Research Article

Some New Existence and Uniqueness Results for an Integral Boundary Value Problem of Caputo Fractional Differential Equations

Haixing Feng and Chengbo Zhai

1College of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan 030031, China
2School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, China

Correspondence should be addressed to Chengbo Zhai; cbzhai@sxu.edu.cn

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In this work, we consider an integral boundary value problem of Caputo fractional differential equations. Based on a fixed-point theorem of generalized concave operators, we obtain the existence and uniqueness of positive solutions. As applications of main results, we give two examples in the end.

1. Introduction

The fractional differential equations appear naturally in fields of physics, chemistry, electrodynamics of complex medium, control of dynamical systems, and so on. So differential equations with fractional order have been studied by many researchers, and then the theory of fractional boundary value problems has been noticed more and more in recent years; see [1–26], for example.

In [25], the author investigated the existence of positive solutions for an integral boundary value problem of fractional differential equations by using a fixed-point theorem on cones. In [21, 22], Yang gave the existence and nonexistence results for fractional differential equation integral boundary value problems. In these papers, we can find that there are no uniqueness results on positive solutions. Moreover, there are still very few results on the uniqueness of positive solutions for an integral boundary value problem of Caputo fractional differential equations. To fill this gap, we study the existence and uniqueness of positive solutions for the following integral boundary value problem for fractional differential equations

\[ \begin{align*}
C^\alpha D_0^\alpha x(t) + f(t, x(t)) &= 0, & t &\in (0, 1), \\
ax(0) - bx'(0) &= 0, \\
cx(1) + dx'(1) &= 0, \\
x''(0) + x'''(0) &= \int_0^1 x''(\tau)dp(\tau), \\
x''(1) + x'''(1) + \int_0^1 x''(\tau)dq(\tau) &= 0,
\end{align*} \]

where \( 3 < \alpha \leq 4, \ a, b, c, d \geq 0, \ \rho = ad + ac + bc > 0. \)

In [19], the authors studied the existence of positive solutions for (1) by the fixed-point theorems on cones. We also do not find the uniqueness results on positive solutions in [19]. In this paper, we will show the existence and uniqueness of positive solutions for problem (1). Our main tool is a fixed-point theorem of generalized concave operators in ordered Banach spaces.

The paper is organized as follows: in Section 2, we will present some useful definitions, preliminaries, and lemmas. The existence and uniqueness results are proved in Section 3. In Section 4, we finish the paper with two examples.

2. Preliminaries

Now we list some conditions for convenience.
(A₁) \( f(t, x) \geq 0 \), \( f(t, x) \) is increasing in \( x \in [0, +\infty) \) for each \( t \in [0, 1] \).

(A₂) For \( \lambda \in (0, 1) \), there exists \( \alpha(\lambda) \in (\lambda, 1) \) such that \( f(t, \lambda x) \geq \alpha(\lambda) f(t, x) \), \( t \in [0, 1] \), \( x \geq 0 \).

(A₃) \( p(t), q(t) \) are right continuous on \([0, 1]\), left continuous at \( t = 0 \), and are represented by

\[
\int_{0}^{1} dp(t) < 1, \quad 0 \leq \int_{0}^{1} dq(t) < 1, \quad 0 \leq \int_{0}^{1} \tau dq(t) < 1, \quad 0 \leq \int_{0}^{1} \tau dp(t) < 1, \tag{2}
\]

\[
\frac{2 + \int_{0}^{1} \tau dq(t)}{1 + \int_{0}^{1} dq(t)} > \frac{1 - \int_{0}^{1} \tau dp(t)}{1 - \int_{0}^{1} dp(t)}.
\]

(A₄) \( f(t, x) \) is continuous with \( \int_{0}^{1} (1 - s)^{\alpha - 4} f(s, x(s)) ds \) existing.

(A₄') \( f(t, x) \) is decreasing in \( x \in [0, +\infty) \) for each \( t \in [0, 1] \) and \( f(t, x) \geq 0 \) for \( [0, 1] \times [0, +\infty) \).

(A₄”) For any \( \lambda \in (0, 1) \), there exists \( \beta(\lambda) \in (0, 1) \) such that

\[
f(t, \lambda x) \leq \lambda^{-\beta(\lambda)} f(t, x), \quad t \in [0, 1], \quad x \geq 0.
\]

In the following, we present some definitions and lemmas.

**Definition 1** (see [12]). The Riemann-Liouville fractional integral of order \( \alpha > 0 \) for function \( y \) is

\[
I_{0+}^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t - s)^{\alpha - 1} y(s) ds, \quad \alpha > 0, \tag{4}
\]

provided that the right side is point-wise defined on \([0, +\infty)\).

**Definition 2** (see [12]). The Riemann-Liouville fractional derivative of order \( \alpha > 0 \) for function \( y \) is

\[
D_{0+}^\alpha y(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_{0}^{t} \frac{y(s)}{(t - s)^{\alpha-n+1}} ds, \quad \alpha > 0, \tag{5}
\]

where \( n = [\alpha] + 1 \), provided that the right side is point-wise defined on \([0, +\infty)\).

**Definition 3** (see [23]). If \( y \in AC^n[0, 1] \), then the Caputo fractional derivatives of order \( \alpha > 0 \) exist almost everywhere on \([0, 1]\) and are represented by

\[
\begin{align*}
C D_{0+}^\alpha y(t) &= \frac{1}{\Gamma(n - \alpha)} \int_{0}^{t} y^{(n)}(s) (t - s)^{\alpha-n+1} ds, \\
&= \left( I_{0+}^{\alpha-n} D^n y \right)(t).
\end{align*}
\]

Next we recall some concepts. Let \( E \) be a real Banach space with norm \( \| \cdot \| \), and let \( \theta \) be the zero element of \( E \). \( P \) is a cone in \( E \). For all \( x, y \in E \), the notation \( x \sim y \) means that there exist \( \lambda > 0 \), \( \mu > 0 \) such that \( \lambda x \leq y \leq \mu x \). Clearly, \( \sim \) is an equivalence relation. Given \( h > \Theta \), we denote by \( P_h \) the set \( P_h := \{ x \in E \mid x \sim y \} \). Clearly, \( P_h \subset P \) is convex and \( \lambda P_h = P_h \) for all \( \lambda > 0 \).

