tau_2 G_4\lambda^4\right. \\
&\quad + \left(3G_3 - \tau_2 G_2\right)\lambda^2 + 2\left(G_2 - \tau_2 G_1\right)\lambda + G_1 \\
&\quad \left. - 2\tau_2 G_0\right) e^{\lambda(\tau_1 - 2\tau_2)} + \left(2\left(I_2 - \tau_2 I_1\right)\lambda - 2\tau_2 I_2\lambda^2 + I_1\right. \\
&\quad \left. - 2\tau_2 I_0\right) e^{-\lambda(\tau_1 + 2\tau_2)},
\end{aligned} \tag{42}$$

$$\begin{aligned}
F_{42}(\lambda) &= \left(F_4\lambda^5 + F_3\lambda^4 + F_2\lambda^3 + F_1\lambda^2 + F_0\lambda\right) e^{-\lambda\tau_1} \\
&\quad + \left(H_3\lambda^4 + H_2\lambda^3 + H_1\lambda^2 + H_0\lambda\right) e^{-\lambda(\tau_1 + \tau_2)} + \left(I_2\lambda^3\right. \\
&\quad \left. + I_1\lambda^2 + I_0\lambda\right) e^{-\lambda(\tau_1 + 2\tau_2)} - \left(\lambda^7 + A_5\lambda^6 + A_4\lambda^5\right. \\
&\quad \left. + A_3\lambda^4 + A_2\lambda^3 + A_1\lambda^2 + A_0\lambda\right) e^{\lambda\tau_1} - \left(C_5\lambda^6\right. \\
&\quad \left. + C_4\lambda^5 + C_3\lambda^4 + C_2\lambda^3 + C_1\lambda^2 + C_0\lambda\right) e^{\lambda(\tau_1 - \tau_2)} \\
&\quad - \left(G_4\lambda^5 + G_3\lambda^4 + G_2\lambda^3 + G_1\lambda^2 + G_0\lambda\right) e^{\lambda(\tau_1 - 2\tau_2)}.
\end{aligned}$$

Define

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\lambda = i\omega_{10}^*} = \frac{G_{4R} \times H_{4R} + G_{4I} \times H_{4I}}{H_{4R}^2 + H_{4I}^2}. \tag{43}$$

Similar to Case 2, we know that if condition (H_{42}) $G_{4R} \times H_{4R} + G_{4I} \times H_{4I} \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_1]_{\lambda = i\omega_{10}^*} \neq 0$. Thus, we have the following results.

Theorem 3. *Let $\tau_2 \in (0, \tau_{20})$ and suppose that conditions (H_1), (H_{41}), and (H_{42}) hold for system (2). The viral equilibrium $P_*(S_*, E_*, I_*, Q_*, R_*, V_*)$ is locally asymptotically stable when $\tau_1 \in [0, \tau_{10}^*)$ and a Hopf bifurcation occurs at the viral equilibrium $P_*(S_*, E_*, I_*, Q_*, R_*, V_*)$ when $\tau_1 = \tau_{10}^*$.*

Case 5 ($\tau_1 \in (0, \tau_{10})$; $\tau_2 > 0$). Regarding τ_2 as the bifurcation parameter when $\tau_1 \in (0, \tau_{10})$, multiplying by $e^{\lambda\tau_2}$, (6) becomes

$$\begin{aligned}
&C_5\lambda^5 + C_4\lambda^4 + C_3\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0 + \left(D_4\lambda^4\right. \\
&\quad \left. + D_3\lambda^3 + D_2\lambda^2 + D_1\lambda + D_0\right) e^{-\lambda\tau_1} + \left(H_3\lambda^3\right. \\
&\quad \left. + H_2\lambda^2 + H_1\lambda + H_0\right) e^{-2\lambda\tau_1} + \left(G_4\lambda^4 + G_3\lambda^3\right. \\
&\quad \left. + G_2\lambda^2 + G_1\lambda + G_0\right) e^{-\lambda\tau_2} + \left(\lambda^6 + A_5\lambda^5 + A_4\lambda^4\right. \\
&\quad \left. + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0\right) e^{\lambda\tau_2} + \left(B_5\lambda^5 + B_4\lambda^4\right. \\
&\quad \left. + B_3\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0\right) e^{\lambda(\tau_2 - \tau_1)} + \left(E_3\lambda^3\right. \\
&\quad \left. + E_2\lambda^2 + E_1\lambda + E_0\right) e^{-\lambda(\tau_2 + \tau_1)} + \left(F_4\lambda^4 + F_3\lambda^3\right. \\
&\quad \left. + F_2\lambda^2 + F_1\lambda + F_0\right) e^{\lambda(\tau_2 - 2\tau_1)} + \left(I_2\lambda^2 + I_1\lambda + I_0\right) \\
&\quad \cdot e^{-\lambda(\tau_2 + 2\tau_1)}.
\end{aligned} \tag{44}$$

Let $\lambda = i\omega_2^*$ ($\omega_2^* > 0$) be the root of (44), and for the convenience we still denote ω_2^* as ω_2; then,

$$\begin{aligned}
L_{51}(\omega_2)\cos\tau_2\omega_2 - L_{52}(\omega_1)\sin\tau_2\omega_2 &= L_{53}(\omega_2), \\
L_{54}(\omega_2)\sin\tau_2\omega_2 + L_{55}(\omega_1)\cos\tau_2\omega_2 &= L_{56}(\omega_2),
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
&L_{51}(\omega_2) \\
&\quad = \left(A_4 + G_4\right)\omega_2^4 - \omega_2^6 - \left(A_2 + G_2\right)\omega_2^2 + A_0 + G_0 \\
&\qquad + \left(B_4\omega_2^4 - \left(B_2 + E_2\right)\omega_2^2 + B_0 + E_0\right)\cos\tau_1\omega_2 \\
&\qquad + \left(B_5\omega_2^5 - \left(B_3 + E_3\right)\omega_2^3 + \left(B_1 - E_1\right)\omega_2\right)\sin\tau_1\omega_2 \\
&\qquad + \left(F_4\omega_2^4 - \left(F_2 + I_2\right)\omega_2^2 + F_0 + I_0\right)\cos 2\tau_1\omega_2 \\
&\qquad + \left(\left(F_1 + I_1\right)\omega_2 - F_3\omega_2^3\right)\sin 2\tau_1\omega_2, \\
&L_{52}(\omega_2) \\
&\quad = A_5\omega_2^5 - \left(A_3 - G_3\right)\omega_2^3 + \left(A_1 - G_1\right)\omega_2 \\
&\qquad + \left(\left(B_2 - E_2\right)\omega_2^2 - B_4\omega_2^4 + E_0 - B_0\right)\sin\tau_1\omega_2 \\
&\qquad + \left(B_5\omega_2^5 - \left(B_3 - E_3\right)\omega_2^3 + \left(B_1 - E_1\right)\omega_2\right)\cos\tau_1\omega_2 \\
&\qquad - \left(F_4\omega_2^4 - \left(F_2 - I_2\right)\omega_2^2 + F_0 - I_0\right)\sin 2\tau_1\omega_2 \\
&\qquad + \left(\left(F_1 - I_1\right)\omega_2 - F_3\omega_2^3\right)\cos 2\tau_1\omega_2,
\end{aligned}$$

