Research Article

Exponential Stability of Traveling Waves for a Reaction Advection Diffusion Equation with Nonlinear-Nonlocal Functional Response

Rui Yan¹ and Guirong Liu²

¹*School of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan, Shanxi 030006, China*

²*School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, China*

Correspondence should be addressed to Guirong Liu; lgr5791@sxu.edu.cn

Received 23 June 2017; Revised 26 September 2017; Accepted 9 October 2017; Published 6 November 2017

Academic Editor: Charalampos Skokos

Copyright © 2017 Rui Yan and Guirong Liu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The stability of a reaction advection diffusion equation with nonlinear-nonlocal functional response is concerned. By using the technical weighted energy method and the comparison principle, the exponential stability of all noncritical traveling waves of the equation can be obtained. Moreover, we get the rates of convergence. Our results improve the previous ones. At last, we apply the stability result to some real models, such as an epidemic model and a population dynamic model.

1. Introduction

As an important nonlinear parabolic equation, reaction (advection) diffusion equations are usually used to describe the development of practical problems related to time and space distribution. The large number of problems involved in reaction-diffusion equations arises from a large number of mathematical models in chemistry, physics, ecology, biology, and infectious diseases, for instance, [1–4]. The traveling wave solutions of reaction-diffusion equations refer to the solutions of the shape, such as $u(t, x) = u(x + ct)$, where the wave velocity c is a real constant. In practical applications, traveling wave solutions can well explain oscillations and finite velocity propagation phenomena in nature such as the change of the concentration of a reactant in a chemical reaction, the invasion of species in biology, the mutual transformation of two states of matter in condensed matter physics, the propagation of nerve impulses in neural networks, and the spread of infectious diseases. The stability of traveling wave solutions based on the existence of traveling wave equations is also very important and has great theoretical and practical value. In the phase transition process, a necessary condition that the state change can be observed is that the traveling wave

solutions are stable. For example, in the sense of epidemiology, the traveling wave solutions describe the transition from a disease-free equilibrium to an endemic equilibrium, while the existence and nonexistence of nontrivial traveling wave solutions suggest whether or not the disease can spread. The results will help predict the developing trend of infectious diseases, identify the key factors for the spread of infectious diseases, and seek the best strategies to prevent and control the spread of infectious diseases.

Hence, the study of traveling wave solutions has become a very interesting problem, in which the stability of traveling waves is significant and difficult, and this topic attracts enormous attention. For the stability of reaction-diffusion equations without time delay, there is a great deal of literature, for example, [5–9]. For the delayed reaction-diffusion equations, Schaaf [10] and the authors [11] studied the stability of the traveling waves. Then, by applying the upper and lower solutions method combined with the squeezing technique, Wang et al. [12] presented the exponential stability of traveling wavefronts for a reaction advection diffusion equation with spatiotemporal delays under the bistable assumptions. Mei et al. [13, 14] proved that the large waves were exponentially stable when an initial solution is close

enough to the traveling wave solution under a weighted L^2 norm. Later, Mei and coauthors [15, 16] obtained the globally exponential stability of the traveling waves by using the similar methods in [17] with some improvement. Recently, Mei et al. [18] gave the globally exponential stability of all noncritical wavefronts and the globally algebraic stability of critical wavefronts for a nonlocal time-delayed reaction-diffusion equations by using the weighted energy method and the Fourier transform. Immediately, Mei and Wang [19] obtained the similar results for a class of nonlocal time-delayed Fisher-KPP type reaction-diffusion equations in n -dimensional space by employing the similar methods, in which the convergence rates were more accurate than previous results.

Motivated by these previous works, in this paper, we consider the stability of the traveling waves of the following reaction advection diffusion equation with nonlinear-nonlocal functional response

$$\begin{aligned} \frac{\partial u(t, x)}{\partial t} &= D \frac{\partial^2 u(t, x)}{\partial^2 x} + B \frac{\partial u(t, x)}{\partial x} \\ &+ F(u(t, x), (\mathcal{K}^* f(u))(t, x)), \end{aligned} \quad (1)$$

$$t \geq 0, x \in \mathbb{R},$$

with the initial data

$$u(s, x) = u_0(s, x), \quad s \in [-r, 0], x \in \mathbb{R}, \quad (2)$$

where $B \in \mathbb{R}, r \geq 0, D > 0$ is a diffusion coefficient, and the convolution is defined by

$$\begin{aligned} (\mathcal{K}^* f(u))(t, x) \\ = \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(s, y) f(u(t-s, x-y)) dy ds. \end{aligned} \quad (3)$$

Some well-known models can be obtained by choosing proper cases of F, f and $\mathcal{K}$. For example, some of them can be generalized as below.

(i) Set $B = 0, f(u) = u$ and $\mathcal{K}(s, y) = \delta(s-r)\delta(y)$; then (1) turns to the following local reaction-diffusion equation with discrete delay

$$u_t(t, x) = D\Delta u(t, x) + F(u(t, x), u(t-r, x)). \quad (4)$$

Schaaf [10] studied the stability of the traveling waves of (4) by a phase plane analysis method. Then, the authors in [11] showed the global asymptotic stability, Lyapunov stability, and uniqueness of traveling waves of (4) by using the elementary super- and subsolution comparison and squeezing methods.