**Definition 4.** A cone \( P \in E \) is said to be normal if and only if there exists a constant \( N > 0 \) such that

\[
\theta \leq x \leq y \implies \| x \| \leq N \| y \|, \tag{7}
\]

\[
x, y \in E,
\]

where \( N \) is called the normality constant of \( P \).

**Definition 5.** An operator \( A : E \to E \) is increasing (decreasing) if \( x \leq y \) implies \( Ax \leq Ay \) (\( Ax \geq Ay \)).

**Lemma 6** (see [19]). Suppose that (A₃), (A₄) hold and \( 3 < \alpha \leq 4 \); then the integral boundary value problem

\[
C D_{0+}^\alpha x(t) + f(t, x(t)) = 0, \quad t \in (0, 1),
\]

\[
ax(0) - bx'(0) = 0,
\]

\[
 cx(1) + dx'(1) = 0,
\]

\[
x''(0) + x'''(0) = \frac{1}{0} x''( \tau ) dp( \tau ),
\]

\[
x''(1) + x'''(1) + \frac{1}{0} x''( \tau ) dq( \tau ) = 0
\]

has a solution

\[
x(t) = \int_{0}^{1} G_1(t, \xi) \int_{0}^{1} G_2(\xi, s) f(s, x(s)) ds d\xi, \tag{9}
\]

where

\[
G_1(t, s) = \begin{cases} 
\frac{1}{\rho} (a\tau + b)[c(1 - \tau) + d], & 0 \leq s \leq t \leq 1, \\
\frac{1}{\rho} (a\tau + b)[c(1 - s) + d], & 0 \leq t \leq s \leq 1,
\end{cases} \tag{10}
\]

\[
G_2(t, s) = G_a(t, s) + G_{\rho}(t, s), \quad t, s \in [0, 1], \tag{11}
\]

\[
G_a(t, s) = \begin{cases} 
\frac{(1 - \tau)^{\alpha-3}(1 - \tau) + (s - \tau)^{\alpha-3}(1 - \tau)}{\Gamma(\alpha - 2)}, & s \leq t, \\
\frac{(1 - \tau)^{\alpha-3}(1 - \tau) + (s - \tau)^{\alpha-4}(1 - \tau)}{\Gamma(\alpha - 3)}, & t \leq s,
\end{cases}
\]
\[ G_{pq}(t,s) = \frac{1}{\delta} \left[ (2 + \int_0^1 t \, dq(r)) - t \left( 1 + \int_0^1 dq(r) \right) \right] \]
\[ \cdot \int_0^1 G_a(r,s) \, dp(r) \]
\[ + \frac{1}{\delta} \left[ (1 - \int_0^1 t \, dp(r)) - t \left( 1 + \int_0^1 dp(r) \right) \right] \]
\[ \cdot \int_0^1 G_a(r,s) \, dq(r), \quad t, s \in [0,1]. \]
\[ \delta = \frac{1 - \int_0^1 t \, dp(r) \cdot \int_0^1 dq(r)}{1 + \int_0^1 dq(r) \cdot \int_0^1 t \, dp(r)} > 0. \]

\[ G_{pq}(t,s) = \frac{1}{\delta} \left[ (2 + \int_0^1 t \, dq(r)) - t \left( 1 + \int_0^1 dq(r) \right) \right] \]
\[ \cdot \int_0^1 G_a(r,s) \, dp(r) \]
\[ + \frac{1}{\delta} \left[ (2 + \int_0^1 t \, dq(r)) - t \left( 1 + \int_0^1 dq(r) \right) \right] \]
\[ \cdot \int_0^1 G_a(r,s) \, dp(r) \]
\[ + \frac{1}{\delta} \left[ (1 - \int_0^1 t \, dp(r)) - t \left( 1 + \int_0^1 dp(r) \right) \right] \]
\[ \cdot \int_0^1 G_a(r,s) \, dq(r), \quad t, s \in [0,1]. \]
\[ \delta = \frac{1 - \int_0^1 t \, dp(r)}{1 + \int_0^1 dp(r)} > 0. \] (12)

**Lemma 7** (see [19]). Let \( 3 < \alpha \leq 4 \). Then Green's function \( G_2(t,s) \) has the following properties:

(i) \( G_2(t,s) \in C([0,1] \times [0,1]), \)
\[ G_2(t,s) > 0, \quad t, s \in (0,1). \] (13)

(ii) \( G_2(t,s) \geq m(t) M_{pq} M_a(s), \)
\[ \max_{0 \leq t \leq 1} G_2(t,s) \leq M_{pq} M_a(s), \]
\[ t \in [0,1], \quad s \in (0,1), \] (14)

where
\[ M_{pq} = 1 + \frac{1}{\delta} \left[ (2 + \int_0^1 \tau \, dq(r)) \int_0^1 dp(r) \right] \]
\[ + \int_0^1 dq(r), \quad \delta \]
\[ m(t) = \frac{1}{M_{pq}} \left( 1 - t \right) \]
\[ + \frac{1}{\delta} \left[ (2 + \int_0^1 \tau \, dq(r)) - t \left( 1 + \int_0^1 dq(r) \right) \right] \]
\[ \cdot \int_0^1 \frac{1}{2} dq(r) \]
\[ + \frac{1}{\delta} \left[ (1 - \int_0^1 t \, dp(r)) - t \left( 1 + \int_0^1 dp(r) \right) \right] \]
\[ \cdot \int_0^1 \frac{1}{2} dp(r), \quad t \in [0,1]. \] (15)

**Lemma 8.** Let \( g(t) = \int_0^1 G_1(t,s) ds; \) then
\[ g(t) = \frac{1}{\rho} \left[ -t^2 \left( bc + ac + ad \right) + a \left( \frac{c}{2} + d \right) t \right] \]
\[ + b \left( \frac{c}{2} + d \right), \quad t \in [0,1] \] (18)

and \( g(t) \geq g_0, \quad t \in [0,1], \) where \( g_0 = \min[(b/\rho)(c/2 + d), \alpha(a + b)]. \)

We now present a fixed-point theorem of generalized concave operators which will be used in the later proofs.