$$
\begin{aligned}
&L_{53}(\omega_2)\\
&\quad = C_2\omega_2^2 - C_4\omega_2^4 - C_0 + \left(D_3\omega_2^3 - D_1\omega_2\right)\sin\tau_1\omega_2\\
&\qquad + \left(D_2\omega_2^2 - D_4\omega_2^4 - D_0\right)\cos\tau_1\omega_2\\
&\qquad + \left(H_3\omega_2^3 - H_1\omega_2\right)\sin 2\tau_1\omega_2\\
&\qquad + \left(H_2\omega_2^2 - H_0\right)\cos 2\tau_1\omega_2,\\
&L_{54}(\omega_2)\\
&\quad = (A_4 - G_4)\omega_2^4 - \omega_2^6 - (A_2 - G_2)\omega_2^2 + A_0 - G_0\\
&\qquad + \left(B_4\omega_2^4 - (B_2 - E_2)\omega_2^2 + B_0 - E_0\right)\cos\tau_1\omega_2\\
&\qquad + \left(B_5\omega_2^5 - (B_3 - E_3)\omega_2^3 + (B_1 - E_1)\omega_2\right)\sin\tau_1\omega_2\\
&\qquad + \left(F_4\omega_2^4 - (F_2 - I_2)\omega_2^2 + F_0 - I_0\right)\cos 2\tau_1\omega_2\\
&\qquad + \left((F_1 - I_1)\omega_2 - F_3\omega_2^3\right)\sin 2\tau_1\omega_2,\\
&L_{55}(\omega_2)\\
&\quad = A_5\omega_2^5 - (A_3 + G_3)\omega_2^3 + (A_1 + G_1)\omega_2\\
&\qquad - \left(B_4\omega_2^4 - (B_2 + E_2)\omega_2^2 + E_0 + B_0\right)\sin\tau_1\omega_2\\
&\qquad + \left(B_5\omega_2^5 - (B_3 + E_3)\omega_2^3 + (B_1 + E_1)\omega_2\right)\cos\tau_1\omega_2\\
&\qquad - \left(F_4\omega_2^4 - (F_2 + I_2)\omega_2^2 + F_0 - I_0\right)\sin 2\tau_1\omega_2\\
&\qquad + \left((F_1 + I_1)\omega_2 - F_3\omega_2^3\right)\cos 2\tau_1\omega_2,\\
&L_{56}(\omega_2)\\
&\quad = C_3\omega_2^3 - C_5\omega_2^5 - C_1\omega_2\\
&\qquad + \left(D_3\omega_2^3 - D - 1\omega_2\right)\cos\tau_1\omega_2\\
&\qquad - \left(D_2\omega_2^2 - D_4\omega_2^4 - D_0\right)\sin\tau_1\omega_2\\
&\qquad + \left(H_3\omega_2^3 - H_1\omega_2\right)\cos 2\tau_1\omega_2\\
&\qquad - \left(H_2\omega_2^2 - H_0\right)\sin 2\tau_1\omega_2.
\end{aligned}
\tag{46}
$$

Thus,

$$
\begin{aligned}
&\cos\tau_2\omega_2\\
&\quad = \frac{L_{52}(\omega_2)\times L_{56}(\omega_2) + L_{53}(\omega_2)\times L_{54}(\omega_2)}{L_{51}(\omega_2)\times L_{54}(\omega_2) + L_{52}(\omega_2)\times L_{55}(\omega_2)},\\
&\sin\tau_2\omega_2\\
&\quad = \frac{L_{51}(\omega_2)\times L_{56}(\omega_2) - L_{53}(\omega_2)\times L_{55}(\omega_2)}{L_{51}(\omega_2)\times L_{54}(\omega_2) + L_{52}(\omega_2)\times L_{55}(\omega_2)}.
\end{aligned}
\tag{47}
$$

Then, we get

$$
\cos^2\tau_2\omega_2 + \sin^2\tau_2\omega_2 = 1. \tag{48}
$$

If (H_{51}) holds, then there exists $\omega_{20}^* > 0$ such that (35) has a pair of purely imaginary roots $\pm i\omega_{20}^*$. For ω_{20}^*,

$$
\begin{aligned}
\tau_{20}^* &= \frac{1}{\omega_{20}^*}\\
&\quad \times \arccos\left\{\frac{L_{52}(\omega_{20}^*)\times L_{56}(\omega_{20}^*) + L_{53}(\omega_{20}^*)\times L_{54}(\omega_{20}^*)}{L_{51}(\omega_{20}^*)\times L_{54}(\omega_{20}^*) + L_{52}(\omega_{20}^*)\times L_{55}(\omega_{20}^*)}\right\}.
\end{aligned}
\tag{49}
$$

Differentiating (25) with respect to τ_2, one can get

$$
\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = \frac{F_{51}(\lambda)}{F_{52}(\lambda)} - \frac{\tau_2}{\lambda}, \tag{50}
$$