(ii) Set $B = 0$ and $\mathcal{K}(s, y) = \delta(s-r)J(y)$; then (1) turns to the following nonlocal reaction-diffusion equation with discrete time delay

$$\begin{aligned} u_t(t, x) \\ = D\Delta u(t, x) \\ + F\left(u(t, x), \int_{-\infty}^{\infty} J(y) f(u(t-r, x-y)) dy\right). \end{aligned} \quad (5)$$

In [20], the existence of traveling wavefronts of (5) was investigated by using the sub- and supersolution method. The authors also proved the asymptotic stability and uniqueness of traveling wave fronts by applying the comparison principle and squeezing technique. Later, Wang [21] proved the existence of the traveling wave solutions of (5) by using the upper and lower solutions method and the Schauder fixed point theorem when the term f may not be monotone or quasimonotone. Recently, the authors [22] got the exponential stability of all noncritical traveling waves of (5) and the algebraic stability of critical traveling waves of (5) by using the weighted energy method coupled with the comparison principle.

It follows from the above examples that these local or nonlocal equations with discrete time delay are studied intensively. In some cases, a discrete delay is a good approximation, but the distributed delay is necessary in others. On the other hand, reaction advection diffusion equations have been usually applied to describe the dispersion courses in the mobile media such as fluids; see, for example, [12, 23]. In fact, Wu [24] proved the exponential stability of traveling wavefronts with large wave velocity of (1) by using the comparison principle and the technical weighted energy method. By choosing a different weighted energy function from [24] and adopting different estimates, then we can get the stability of all noncritical traveling wavefronts of (1), not just the traveling waves with large wave velocity.

Now we impose some assumptions on (1) as follows:

- (A₁) $\mathcal{K}(s, y)$ is a continuous nonnegative function with $\mathcal{K}(s, -y) = \mathcal{K}(s, y)$ for $s \geq 0, y \in \mathbb{R}$ and $\int_0^r \int_{-\infty}^{\infty} \mathcal{K}(s, y) dy ds = 1, \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(s, y) e^{-\lambda(y+cs)} dy ds < \infty$ for any $c \geq 0$ and $\lambda \geq 0$.
- (A₂) For any $t \in [0, r], x \in \mathbb{R}, \mathcal{K}(t, x) = \overline{\mathcal{K}}(t, Bt + x)$, where $\overline{\mathcal{K}}$ satisfies $\overline{\mathcal{K}}(t, x) = \overline{\mathcal{K}}(t, -x)$.
- (A₃) Let $K > 0, F \in C^2([0, K] \times [0, f(K)], \mathbb{R}), F(0, f(0)) = F(K, f(K)) = 0, F(x, f(x)) > 0$, for $x \in (0, K), \partial_2 F(x, y) > 0, \partial_1 F(0, 0) < 0, \partial_{ij} F(x, y) \leq 0$ ($i, j = 1, 2$) for any $(x, y) \in [0, K] \times [0, f(K)]$, where $\partial_1 F(x, y) = \partial F(x, y)/\partial x$ and $\partial_2 F(x, y) = \partial F(x, y)/\partial y$.
- (A₄) $f \in C^2([0, K], \mathbb{R}_+), f(0) = 0, f'(x) \geq 0, f''(x) \leq 0$ for any $x \in [0, K]$.
- (A₅) $f'(0)f'(K)\partial_2 F(0, 0)\partial_2 F(K, 0) - (\partial_1 F(K, f(K)))^2 < 0$.
- (A₆) $\partial_2 F(K, f(K))f'(K) + \partial_1 F(K, f(K)) < 0$.

From (A₃), it is easy to see that (1) has two constant equilibria $u_{\pm}$, where

$$\begin{aligned} u_- &= 0, \\ u_+ &= K. \end{aligned} \quad (6)$$

Throughout this paper, a traveling wave solution of (1) always refers to a pair (ϕ, c) , where $\phi = \phi(\xi)$ ($\xi = x + ct$) is a function on $\mathbb{R}$ satisfying the following ordinary differential equation:

$$(c - B) \phi'(\xi) = D\phi''(\xi) + F(\phi(\xi)),$$

$$\int_0^r \int_{-\infty}^{\infty} \mathcal{K}(s, y) f(\phi(\xi - y - cs)) dy ds, \quad (7)$$

$$\phi(\pm\infty) = u_{\pm}.$$

We call c the traveling wave speed.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries and present our main result on the exponential stability of traveling waves of (1). In Section 3, we will give the proof the main result by using the weighted energy method. In Section 4, we apply our results to some models, such as an epidemic model and a population dynamic model.