**Theorem 9 (see [27]).** Let \( h > \theta \) and \( P \) be a normal cone. Assume that

\[ (D_1) \quad A : P \to P \text{ is increasing and } Ah \in P_h, \]
\[ (D_2) \quad \text{for any } x \in P \text{ and } t \in (0,1), \text{ there exists } \alpha(t) \in (t,1) \text{ with respect to } t \text{ such that} \]
\[ A(tx) \geq \alpha(t) A x. \] (19)

Then

(i) there are \( u_0, v_0 \in P_h \) and \( r \in (0,1) \) such that
\[ r v_0 \leq u_0 < v_0, \]
\[ u_0 \leq A u_0 \leq A v_0 \leq v_0; \] (20)

(ii) operator equation \( x = Ax \) has a unique solution in \( P_h. \)

**Remark 10.** An operator \( A \) is said to be generalized concave if \( A \) satisfies condition \( (D_2). \)

3. Existence and Uniqueness of Positive Solutions for Problem (1)

In this section, we use Theorem 9 to study problem (1) and we obtain some new results on the existence and uniqueness of positive solutions. This is also the main motivation for the study of (1) in the present work.

Set \( J = [0,1], C[J,R] = \{ x : J \to R \text{ is continuous} \} \) is a Banach space. Evidently, \( C[J,R] \) is a Banach space with the norm \( \|x\|_C = \sup \|x(t)\| : t \in J \), \( P = \{ x \in C[J,R] : x(t) \geq 0, \ t \in J \} \), the standard cone, and it is normal. Our main result is summarized in the following theorem.

**Theorem 11.** Assume that \( (A_1)-(A_4) \) hold and \( f(t, bc + bd) > 0, f(t, ad + bd) > 0, \ t \in [0,1]. \) Then

(i) there exist \( u_0, v_0 \in P_h \) such that
\[ u_0(t) \leq \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, u_0(s)) \, ds \, d\xi, \]
\[ t \in J, \]
\[ v_0(t) \geq \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, v_0(s)) \, ds \, d\xi, \]
\[ t \in J; \] (21)
(ii) the integral boundary value problem (1) has a unique positive solution $x^*$ in $P_h$, where $h(t) = (at + b)[c(1 - t) + d]$, $t \in J$.

Proof. Define an operator $A : C[J, R] \rightarrow C[J, R]$ by

$$
Ax(t) = \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, x(s)) ds d\xi.
$$

(22)

We know that $x$ is a solution of problem (1) if and only if $x$ is a fixed point of the operator $A$.

Firstly, we show that $A : P \rightarrow P$ is increasing, generalized concave. From $(A_1)$, we know that $f(t, x(t)) \geq 0$, and from (10) and (13) we have $G_1(t, s) \geq 0$, $G_2(t, s) \geq 0$, so we have $Ax(t) \geq 0$ for $x \in P$. It follows from $(A_1)$ that $A : P \rightarrow P$ is increasing.

Now we prove that $A : P \rightarrow P$ is generalized concave. For any $x \in P$ and $\lambda \in (0, 1)$, from $(A_2)$ we have

$$
A(\lambda x)(t) = \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, \lambda x(s)) ds d\xi
\geq \alpha(\lambda) \int_0^1 (t, \xi) \int_0^1 G_2(\xi, s) f(s, x(s)) ds d\xi
= \alpha(\lambda) Ax(t).
$$

(23)

Thus, we have $A(\lambda x) \geq \alpha(\lambda) Ax$, $x \in P$, $\lambda \in (0, 1)$.

Secondly, we prove that $A : P \rightarrow P$ is generalized concave. For any $x \in P$ and $\lambda \in (0, 1)$, from $(A_2)$ we have

$$
A(\lambda x)(t) = \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, \lambda x(s)) ds d\xi
\leq \frac{h(t)}{\rho} \int_0^1 M_\alpha(s) ds.
$$

(24)

From (10), it is easy to obtain that

$$
\frac{h(t)}{(a + b)(c + d)} G_1(\xi, s) \leq G_1(1, s) \leq \frac{h(t)}{\rho},
$$

(25)

$\forall \xi \in [0, 1], t, s \in [0, 1]$.

From (14), we have

$$
m(t) M_p q M_\alpha(s) \leq G_2(t, s) \leq M_p q M_\alpha(s).
$$

(26)

We know by (16) that $m(t)$ is decreasing, so when $t = 1$, $m(t)$ obtains the minimum, and $m(1) > 0$, so we have

$$
m(1) M_p q M_\alpha(s) \leq G_2(t, s) \leq M_p q M_\alpha(s).
$$

(27)

and from (17), we have

$$
\int_0^1 M_\alpha(s) ds = \int_0^1 \left[ \frac{2(1 - s)^{\alpha - 3}}{\Gamma(\alpha - 2)} + \frac{(1 - s)^{\alpha - 4}}{\Gamma(\alpha - 3)} \right] ds
= \frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)}.
$$

(28)

So from (25), (27), and (28), (22) has the following form:

$$
Ah(t) = \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, h(s)) ds d\xi
\leq \frac{h(t)}{\rho} \int_0^1 G_2(\xi, s) f(s, h(s)) ds
\leq \frac{h(t)}{\rho} M_p q \int_0^1 M_\alpha(s) ds.
$$

(29)