where

$$
\begin{aligned}
F_{51}(\lambda) &= 5C_5\lambda^4 + 4C_4\lambda^3 + 3C_3\lambda^2 + 2C_2\lambda + C_1\\
&\quad + \left((4D_4 - \tau_1 D_3)\lambda^3 - \tau_1 D_4\lambda^4 + (3D_3 - \tau_1 D_2)\lambda^2\right.\\
&\quad \left. + (2D_2 - \tau_1 D_1)\lambda + D_1 - \tau_1 D_0\right)e^{-\lambda\tau_1}\\
&\quad + \left((3H_3 - 2\tau_1 H_2)\lambda^2 - 2\tau_1 H_3\lambda^3\right.\\
&\quad \left. + 2(H_2 - \tau_1 H_1)\lambda + H_1 - 2\tau_1 H_0\right)e^{-2\lambda\tau_1} + \left(4G_4\lambda^3\right.\\
&\quad \left. + 3G_3\lambda^2 + 2G_2\lambda + G_1\right)e^{-\lambda\tau_2} + \left(6\lambda^5 + 5A_5\lambda^4\right.\\
&\quad \left. + 4A_4\lambda^3 + 3A_3\lambda^2 + 2A_2\lambda + A_1\right)e^{\lambda\tau_2}\\
&\quad + \left((5B_5 - \tau_1 B_4)\lambda^4 - \tau_1 B_5\lambda^5 + (4B_4 - \tau_1 B_3)\lambda^3\right.\\
&\quad \left. + (3B_3 - \tau_1 B_2)\lambda^2 + (2B_2 - \tau_1 B_1)\lambda + B_1 - \tau_1 B_0\right)\\
&\quad \cdot e^{\lambda(\tau_2 - \tau_1)} + \left((3E_3 - \tau_1 E_2)\lambda^2 - \tau_1 E_3\lambda^3\right.\\
&\quad \left. + (2E_2 - \tau_1 E_1)\lambda + E_1 - \tau_1 E_0\right)e^{-\lambda(\tau_2 + \tau_1)}\\
&\quad + \left((4F_4 - 2\tau_1 F_3)\lambda^3 - 2\tau_1 F_4\lambda^4 + (3F_3 - 2\tau_1 F_2)\lambda^2\right.\\
&\quad \left. + (2F_2 - 2\tau_1 F_1)\lambda + F_1 - 2\tau_1 F_0\right)e^{\lambda(\tau_2 - 2\tau_1)}\\
&\quad + \left(2(I_2 - \tau_1 I_1)\lambda - 2\tau_1 I_2\lambda^2 + I_1 - 2\tau_1 I_0\right)e^{\lambda(\tau_2 + 2\tau_1)},\\
F_{52}(\lambda) &= \left(G_4\lambda^5 + G_3\lambda^4 + G_2\lambda^3 + G_1\lambda^2 + G_0\lambda\right)e^{-\lambda\tau_2}\\
&\quad + \left(E_3\lambda^4 + E_2\lambda^3 + E_1\lambda^2 + E_0\lambda\right)e^{-\lambda(\tau_2 + \tau_1)} + \left(I_2\lambda^3\right.\\
&\quad \left. + I_1\lambda^2 + I_0\lambda\right)e^{-\lambda(\tau_2 + 2\tau_1)} - \left(F_4\lambda^5 + F_3\lambda^4 + F_2\lambda^3\right.\\
&\quad \left. + F_1\lambda^2 + F_0\lambda\right)e^{\lambda(\tau_2 - 2\tau_1)} - \left(B_5\lambda^6 + B_4\lambda^5 + B_3\lambda^4\right.\\
&\quad \left. + B_2\lambda^3 + B_1\lambda^2 + B_0\lambda\right)e^{\lambda(\tau_2 - \tau_1)} - \left(\lambda^7 + A_5\lambda^6\right.\\
&\quad \left. + A_4\lambda^5 + A_3\lambda^4 + A_2\lambda^3 + A_1\lambda^2 + A_0\lambda\right)e^{\lambda\tau_2}.
\end{aligned}
\tag{51}
$$

Thus,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]^{-1}_{\lambda=i\omega_{20}^*} = \frac{G_{3R}\times H_{3R}+G_{3I}\times H_{3I}}{H_{3R}^2+H_{3I}^2}. \tag{52}$$

Therefore, we know that if condition (H_{52}) $G_{5R}\times H_{5R}+G_{5I}\times H_{5I}\neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_2]_{\lambda=i\omega_{20}^*}\neq 0$. Then, we have the following results.

Theorem 4. *Let $\tau_1\in(0,\tau_{10})$ and suppose that conditions (H_1), (H_{51}), and (H_{52}) hold for system (2). The viral equilibrium $P_*(S_*,E_*,I_*,Q_*,R_*,V_*)$ is locally asymptotically stable when $\tau_2\in[0,\tau_{20}^*)$ and a Hopf bifurcation occurs at the viral equilibrium $P_*(S_*,E_*,I_*,Q_*,R_*,V_*)$ when $\tau_2=\tau_{20}^*$.*

3. Properties of the Hopf Bifurcation

In this section, we shall investigate direction and stability of the Hopf bifurcation under the case where $\tau_1\in(0,\tau_{10})$ and $\tau_2>0$. Set $u_1(t)=S(t)-S_*$, $u_2(t)=E(t)-E_*$, $u_3(t)=I(t)-I_*$, $u_4(t)=Q(t)-Q_*$, $u_5(t)=R(t)-R_*$, $u_6(t)=V(t)-V_*$, and $t\to(t/\tau_2)$. For convenience, we assume that $\tau_1^*\in(0,\tau_{10})<\tau_{20}^*$ throughout this section. Then, system (2) becomes functional differential equations in $C=C([-1,0],R^6)$:

$$\dot{u}(t)=L_\mu u_t+F(\mu,u_t), \tag{53}$$

with

$$L_\mu\phi=(\tau_{20}^*+\mu)\left(A\phi(0)+B\phi\left(-\frac{\tau_1^*}{\tau_{20}^*}\right)+C\phi(-1)\right),$$
$$F(\mu,\phi)=(\tau_{20}^*+\mu)\begin{pmatrix}-\beta\phi_1(0)\phi_3(0)\\ \beta\phi_1(0)\phi_3(0)\\ 0\\ 0\\ 0\\ 0\end{pmatrix}, \tag{54}$$

where

$$A=\begin{pmatrix}a_1&0&a_2&0&0&0\\ a_3&a_4&a_5&0&0&0\\ 0&a_6&a_7&0&0&0\\ 0&0&a_8&a_9&0&0\\ 0&0&0&0&a_{10}&0\\ a_{11}&0&0&0&0&a_{12}\end{pmatrix},$$

$$B=\begin{pmatrix}0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&b_1&0&0&0\\ 0&0&0&b_2&0&0\\ 0&0&b_3&b_4&0&0\\ 0&0&0&0&0&0\end{pmatrix},$$

$$C=\begin{pmatrix}0&0&0&0&c_1&c_2\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&c_3&0&0\\ 0&0&0&0&0&c_4\end{pmatrix}. \tag{55}$$

Based on the Riesz representation theorem, there exists a 6×6 function $\eta(\theta,\mu):[-1,0]\to R^{6\times6}$ such that

$$L_\mu\phi=\int_{-1}^{0}d\eta(\theta,\mu)\phi(\theta),\quad \phi\in C. \tag{56}$$

In fact, we choose

$$\eta(\theta,\mu)=\begin{cases}(\tau_{20}^*+\mu)(A+B+B), & \theta=0,\\ (\tau_{20}^*+\mu)(B+C), & \theta\in\left[-\dfrac{\tau_1^*}{\tau_{20}^*},0\right),\\ (\tau_{20}^*+\mu)B, & \theta\in\left(-1,-\dfrac{\tau_1^*}{\tau_{20}^*}\right),\\ 0, & \theta=-1.\end{cases} \tag{57}$$

For $\phi\in C([-1,0],R^6)$, we define

$$A(\mu)\phi=\begin{cases}\dfrac{d\phi(\theta)}{d\theta}, & -1\le\theta<0,\\ \displaystyle\int_{-1}^{0}d\eta(\theta,\mu)\phi(\theta), & \theta=0,\end{cases}$$
$$R(\mu)\phi=\begin{cases}0, & -1\le\theta<0,\\ F(\mu,\phi), & \theta=0.\end{cases} \tag{58}$$

Then, system (53) becomes

$$\dot{u}(t)=A(\mu)u_t+R(\mu)u_t, \tag{59}$$

where $u_t(\theta)=u(t+\theta)$ for $\theta\in[-1,0]$.

Define A^* as follows:

$$A^*(\varphi)=\begin{cases}-\dfrac{d\varphi(s)}{ds}, & 0<s\le1,\\ \displaystyle\int_{-1}^{0}d\eta^T(s,0)\varphi(-s), & s=0,\end{cases} \tag{60}$$

and a bilinear form

$$\langle\varphi(s),\phi(\theta)\rangle=\overline{\varphi}(0)\phi(0)-\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\overline{\varphi}(\xi-\theta)\,d\eta(\theta)\phi(\xi)\,d\xi, \tag{61}$$

where $\eta(\theta)=\eta(\theta,0)$.

Let $q(\theta) = (1, q_2, q_3, q_4, q_5, q_6)^T e^{i\omega_{20}^*\tau_{20}^*\theta}$ be the eigenvector of $A(0)$ with $+i\omega_{20}^*\tau_{20}^*$ and let $q^*(s) = D(1, q_2^*, q_3^*, q_4^*, q_5^*)e^{i\omega_{20}^*\tau_{20}^* s}$ be the eigenvector of $A^*(0)$ with $-i\omega_{20}^*\tau_{20}^*$. Then, according to the definition of $A(0)$ and $A^*(0)$, we obtain

$$\begin{aligned}
q_2 &= \frac{a_3 + a_5 q_3}{i\omega_{20}^* - a_3},\\
q_3 &= \frac{a_3 a_6}{(i\omega_{20}^* - a_3)(i\omega_{20}^* - a_7 - b_1 e^{-i\tau_1^*\omega_{20}^*}) - a_5 a_6},\\
q_4 &= \frac{a_8 q_3}{i\omega_{20}^- a_9 - b_2 e^{-i\tau_1^*\omega_{20}^*}},\\
q_5 &= \frac{i\omega_{20}^* - a_1 - a_2 q_3 - c_2 e^{-i\tau_{20}^*\omega_{20}^*}}{c_1 e^{-i\tau_{20}^*\omega_{20}^*}},\\
q_6 &= \frac{i\omega_{20}^* - a_{11}}{a_{12} + c_4 e^{-i\tau_{20}^*\omega_{20}^*}},\\
q_2^* &= -\frac{i\omega_{20}^* + a_1}{a_3},\\
q_3^* &= \frac{(i\omega_{20}^* + a_1)(i\omega_{20}^* + a_4)}{a_3 a_6},\\
q_4^* &= -\frac{\left(i\omega_{20}^* + a_7 + b_1 e^{i\tau_1^*\omega_{20}^*}\right) q_3}{a_8}\\
&\quad + \frac{a_2 a_3 + (i\omega_{20}^* + a_1) a_5}{a_3 a_8}\\
&\quad + \frac{b_3\left(i\omega_{20}^* + a_9 + b_2 e^{i\tau_1^*\omega_{20}^*}\right)}{a_8 b_4},\\
q_5 &= -\frac{\left(i\omega_{20}^* + a_9 + b_2 e^{i\tau_1^*\omega_{20}^*}\right) q_4}{b_4 e^{i\tau_1^*\omega_{20}^*}},\\
q_6^* &= -\frac{c_2 e^{i\tau_{20}^*\omega_{20}^*}}{i\omega_{20}^* + a_{12} + c_4 e^{i\tau_{20}^*\omega_{20}^*}}.
\end{aligned} \tag{62}$$

In addition, from (61), we have

$$\begin{aligned}
\langle q^*(s), q(\theta)\rangle &= \overline{q}(0)\, q(0)\\
&\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{q}^*(\xi - \theta)\, d\eta(\theta)\, q(\xi)\, d\xi\\
&= \overline{D}(1, \overline{q}_2^*, \overline{q}_3^*, \overline{q}_4^*, \overline{q}_5^*, \overline{q}_6^*)(1, q_2, q_3, q_4, q_5, q_6)^T\\
&\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{D}(1, \overline{q}_2^*, \overline{q}_3^*, \overline{q}_4^*, \overline{q}_5^*, \overline{q}_6^*)\\
&\quad \cdot e^{-i\tau_1^*\omega_{20}^*(\xi-\theta)} d\eta(\theta)(1, q_2, q_3, q_4, q_5, q_6)^T e^{i\tau_1^*\omega_{20}^*\xi} d\xi\\
&\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{D}(1, \overline{q}_2^*, \overline{q}_3^*, \overline{q}_4^*, \overline{q}_5^*, \overline{q}_6^*)\\
&\quad \cdot e^{-i\tau_{20}^*\omega_{20}^*(\xi-\theta)} d\eta(\theta)(1, q_2, q_3, q_4, q_5, q_6)^T e^{i\tau_{20}^*\omega_{20}^*\xi} d\xi\\
&= \overline{D}\Big[1 + q_2\overline{q}_2^* + q_3\overline{q}_3^* + q_4\overline{q}_4^* + q_5\overline{q}_5^* + q_6\overline{q}_6^*\\
&\quad + \tau_1^* e^{i\tau_1^*\omega_{20}^*}\int_{-1}^{0}(1, \overline{q}_2^*, \overline{q}_3^*, \overline{q}_4^*, \overline{q}_5^*, \overline{q}_6^*)\\
&\quad \cdot B(1, q_2, q_3, q_4, q_5, q_6)^T\\
&\quad + \tau_1^* e^{i\tau_2^*\omega_{20}^*}\int_{-1}^{0}(1, \overline{q}_2^*, \overline{q}_3^*, \overline{q}_4^*, \overline{q}_5^*, \overline{q}_6^*)\\
&\quad \cdot C(1, q_2, q_3, q_4, q_5, q_6)^T\Big] = \overline{D}\Big[1 + q_2\overline{q}_2^* + q_3\overline{q}_3^*\\
&\quad + q_4\overline{q}_4^* + q_5\overline{q}_5^* + q_6\overline{q}_6^*\\
&\quad + \tau_1^* e^{-i\tau_1^*\omega_{20}^*}\left(b_3\left(b_1\overline{q}_3^* + b_3\overline{q}_5^*\right) + b_4\left(b_2\overline{q}_4^* + b_4\overline{q}_5^*\right)\right)\\
&\quad + \tau_{20}^* e^{-i\tau_{20}^*\omega_{20}^*}\left(c_1\overline{q}_5^* + c_2\overline{q}_6^* + c_3 q_5\overline{q}_5^* + c_4 q_6\overline{q}_6^*\right)\Big].
\end{aligned} \tag{63}$$