After simple calculation, we get $h'(t_0) = 0$, where $t_0 = (ac + ad - bc)/2ac$. So we need to discuss several cases:

(i) When $0 \leq t_0 \leq 1$, we have

$$
f(t, h(t)) \leq f(t, h(t_0)), t \in [0, 1].
$$

(30)

Let $f_{01} = \max\{f(t, h(t)), t \in [0, 1]\};$ then $f_{01} \geq f(t, h(t_0)) \geq f(t, h(0)) = f(t, bc + bd) > 0$. From (28) and (29), we obtain that

$$
Ah(t) \leq \frac{h(t)}{\rho} M_p q f_{01} \int_0^1 M_\alpha(s) ds
= \frac{M_p q f_{01}}{\rho} \left[ \frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \right] \cdot h(t).
$$

(31)

(ii) When $t_0 > 1$, we have

$$
f(t, h(t)) \leq f(t, h(1)) = f(t, ad + bd), t \in [0, 1].
$$

(32)

Let $f_{02} = \max\{f(t, ad + bd), t \in [0, 1]\};$ then $f_{02} \geq f(t, h(1)) = f(t, ad + bd) > 0$. From (28) and (29), we obtain that

$$
Ah(t) \leq \frac{h(t)}{\rho} M_p q f_{02} \int_0^1 M_\alpha(s) ds
= \frac{M_p q f_{02}}{\rho} \left[ \frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \right] \cdot h(t).
$$

(33)

(iii) When $t_0 < 0$, we have

$$
f(t, h(t)) \leq f(t, h(0)) = f(t, bc + bd), t \in [0, 1].
$$

(34)

Let $f_{03} = \max\{f(t, bc + bd), t \in [0, 1]\};$ then $f_{03} \geq f(t, h(0)) = f(t, bc + bd) > 0$. From (28) and (29), we obtain that

$$
Ah(t) \leq \frac{h(t)}{\rho} M_p q f_{03} \int_0^1 M_\alpha(s) ds
= \frac{M_p q f_{03}}{\rho} \left[ \frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \right] \cdot h(t).
$$

(35)
Let

\[ \mu_1 = \frac{M_{pq}}{\rho} \left[ \frac{2}{(\alpha-2) \Gamma(\alpha-2)} + \frac{1}{(\alpha-3) \Gamma(\alpha-3)} \right] \]

\[
\begin{align*}
&f_{01}, \quad t_0 \in [0, 1], \\
&f_{02}, \quad t_0 > 1, \\
&f_{03}, \quad t_0 < 0.
\end{align*}
\]

(36)

Then from (31), (33), and (35), we have

\[ A \mathcal{h}(t) \leq \mu_1 \mathcal{h}(t), \quad t \in [0, 1]. \]  

(37)

On the other hand, from (25), (27), and (28), we have

\[
A \mathcal{h}(t) = \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, \mathcal{h}(s)) ds d\xi
\geq \frac{h(t)}{(a+b)(c+d)} m(1) M_{pq} \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, \mathcal{h}(s)) ds d\xi.
\]

(38)

Also, we need to discuss the following cases:

(i) When \(0 \leq t_0 \leq 1\), we have

\[
f(t, h(t)) \geq \min \{f(t, h(0)), f(t, h(1))\}, \quad t \in [0, 1]
\]

(39)

We can see that \(f_{04} = \min\{f(t, h(0)), f(t, h(1))\}, \quad t \in [0, 1]\) > 0. From (28), (38), and Lemma 8, we obtain that

\[
A \mathcal{h}(t) \geq \frac{h(t)}{(a+b)(c+d)} m(1) 
\cdot M_{pq} f_{04} \left[ \frac{2}{(\alpha-2) \Gamma(\alpha-2)} + \frac{1}{(\alpha-3) \Gamma(\alpha-3)} \right] 
\geq m(1) M_{pq} f_{04} g_0 \left[ \frac{1}{(a+b)(c+d)} \right] h(t).
\]

(40)

(ii) When \(t_0 > 1\), we have

\[ f(t, h(t)) \geq f(t, h(0)) = f(t, bc + bd), \quad t \in [0, 1]. \]  

(41)

Let \(f_{05} = \min\{f(t, bc + bd), \quad t \in [0, 1]\}\), and then \(f_{05} > 0\). From (28), (38), and Lemma 8, we obtain that

\[
A \mathcal{h}(t) \geq \frac{h(t)}{(a+b)(c+d)} m(1)
\cdot M_{pq} f_{05} \left[ \frac{2}{(\alpha-2) \Gamma(\alpha-2)} + \frac{1}{(\alpha-3) \Gamma(\alpha-3)} \right] 
\geq m(1) M_{pq} f_{05} g_0 \left[ \frac{2}{(a+b)(c+d)} \right] h(t).
\]

(42)

Let \(f_{06} = \min\{f(t, ad + bd), \quad t \in [0, 1]\}\), and then \(f_{06} > 0\). From (28), (38), and Lemma 8, we obtain that

\[
A \mathcal{h}(t) \geq \frac{h(t)}{(a+b)(c+d)} m(1)
\cdot M_{pq} f_{06} \left[ \frac{2}{(\alpha-2) \Gamma(\alpha-2)} + \frac{1}{(\alpha-3) \Gamma(\alpha-3)} \right] 
\geq m(1) M_{pq} f_{06} g_0 \left[ \frac{2}{(a+b)(c+d)} \right] h(t).
\]

(44)

So, from (37) and (46) we have

\[
\mu_2 h(t) \leq A \mathcal{h}(t) \leq \mu_1 h(t), \quad t \in [0, 1].
\]

(47)
that is, $Ah \in P_h$. Finally, an application of Theorem 9 implies that 

(i) there exist $u_0, v_0 \in P_h$ such that $u_0 \leq Au_0$, $v_0 \geq Av_0$, that is,

$$u_0(t) \leq \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, u_0(s)) \, ds \, d\xi,$$

$$v_0(t) \geq \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, v_0(s)) \, ds \, d\xi,$$