Thus, we can choose

$$\begin{aligned}
\overline{D} = \Big[&1 + q_2\overline{q}_2^* + q_3\overline{q}_3^* + q_4\overline{q}_4^* + q_5\overline{q}_5^* + q_6\overline{q}_6^*\\
&+ \tau_1^* e^{-i\tau_1^*\omega_{20}^*}\left(b_3\left(b_1\overline{q}_3^* + b_3\overline{q}_5^*\right) + b_4\left(b_2\overline{q}_4^* + b_4\overline{q}_5^*\right)\right)\\
&+ \tau_{20}^* e^{-i\tau_{20}^*\omega_{20}^*}\left(c_1\overline{q}_5^* + c_2\overline{q}_6^* + c_3 q_5\overline{q}_5^* + c_4 q_6\overline{q}_6^*\right)\Big]^{-1},
\end{aligned} \tag{64}$$

such that $\langle q^*, q\rangle = 1$, $\langle q^*, \overline{q}\rangle = 0$.

Then, using the algorithms from Hassard et al. [25] and the similar computation process in [26–29], we obtain

$$\begin{aligned}
g_{20} &= 2\beta\tau_{20}^*\overline{D} q_3(\overline{q}_2^* - 1),\\
g_{11} &= \beta\tau_{20}^*\overline{D}\,\mathrm{Re}\{q_3\}(\overline{q}_2^* - 1),\\
g_{02} &= 2\beta\tau_{20}^*\overline{D}\overline{q}_3(\overline{q}_2^* - 1),\\
g_{21} &= 2\beta\tau_{20}^*\overline{D}(\overline{q}_2^* - 1)\Big(W_{11}^{(1)}(0)\, q_3 + \frac{1}{2}W_{20}^{(1)}(0)\,\overline{q}_3\\
&\quad + W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0)\Big),
\end{aligned} \tag{65}$$

with

$$\begin{aligned}
W_{20}(\theta) &= \frac{i g_{20} q(0)}{\tau_{20}^*\omega_{20}^*} e^{i\tau_{20}^*\omega_{20}^*\theta} + \frac{i\overline{g}_{02}\overline{q}(0)}{3\tau_{20}^*\omega_{20}^*} e^{-i\tau_{20}^*\omega_{20}^*\theta} + E_1 e^{2i\tau_{20}^*\omega_{20}^*\theta},\\
W_{11}(\theta) &= -\frac{i g_{11} q(0)}{\tau_{20}^*\omega_{20}^*} e^{i\tau_{20}^*\omega_{20}^*\theta} + \frac{i\overline{g}_{11}\overline{q}(0)}{\tau_{20}^*\omega_{20}^*} e^{-i\tau_{20}^*\omega_{20}^*\theta} + E_2,
\end{aligned}$$

$$E_1 = 2\begin{pmatrix} a_1' & 0 & -a_2 & 0 & -c_1 e^{-2i\tau_{20}^*\omega_{20}^*} & -c_2 e^{-2i\tau_{20}^*\omega_{20}^*} \\ -a_3 & a_2' & -a_5 & 0 & 0 & 0 \\ 0 & -a_6 & a_3' & 0 & 0 & 0 \\ 0 & 0 & -a_8 & a_4' & 0 & 0 \\ 0 & 0 & -b_3 e^{-2i\tau_1^*\omega_{20}^*} & -b_4 e^{-2i\tau_1^*\omega_{20}^*} & a_5' & 0 \\ -a_{11} & 0 & 0 & 0 & 0 & a_6' \end{pmatrix}^{-1} \times \begin{pmatrix} -\beta q_3 \\ \beta q_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$E_2 = -\begin{pmatrix} a_1 & 0 & a_2 & 0 & c_1 & c_2 \\ a_3 & a_4 & a_5 & 0 & 0 & 0 \\ 0 & a_6 & a_7 + b_1 & 0 & 0 & 0 \\ 0 & 0 & a_8 & a_9 + b_2 & 0 & 0 \\ 0 & 0 & b_3 & b_4 & a_{10} + c_3 & 0 \\ a_{11} & 0 & 0 & 0 & 0 & a_{12} + c_4 \end{pmatrix}^{-1} \times \begin{pmatrix} -\beta \operatorname{Re}\{q_3\} \\ \beta \operatorname{Re}\{q_3\} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{66}$$

where

$$\begin{aligned} a_1' &= 2i\omega_{20}^* - a_1, \\ a_2' &= 2i\omega_{20}^* - a_4, \\ a_3' &= 2i\omega_{20}^* - a_7 - b_1 e^{-2i\tau_1^*\omega_{20}^*}, \\ a_4' &= 2i\omega_{20}^* - a_9 - b_2 e^{-2i\tau_1^*\omega_{20}^*}, \\ a_5' &= 2i\omega_{20}^* - a_{10} - c_3 e^{-2i\tau_{20}^*\omega_{20}^*}, \\ a_6' &= 2i\omega_{20}^* - a_{12} - c_4 e^{-2i\tau_{20}^*\omega_{20}^*}. \end{aligned} \tag{67}$$

Then, we can get the following coefficients:

$$\begin{aligned} C_1(0) &= \frac{i}{2\tau_{20}^*\omega_{20}^*}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) \\ &\quad + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_{20}^*)\}}, \\ \rho_2 &= 2\operatorname{Re}\{C_1(0)\}, \\ T_2 &= -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_{20}^*)\}}{\tau_{20}^*\omega_{20}^*}. \end{aligned} \tag{68}$$

Thus, we have the following results.