$t \in J$; \tag{48}

(ii) operator equation $x = Ax$ has a unique solution $x^*$ in $P_h$. That is, $x^*(t)$ is the unique positive solution for problem (1) in $P_h$. \hfill \square

**Theorem 12.** Assume that $(A_1^t)$, $(A_2^t)$, $(A_3)$, $(A_4)$ hold and $f(t, x) > 0$ for $x > 0$, $t \in [0, 1]$. Then

(i) there exist $u_0, v_0 \in P_h$ such that

$$u_0(t) \leq \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, u_0(s)) \, ds \, d\xi,$$

$$v_0(t) \geq \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, v_0(s)) \, ds \, d\xi,$$

$t \in J$; \tag{49}

where

$$\pi_0(t) = \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, u_0(s)) \, ds \, d\xi,$$

$$\tau_0(t) = \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, v_0(s)) \, ds \, d\xi,$$

$t \in J$; \tag{50}

(ii) the integral boundary value problem (1) has a unique positive solution $x^*$ in $P_h$, where $h(t) = (at + b)(c(1-t) + d)$, $t \in J$.

**Proof.** Similar to the proof of Theorem 11, we consider the operator $A : C[I, R] \to C[I, R]$ by

$$Ax(t) = \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, x(s)) \, ds \, d\xi.$$ \tag{51}

From $(A_1^t)$, we know that $Ax(t) \geq 0$, $t \in [0, 1]$. It follows from $(A_3^t)$ that $A : P \to P$ is decreasing. For any $x \in P$, $\lambda \in (0, 1)$ and from $(A_2^t)$, we have

$$A(\lambda x)(t) = \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, x(s)) \, ds \, d\xi$$

$$\leq \lambda^{\beta(\lambda)} \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, x(s)) \, ds \, d\xi$$

$$= \lambda^{\beta(\lambda)} A(x(t)).$$ \tag{52}

Thus, we have $A(\lambda x) \leq \lambda^{\beta(\lambda)} A(x(t))$, $x \in P$, $\lambda \in (0, 1)$. Further, for $\lambda \in (0, 1)$, $x \in P$, we have

$$Ax = A\left(\lambda \cdot \frac{1}{\lambda} x\right) \leq \lambda^{\beta(\lambda)} A\left(\frac{1}{\lambda} x\right).$$ \tag{53}

So we obtain $A((1/\lambda)x) \geq \lambda^{\beta(\lambda)} A(x(t))$, $x \in P$, $\lambda \in (0, 1)$. Consequently, $A^2 : P \to P$ is increasing and for $x \in P$, $\lambda \in (0, 1)$,

$$A^2(\lambda x) = A(A(\lambda x)) \geq A(\lambda^{\beta(\lambda)} A(x))$$

$$= A\left(\frac{1}{\lambda^{\beta(\lambda)}} A^2(x)\right) \geq \lambda^{\beta(\lambda)} A^2 x.$$ \tag{54}

Let $\alpha(t) = t^{\beta(\lambda)}$, $t \in (0, 1)$. Then $t < \alpha(t) < 1$ and $A^2(\lambda x) \geq \alpha(\lambda) A^2 x$, $\lambda \in (0, 1)$, $x \in P$. So the operator $A^2 : P \to P$ is generalized concave.

Next we prove that $A^2 h \in P_h$. Note that $h(t) = (at + b)(c(1-t) + d)$, and we can obtain

$$Ah(t) = \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, h(s)) \, ds \, d\xi,$$

$$A^2h(t) = A(A(t))$$

$$= \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, A(h(s))) \, ds \, d\xi.$$ \tag{55}

Because $h'(t_o) = 0$, where $t_o = (ac + ad - bc)/2ac$, we need to discuss several cases:

(i) When $0 \leq t_o \leq 1$, we have

$$f(t, h(t)) \leq \min\{f(t, h(0)), f(t, h(1))\}, t \in [0, 1]$$

$$= f_{04},$$ \tag{56}

and we can see that $f_{04} = \min\{f(t, h(0)), f(t, h(1))\}, t \in [0, 1] > 0$. From (28) and (29), we obtain that

$$Ah(t) \leq \frac{h(t)}{\rho} M_{pq} f_{04} \int_0^1 M_\alpha(s) \, ds$$

$$= \frac{M_{pq} f_{04}}{\rho} \left[\frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)}\right] h(t).$$ \tag{57}
(ii) When \( t_0 > 1 \), we have
\[
 f(t, h(t)) \leq f(t, h(0)) = f(t, bc + bd), \quad t \in [0,1],
\] (58)

Note that \( f_{03} = \max \{ f(t, bc + bd), \ t \in [0,1] \} \); then \( f_{03} \geq f(t, h(0)) = f(t, bc + bd) > 0 \). From (28) and (29), we obtain that
\[
 A h(t) \leq \frac{h(t)}{\rho} M_{pq} f_{03} \int_0^1 M_a(s) \, ds \\
= \frac{M_{pq} f_{03}}{\rho} \left[ \frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \right] \cdot h(t).
\] (59)

(iii) When \( t_0 < 0 \), we have
\[
 f(t, h(t)) \leq f(t, h(1)) = f(t, ad + bd), \quad t \in [0,1].
\] (60)

Note that \( f_{02} = \max \{ f(t, ad + bd), \ t \in [0,1] \} \); then \( f_{02} \geq f(t, h(1)) = f(t, ad + bd) > 0 \). From (28) and (29), we obtain that
\[
 A h(t) \leq \frac{h(t)}{\rho} M_{pq} f_{02} \int_0^1 M_a(s) \, ds \\
= \frac{M_{pq} f_{02}}{\rho} \left[ \frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \right] \cdot h(t).
\] (61)

Let
\[
 \mu_3 = \frac{M_{pq}}{\rho} \left[ \frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \right]
\]
\[
\begin{cases} 
 f_{04}, & t_0 \in [0,1], \\
 f_{03}, & t_0 > 1, \\
 f_{02}, & t_0 < 0.
\end{cases}
\] (62)

Then from (57), (59), and (61), we have
\[
 A h(t) \leq \mu_3 h(t), \quad t \in [0,1].
\] (63)

Further, we need to discuss the following cases:

(i) When \( 0 \leq t_0 \leq 1 \), we have
\[
 f(t, h(t)) \geq f(t, h(t_0)), \quad t \in [0,1].
\] (64)

Let \( f_{07} = \min \{ f(t, h(t_0)), \ t \in [0,1] \} \), and then \( f_{07} > 0 \). From (28), (38), and Lemma 8, we obtain that
\[
 A h(t) \geq \frac{h(t)}{(a + b) (c + d)} m(1)
\]
\[
\cdot M_{pq} f_{07} \left[ \frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \right]
\]
\[
\cdot \int_0^1 G_1(r, \xi) \, d\xi
\]
\[
\geq m(1) M_{pq} f_{07} \left[ \frac{2}{(a + b) (c + d)} \right] \Gamma(\alpha - 2)
\]
\[
+ \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \cdot h(t).
\] (65)

(ii) When \( t_0 > 1 \), we have
\[
 f(t, h(t)) \geq f(t, h(1)) = f(t, ad + bd), \quad t \in [0,1].
\] (66)

Note that \( f_{06} = \min \{ f(t, ad + bd), \ t \in [0,1] \} \); then \( f_{06} \geq f(t, h(1)) = f(t, ad + bd) > 0 \). From (28) and (38), we obtain that
\[
 A h(t) \geq \frac{h(t)}{(a + b) (c + d)} m(1)
\]
\[
\cdot M_{pq} f_{06} \left[ \frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \right]
\]
\[
\cdot \int_0^1 G_1(r, \xi) \, d\xi
\]
\[
\geq m(1) M_{pq} f_{06} \left[ \frac{2}{(a + b) (c + d)} \right] \Gamma(\alpha - 2)
\]
\[
+ \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \cdot h(t).
\] (67)

(iii) When \( t_0 < 0 \), we have
\[
 f(t, h(t)) \geq f(t, h(0)) = f(t, bc + bd), \quad t \in [0,1].
\] (68)

Note that \( f_{05} = \min \{ f(t, bc + bd), \ t \in [0,1] \} \); then \( f_{05} \geq f(t, h(0)) = f(t, bc + bd) > 0 \). From (28) and (38), we obtain that
\[
 A h(t) \geq \frac{h(t)}{(a + b) (c + d)} m(1)
\]
\[
\cdot M_{pq} f_{05} \left[ \frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \right]
\]
\[
\cdot \int_0^1 G_1(r, \xi) \, d\xi
\]
\[
\geq m(1) M_{pq} f_{05} \left[ \frac{2}{(a + b) (c + d)} \right] \Gamma(\alpha - 2)
\]
\[
+ \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \cdot h(t).
\] (69)
Let
\[
\mu_4 = \frac{m(1) M_{pq} g_0}{(a + b) (c + d)} \left[ \frac{2}{\Gamma(\alpha - 2)} + \frac{1}{\Gamma(\alpha - 3)} \right]
\]
\[
\times \begin{cases} 
  f_{w_0}, & t_0 \in [0, 1], \\
  f_{w_0}, & t_0 > 1, \\
  f_{w_0}, & t_0 < 0.
\end{cases}
\] (70)

Then from (65), (67), and (69), we have
\[
Ah(t) \geq \mu_4 h(t), \quad t \in [0, 1].
\] (71)

So, from (63) and (71) we have
\[
\mu_4 h(t) \leq A h(t) \leq \mu_3 h(t).
\] (72)

Hence, we have
\[
A^2 h(t) \leq \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, \mu_4 h(s)) ds d\xi,
\] (73)
\[
A^2 h(t) \geq \int_0^1 G_1(t, \xi) \int_0^1 G_2(\xi, s) f(s, \mu_4 h(s)) ds d\xi.
\] (74)

Because \( h'(t_0) = 0 \), where \( t_0 = (ac + ad - bc)/2ac \), we need to discuss the following cases:

(i) When \( 0 \leq t_0 \leq 1 \), we have
\[
f(t, \mu_4 h(t)) \leq \min \{ f(t, \mu_4 h(0)), f(t, \mu_4 h(1)), t \in [0, 1] \}
\]
\[= f_{w_0},
\] (75)

and we can see that \( f_{w_0} = \min \{ f(t, \mu_4 h(0)), f(t, \mu_4 h(1)), t \in [0, 1] \} > 0 \). From (28) and (73), we obtain that
\[
A^2 h(t) \leq \frac{h(t)}{\rho} \left[ \frac{2}{\Gamma(\alpha - 2)} + \frac{1}{\Gamma(\alpha - 3)} \right]
\]
\[
\times \int_0^1 M_\alpha(s) ds.
\] (76)

(ii) When \( t_0 > 1 \), we have
\[
f(t, \mu_4 h(t)) \leq f(t, \mu_4 h(1)) = f(t, \mu_3 (bc + bd)),
\] (77)

Let \( f_{w_0} = \max \{ f(t, \mu_4 h(0)), f(t, \mu_4 h(1)) \} = f(t, \mu_3 h(t)) \geq 0 \). From (28) and (73), we obtain that
\[
A^2 h(t) \leq \frac{h(t)}{\rho} \left[ \frac{2}{\Gamma(\alpha - 2)} + \frac{1}{\Gamma(\alpha - 3)} \right]
\]
\[
\times \int_0^1 M_\alpha(s) ds.
\] (78)