Theorem 5. *The sign of μ_2 determines direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical); the sign of ρ_2 determines stability of the bifurcating periodic solutions: if $\rho_2 < 0$ ($\rho_2 > 0$), then the bifurcating periodic solutions are stable (unstable); the sign of T_2 determines period of the bifurcating solutions: if $T_2 > 0$ ($T_2 < 0$), then the period of the bifurcating periodic solutions increases (decreases).*

4. Numerical Simulation

In this section, we present some numerical results of system (2) in order to validate the analytical predictions obtained in Sections 2 and 3. We choose a set of parameters as follows: $A = 100$, $\beta = 0.009$, $d = 0.05$, $\rho = 0.65$, $\theta = 0.05$, $\chi = 0.55$, $\gamma = 0.45$, $\alpha = 0.035$, $\delta = 0.1$, $\eta = 0.35$, and $\varepsilon = 0.07$, and consider the following special case of (2):

$$\begin{aligned} \frac{dS(t)}{dt} &= 100 - 0.009S(t)I(t) - 0.05S(t) - 0.65S(t) \\ &\quad + 0.05R(t-\tau_2) + 0.55V(t-\tau_2), \\ \frac{dE(t)}{dt} &= 0.009S(t)I(t) - 0.05E(t) - 0.45E(t), \\ \frac{dI(t)}{dt} &= 0.45E(t) - 0.05I(t) - 0.035I(t) - 0.1I(t) \\ &\quad - 0.35I(t-\tau_1), \\ \frac{dQ(t)}{dt} &= \delta I(t) - 0.05Q(t) - 0.035Q(t) \\ &\quad - 0.07Q(t-\tau_1), \\ \frac{dR(t)}{dt} &= 0.07Q(t-\tau_1) - 0.05R(t) - 0.05R(t-\tau_2) \\ &\quad + 0.35I(t-\tau_1), \\ \frac{dV(t)}{dt} &= 0.65S(t) - 0.05V(t) - 0.55V(t-\tau_2), \end{aligned} \tag{69}$$

from which we can get the unique viral equilibrium $P_*(66.0494, 277.7978, 233.6617, 150.7495, 923.3406, 71.7439)$.

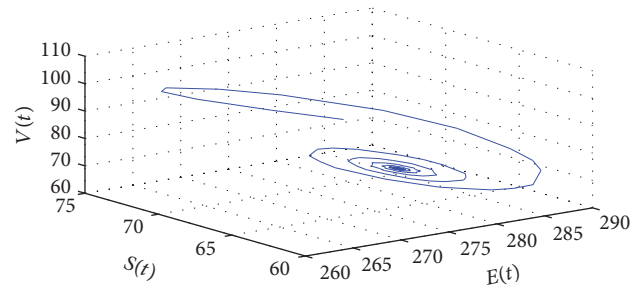

Figure 1: The projection of the phase portrait of system (69) in (S, E, V)-space with $\tau_1 = 3.605$.

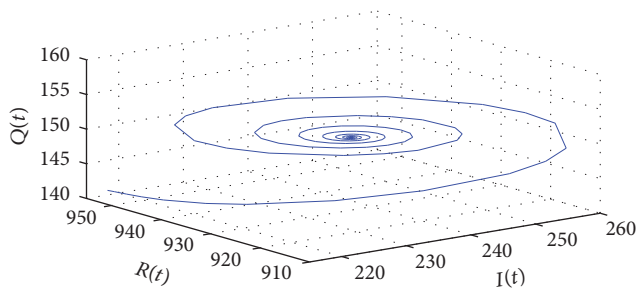

Figure 2: The projection of the phase portrait of system (69) in (I, Q, R)-space with $\tau_1 = 3.605$.

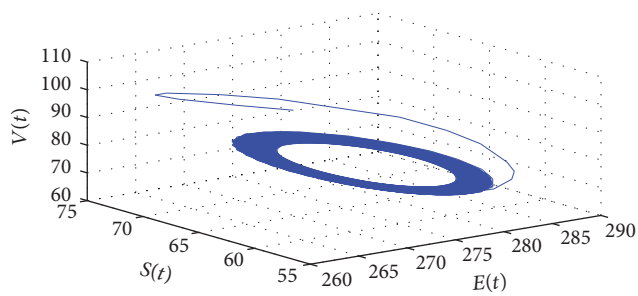

Figure 3: The projection of the phase portrait of system (69) in (S, E, V)-space with $\tau_1 = 4.60$.

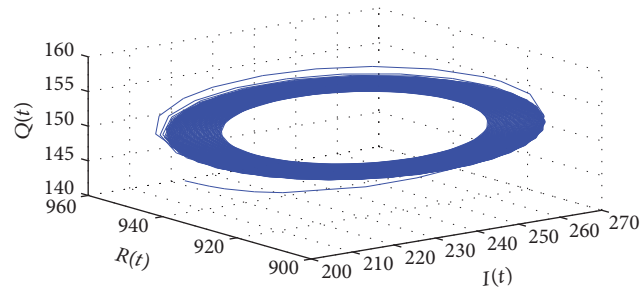

Figure 4: The projection of the phase portrait of system (69) in (I, Q, R)-space with $\tau_1 = 4.60$.

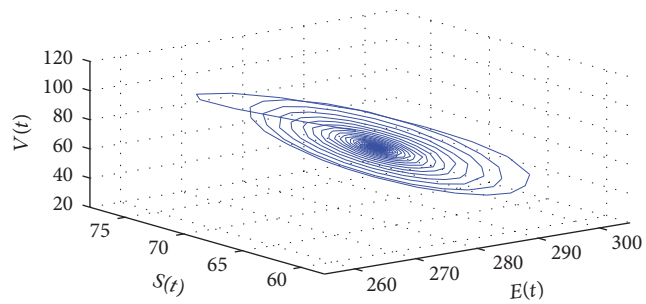

Figure 5: The projection of the phase portrait of system (69) in (S, E, V)-space with $\tau_2 = 3.65$.