(iii) When \( t_0 < 0 \), we have
\[
f(t, \mu_4 h(t)) \leq f(t, \mu_4 h(1)) = f(t, \mu_3 (ad + bd)),
\] (79)
\[t \in [0, 1].
\]

Let \( f_{11} = \max \{ f(t, \mu_3 h(t)), t \in [0, 1] \} \), then \( f_{11} > 0 \). From (28) and (73), we obtain that
\[
A^2 h(t) \geq \frac{h(t)}{\rho} \left[ \frac{2}{\Gamma(\alpha - 2)} + \frac{1}{\Gamma(\alpha - 3)} \right]
\]
\[
\times \int_0^1 M_\alpha(s) ds
\] (84)

Similarly, we need to discuss the following cases:

(i) When \( 0 \leq t_0 \leq 1 \), we have
\[
f(t, \mu_3 h(t)) \geq f(t, \mu_3 h(t_0)), \quad t \in [0, 1].
\] (83)

Let \( f_{11} = \min \{ f(t, \mu_3 h(t_0)), t \in [0, 1] \} \), then \( f_{11} > 0 \). From (28), (74), and Lemma 8, we obtain that
\[
A^2 h(t) \geq \frac{h(t)}{\rho} \left[ \frac{2}{\Gamma(\alpha - 2)} + \frac{1}{\Gamma(\alpha - 3)} \right]
\]
\[
\times \int_0^1 G_1(t, \xi) d\xi
\] (84)

(ii) When \( t_0 > 1 \), we have
\[
f(t, \mu_3 h(t)) \geq f(t, \mu_3 h(1)) = f(t, \mu_5 (ad + bd)),
\] (85)
\[t \in [0, 1].
\]
Let $f_{12} = \min\{f(t, \mu_2(ad + bd)), t \in [0, 1]\}$, and then $f_{12} > 0$. From (28), (74), and Lemma 8, we obtain that

\[ A^2 h(t) \geq \frac{h(t)}{(a + b)(c + d)} m(1) \]

\[ \cdot M_{pq} f_{12} \left[ \frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \right] \]

\[ \cdot \int_0^1 G_1 (t, \xi) d\xi \]  \hspace{1cm} (86)

\[ \geq m(1) M_{pq} f_{12} g_0 \left[ \frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \right] h(t). \]  \hspace{1cm} (87)

(iii) When $t_0 < 0$, we have

\[ f(t, \mu_3 h(t)) \geq f(t, \mu_3 h(0)) = f(t, \mu_3 (bc + bd)), \]  \hspace{1cm} (88)

\[ t \in [0, 1]. \]

Note that $f_{11} = \min\{f(t, \mu_2(bc + bd)), t \in [0, 1]\} > 0$ and from (28), (74), and Lemma 8, we obtain that

\[ A^2 h(t) \geq \frac{h(t)}{(a + b)(c + d)} m(1) \]

\[ \cdot M_{pq} f_{12} \left[ \frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \right] \]

\[ \cdot \int_0^1 G_1 (t, \xi) d\xi \]

\[ \geq m(1) M_{pq} f_{12} g_0 \left[ \frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \right] h(t). \]  \hspace{1cm} (89)

Let

\[ \mu_2' = m(1) M_{pq} g_0 \left[ \frac{2}{(\alpha - 2) \Gamma(\alpha - 2)} + \frac{1}{(\alpha - 3) \Gamma(\alpha - 3)} \right] \]

\[ \left\{ \begin{array}{ll} f_{11}, & t_0 \in [0, 1], \\ f_{12}, & t_0 > 1, \\ f_{13}, & t_0 < 0. \end{array} \right. \]  \hspace{1cm} (90)

Then from (84), (86), and (88), we have

\[ A^2 h(t) \geq \mu_2' h(t), \]  \hspace{1cm} (91)

\[ t \in [0, 1]. \]

From (82) and (90), we can get

\[ \mu_2' h(t) \leq A^2 h(t) \leq \mu_1' h(t), \]  \hspace{1cm} (92)

that is, $A^2 h \in P_h$. Finally, an application of Theorem 9 implies the following: (i) there exist $u_0, v_0 \in P_h$ such that $u_0 \leq A^2 u_0, v_0 \geq A^2 v_0$; (ii) $A^2 x = x$ has a unique solution $x^*$ in $P_h$. Let $u_0 = A u_0, v_0 = A v_0$, and then $u_0 \leq A u_0, v_0 \geq A v_0$. That is,

\[ u_0(t) \leq \int_0^1 G(t, \xi) \int_0^1 G_2 (\xi, s) f(s, \overline{u}_0(s)) ds d\xi, \]  \hspace{1cm} (93)

\[ v_0(t) \geq \int_0^1 G(t, \xi) \int_0^1 G_2 (\xi, s) f(s, \overline{v}_0(s)) ds d\xi, \]  \hspace{1cm} (94)

\[ t \in J, \]  \hspace{1cm} (95)

\[ t \in J. \]  \hspace{1cm} (96)

Moreover, $A^2 x^* = x^*$. Next, we show that $x^*$ is the unique fixed point of $A$ in $P_h$. In view of $A^2(A x^*) = A(A^2 x^*) = A x^*$, by the uniqueness of solutions for the operator equation $x = A^2 x$, we have that $A x^* = x^*$. Suppose that $y$ is another fixed point of $A$ in $P_h$, so we have $A^2 y = A(A y) = A y = y$. Hence, we have $x^* = y$. So problem (1) has a unique positive solution $x^*$ in $P_h$. \hfill \Box

4. Examples

Example 1. Consider the following fractional boundary value problem of form (1):

\[ C_{D_0^{1/3}} x(t) + (1 - t)^{1/3} x^{1/3} + 1 = 0, \]  \hspace{1cm} (97)

\[ x(0) = x(1) = 0, \]  \hspace{1cm} (98)

\[ x''(0) + x'''(0) = \int_0^1 x''(\tau) d\left( \frac{1}{2} \tau \right), \]  \hspace{1cm} (99)

\[ x''(1) + x'''(1) + \int_0^1 x''(\tau) d\left( \frac{1}{6} \tau \right) = 0. \]  \hspace{1cm} (100)

Conclusion 1. Problem (93) has a unique positive solution in $P_h$, where $h(t) = t(1 - t), t \in [0, 1]$.

Proof. From (93), we know $\alpha = 11/3$, $f(t, x) = (1 - t)^{1/3} x^{1/3} + 1$, and it is not difficult to find the following:

\[ (1) \ f(t, bc + bd) = f(t, ad + bd) = f(t, 0) = 1 > 0, \]  \hspace{1cm} (101)

\[ t \in [0, 1] \]  \hspace{1cm} (102)

and $f(t, x)$ is continuous, increasing in $x \in [0, +\infty)$.

\[ (2) \ Let \ \alpha(\lambda) = \lambda^{1/3}, \]  \hspace{1cm} (103)

\[ \text{then } \alpha(\lambda) \in (1, 1), \]  \hspace{1cm} (104)

\[ \text{and we have} \]  \hspace{1cm} (105)

\[ f(t, \lambda x) = (1 - t)^{1/3} (\lambda x)^{1/3} + 1 \]

\[ \geq \lambda^{1/3} \left[ (1 - t)^{1/3} x^{1/3} + 1 \right] = \alpha(\lambda) f(t, x). \]  \hspace{1cm} (106)
(3) From (93), we know
\[ p(t) = \frac{1}{2} t, \quad q(t) = \frac{1}{6} t, \]
\[ \int_0^1 dp(t) = \frac{1}{2}, \quad \int_0^1 dq(t) = \frac{1}{6}. \] (95)

so we can get
\[ \frac{2 + \int_0^1 \tau dq(\tau)}{1 + \int_0^1 dq(\tau)} = \frac{2 + \int_0^1 \tau d((1/6)\tau)}{1 + \int_0^1 d((1/6)\tau)} = \frac{25}{14}, \]
\[ \frac{1 - \int_0^1 \tau dp(\tau)}{1 - \int_0^1 dp(\tau)} = \frac{1 - \int_0^1 \tau d((1/2)\tau)}{1 - \int_0^1 d((1/2)\tau)} = \frac{21}{14}. \] (96)

(4)
\[ \int_0^1 (1-s)^{1/3-4} \left[ (1-s)^{1/3} x^{1/3} (s) + 1 \right] ds \]
\[ = \int_0^1 x^{1/3} (s) ds + \int_0^1 (1-s)^{-1/3} ds \] (97)
exists.

So we claim that the conditions \((A_1)-(A_4)\) hold. Hence, by Theorem 11, we can obtain that (93) has a unique positive solution in \(P_h\), where \(h(t) = t(1-t), \quad t \in [0,1]\). □

Example 2. Consider the following fractional boundary value problem of form (1):
\[ {}^cD_0^{1/3} x(t) + \frac{(1-t)^{1/3}}{1 + x^{1/3}} + 1 = 0, \quad t \in (0,1), \]
\[ x(0) = x(1) = 0, \quad x''(0) + x'''(0) = \int_0^1 x''(\tau) d\left(\frac{1}{2}\tau\right), \quad x''(1) + x'''(1) + \int_0^1 x''(\tau) d\left(\frac{1}{6}\tau\right) = 0. \] (98)

Conclusion 2. Problem (98) has a unique positive solution in \(P_h\), where \(h(t) = t(1-t), \quad t \in [0,1]\).

Proof. From (98), we know \(\alpha = 11/3, \quad f(t,x) = (1-t)^{1/3}/(1+x^{1/3}) + 1\), and it is not difficult to find the following:
(1) \(f(t,x) \geq 1 > 0, \quad t \in [0,1], \quad x \in [0, +\infty)\) and \(f(t,x)\) is decreasing in \(x \in [0, +\infty)\).
(2) Let \(\beta(\lambda) = 1/3, \) and then \(\beta(\lambda) \in (0,1)\), so we have
\[ f(t,\lambda x) = \frac{(1-t)^{1/3}}{1 + (\lambda x)^{1/3}} + 1 \leq \lambda^{-1/3} \left[ \frac{(1-t)^{1/3}}{1 + x^{1/3}} + 1 \right], \] (99)
\[ = \lambda^{-\beta(\lambda)} f(t,x). \]

(3) From (98), we know
\[ p(t) = \frac{1}{2} t, \quad q(t) = \frac{1}{6} t, \]
\[ \int_0^1 dp(t) = \frac{1}{2}, \quad \int_0^1 dq(t) = \frac{1}{6}. \] (100)

so we can get
\[ \frac{2 + \int_0^1 \tau dq(\tau)}{1 + \int_0^1 dq(\tau)} = \frac{2 + \int_0^1 \tau d((1/6)\tau)}{1 + \int_0^1 d((1/6)\tau)} = \frac{25}{14}, \]
\[ \frac{1 - \int_0^1 \tau dp(\tau)}{1 - \int_0^1 dp(\tau)} = \frac{1 - \int_0^1 \tau d((1/2)\tau)}{1 - \int_0^1 d((1/2)\tau)} = \frac{21}{14}. \] (101)

(4)
\[ \int_0^1 (1-s)^{1/3-4} \left[ \frac{(1-s)^{1/3}}{1 + x^{1/3}} (s) + 1 \right] ds \]
\[ = \int_0^1 x^{1/3} (s) ds + \int_0^1 (1-s)^{-1/3} ds \] (102)
exists.

So the conditions \((A'_{1}), (A'_{2}), (A_{3}), (A_{4})\) hold. Hence, by Theorem 12, we can obtain that (98) has a unique positive solution in \(P_h\), where \(h(t) = t(1-t), \quad t \in [0,1]\). □

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

Authors’ Contributions

The authors declare that they share equal responsibility. All authors read and approved the final manuscript.

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