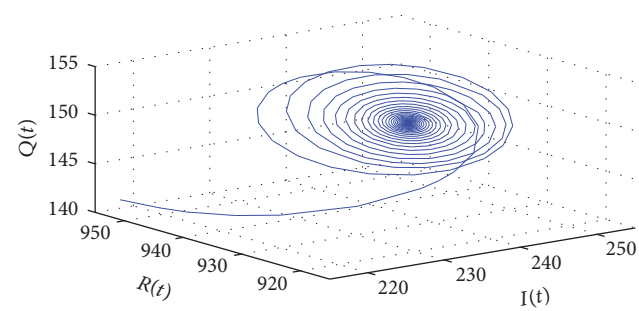

Figure 6: The projection of the phase portrait of system (69) in (I, Q, R)-space with $\tau_2 = 3.65$.

It can be easily verified that condition (H_1) is satisfied when $\tau_1 = \tau_2 = 0$.

By computation, we have $\omega_{10} = 0.8554$ and $\tau_{10} = 4.1056$. Then, we get $\lambda'(\tau_{10}) = 2.3686 + i1.0212$. Thus, we know that conditions (H_{21}) and (H_{22}) hold. We can conclude that all roots that cross the imaginary axis at $i\omega_{10}$ cross from left to right as τ_1 increases by the theory in [22]. According to Theorem 1, $P_*(66.0494, 277.7978, 233.6617, 150.7495, 923.3406, 71.7439)$ is asymptotically stable when $\tau_1 \in (0, \tau_{10})$. This property can be illustrated by Figures 1 and 2. In this case, spreading law of the computer viruses can be predicted and the viruses can be controlled and eliminated. However, once the value of τ_1 passes through the critical value τ_{10}, $P_*(66.0494, 277.7978, 233.6617, 150.7495, 923.3406, 71.7439)$ loses its stability and a Hopf bifurcation occurs, which can be shown in Figures 3 and 4. The occurrence of a Hopf bifurcation means that the state of computer viruses propagation changes from the viral equilibrium point to a limit cycle. This makes spreading of the computer viruses be out of control.

Similarly, we have the following: $\omega_{20} = 1.8255$ and $\tau_{20} = 3.7424$ when $\tau_1 = 0$ and $\tau_2 > 0$; $\omega_{10}^* = 0.9665$ and $\tau_{10}^* = 3.1862$ when $\tau_1 > 0$ and $\tau_2 = 2.25 \in (0, \tau_{20})$; $\omega_{20}^* = 2.4217$ and $\tau_{20}^* = 3.0254$ when $\tau_2 > 0$ and $\tau_1 = 2.45 \in (0, \tau_{10})$. The corresponding phase plots are shown in Figures 5–8, Figures 9–12, and Figures 13–16, respectively. In addition, for $\tau_2 > 0$ and $\tau_1 = 2.45 \in (0, \tau_{10})$, we have $C_1(0) = -17.2982 + i13.5056$ and $\lambda'(\tau_{20}^*) = 0.3796 + i2.0581$ by some complex computation. Based on (68), we get $\mu_2 = 45.5692 > 0$, $\rho_2 = -34.5964 < 0$, and $T_2 = -14.6441 < 0$. Therefore, the Hopf bifurcation is supercritical, the bifurcating periodic solutions are stable, and the period of the bifurcating periodic solutions decreases.

According to the numerical simulation results, we know that the time delay should remain less than the corresponding threshold in order to control and predict the viruses' propagation by decreasing the period that antivirus software uses

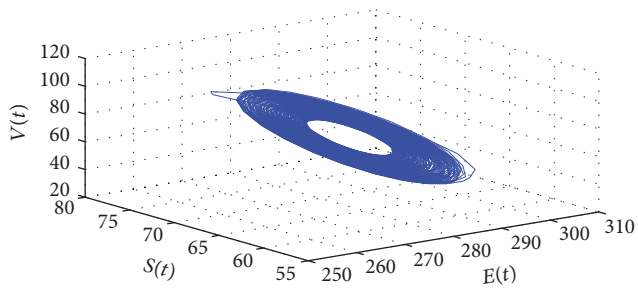

Figure 7: The projection of the phase portrait of system (69) in (S, E, V)-space with $\tau_2 = 3.805$.

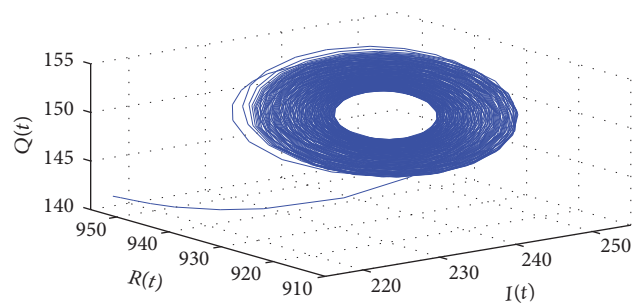

Figure 8: The projection of the phase portrait of system (69) in (I, Q, R)-space with $\tau_2 = 3.805$.

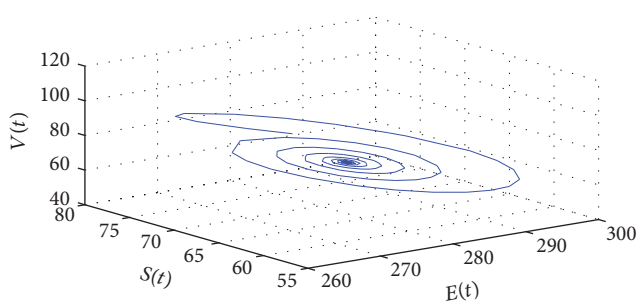

Figure 9: The projection of the phase portrait of system (69) in (S, E, V)-space with $\tau_1 = 2.86$ and $\tau_2 = 2.25 \in (0, \tau_{20})$.

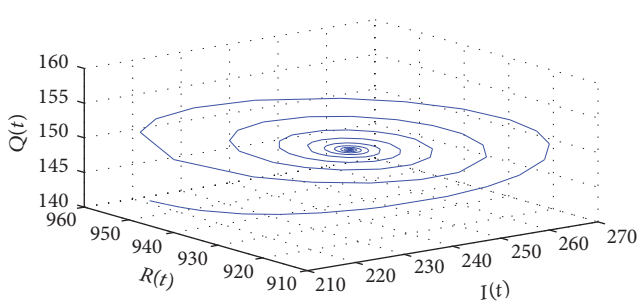

Figure 10: The projection of the phase portrait of system (69) in (I, Q, R)-space with $\tau_1 = 2.86$ and $\tau_2 = 2.25 \in (0, \tau_{20})$.

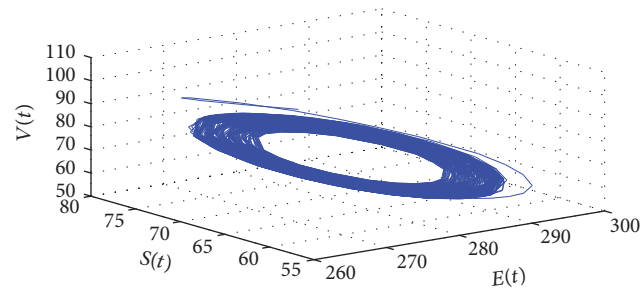

Figure 11: The projection of the phase portrait of system (69) in (S, E, V)-space with $\tau_1 = 3.574$ and $\tau_2 = 2.25 \in (0, \tau_{20})$.

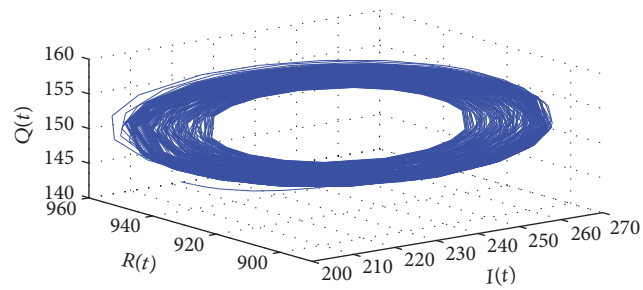

Figure 12: The projection of the phase portrait of system (69) in (I, Q, R)-space with $\tau_1 = 3.574$ and $\tau_2 = 2.25 \in (0, \tau_{20})$.

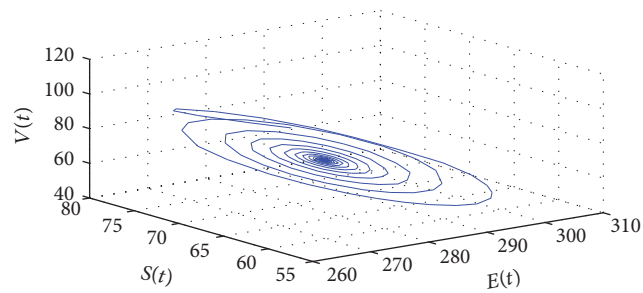

Figure 13: The projection of the phase portrait of system (69) in (S, E, V)-space with $\tau_2 = 2.862$ and $\tau_1 = 2.45 \in (0, \tau_{10})$.

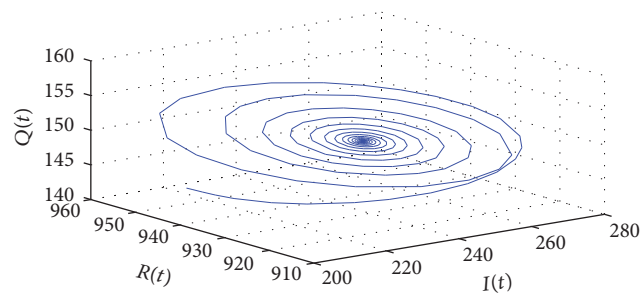

Figure 14: The projection of the phase portrait of system (69) in (I, Q, R)-space with $\tau_2 = 2.862$ and $\tau_1 = 2.45 \in (0, \tau_{10})$.

to clean the computer viruses and the temporary immunity period of the recovered and the vaccinated computers. To this end, we can adjust the parameters of our proposed model in real-world networks, such as timely updating the antivirus software on computers, properly controlling the number of computers attached to the network, and timely disconnecting computers from the network when the connections are unnecessary. Of course, in the next step, we also need to collect large amount of relevant data and estimate the parameters involved in our proposed model through statistical analysis in real-world networks. Namely, we have to adjust the parameters in the model so as to control viruses' propagation effectively if it is necessary.

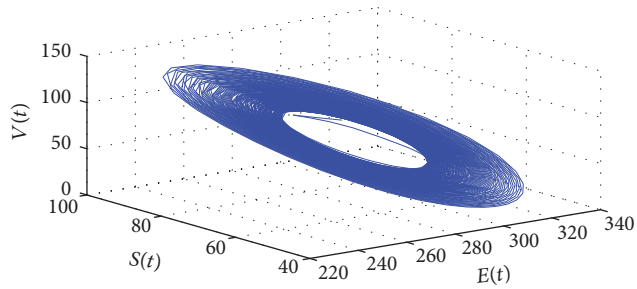

Figure 15: The projection of the phase portrait of system (69) in (S, E, V)-space with $\tau_2 = 3.225$ and $\tau_1 = 2.45 \in (0, \tau_{10})$.

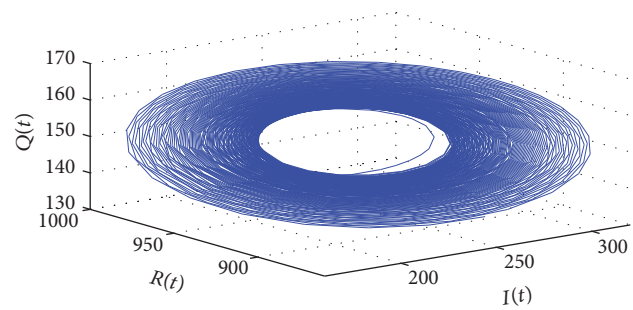

Figure 16: The projection of the phase portrait of system (69) in (I, Q, R)-space with $\tau_2 = 3.225$ and $\tau_1 = 2.45 \in (0, \tau_{10})$.

5. Conclusions

It is definitely an interesting work to consider the effect of delays on dynamical systems, because a stability switch occurs even when an ignored delay is small for a dynamical system. Based on this fact, we introduce the time delay due to the period that antivirus software uses to clean the computer viruses in the infectious and quarantined computers (τ_1) and the time delay due to the temporary immunity period of the recovered and the vaccinated computers (τ_2) into the SEIQRS-V computer virus propagation model considered in [21]. We obtain some conditions for local stability and Hopf bifurcation occurring by analyzing distribution of roots of the associated characteristic equation.

By computation, there exists a corresponding critical value of the time delay below which system (2) is stable and above which system (2) is unstable. When the system is stable, the characteristics of the propagation of computer viruses can be easily predicted and then the computer viruses can get eliminated. Otherwise, the propagation of the computer viruses is out of control. Therefore, stability of the computer virus propagation system must be guaranteed in practice. In addition, we find that the effect of τ_2 on system (2) is marked compared with τ_1, because the critical value of τ_2 is much smaller when we only consider it. At last, we have also derived the explicit formula which can determine direction and stability of the Hopf bifurcation under the case where $\tau_1 \in (0, \tau_{10})$ and $\tau_2 > 0$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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