2. Preliminaries and Main Results

In this paper, we suppose that $C > 0$ represents a generic constant and $C_i > 0$ denotes a particular constant. Set I as an interval; typically $I = \mathbb{R}$. Take

$$L^p(I) = \left\{ f(x) : \int_I f^p(x) dx < \infty, x \in I \right\}, \quad p \geq 1,$$

$$L^p_{\omega}(I) = \left\{ f(x) : \int_I \omega(x) f^p(x) dx < \infty, x \in I \right\},$$

$$p \geq 1, \omega(x) \geq 0, \quad (8)$$

$$H^k(I) = \left\{ f(x) \in L^2(I) : \frac{d^i f}{dx^i} \in L^2(I), i = 1, \dots, k, x \in I \right\}$$

with the norm defined by

$$\|f\|_{L^p_{\omega}} = \left(\int_I \omega(x) |f(x)|^p dx \right)^{1/p}, \quad p \geq 1,$$

$$\|f\|_{H^k(I)} = \left(\sum_{i=0}^k \int_I \omega(x) \left| \frac{d^i f}{dx^i} \right|^2 dx \right)^{1/2}. \quad (9)$$

Moreover, set $T > 0$ as a constant and $\mathcal{B}$ as a Banach space; we denote by $C([0, T]; \mathcal{B})$ the space of the $\mathcal{B}$ -valued continuous functions on $[0, T]$. The corresponding spaces of the $\mathcal{B}$ -valued functions on $[0, \infty)$ are defined similarly.

Now we show the following existence result of traveling wave solutions of (1).

Theorem 1 ([24], existence of traveling waves). *Assume that $(A_1)-(A_4)$ hold; there exist numbers $c_* > 0$ and $\lambda_*(c_*) > 0$ satisfying*

$$\mathcal{P}(\lambda_*, c_* + B) = 0,$$

$$\frac{\partial}{\partial \lambda} \mathcal{P}(\lambda, c_* + B)|_{\lambda_*} = 0, \quad (10)$$

where

$$\mathcal{P}(\lambda, c)$$

$$= D\lambda^2 - (c - B)\lambda + \partial_1 F(0, 0)$$

$$+ \partial_2 F(0, 0) f'(0) \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(s, y) e^{-\lambda(y+cs)} dy ds. \quad (11)$$

For every $c > c_* + B$, (1) has a monotone traveling wave solution $\phi(x + ct)$ such that $\phi(-\infty) = 0$ and $\phi(+\infty) = K$.

From the assumptions we can introduce that $\partial_1 F(0, 0) + \partial_2 F(0, 0) f'(0) > 0$. Then by a simple calculation similar to that of [21] we can get the following conclusions. When $c > c_* + B$, $\mathcal{P}(\lambda, c) = 0$ has two positive real roots $\lambda_1(c)$ and $\lambda_2(c)$ with $0 < \lambda_1(c) < \lambda_2(c)$ which also satisfies

$$\mathcal{P}(\lambda, c) < 0 \quad \text{for } \lambda_1 < \lambda < \lambda_2. \quad (12)$$

When $c = c_* + B$, it holds that

$$\mathcal{P}(\lambda_*, c_* + B) = 0 \quad \text{for } \lambda_1 = \lambda^* = \lambda_2. \quad (13)$$

Then, for $c > c_* + B$, we define a weight function related to such a number $\lambda > 0$,

$$\omega(\xi) = \begin{cases} e^{-\lambda(\xi - \xi_0)}, & \xi \leq \xi_0, \\ 1, & \xi > \xi_0 \end{cases} \quad (14)$$

with a sufficiently large number $\xi_0 \gg 1$, where $\lambda \in (\lambda_1(c), \lambda_*)$. Obviously $\omega(\xi) \geq 1$ for all $\xi \in \mathbb{R}$ and $\omega(\xi) \rightarrow +\infty$ as $\xi \rightarrow -\infty$.

Then we give the following globally exponential stability of the traveling wavefronts of (1).

Theorem 2 (stability of traveling waves). *Suppose that $(A_1)-(A_6)$ hold. For the traveling wave $\phi(\xi)$ of (1) and $\phi(\pm\infty) = u_{\pm}$ with speed $c > c_* + B$, if the initial data holds*

$$0 = u_- \leq u_0(s, x) \leq u_+ = K,$$

$$\text{for } (s, x) \in [-r, 0] \times \mathbb{R} \quad (15)$$

and the initial perturbation is $u_0(s, x) - \phi(x + cs) \in C([-r, 0]; L^1_{\omega}(\mathbb{R}) \cap H^1(\mathbb{R}))$; then the solution of (1) and (2) satisfies

$$0 = u_- \leq u(t, x) \leq u_+ = K,$$

$$\text{for } (t, x) \in [0, +\infty) \times \mathbb{R}, \quad (16)$$

$$u(t, x) - \phi(x + ct) \in C([0, +\infty); L^1_{\omega}(\mathbb{R}) \cap H^1(\mathbb{R})).$$

If $c > c_* + B$, the solution $u(t, x)$ converges to the traveling wave $\phi(x + ct)$ exponentially

$$\sup_{x \in \mathbb{R}} |u(t, x) - \phi(x + ct)| \leq C e^{-\mu t}, \quad t \geq 0, \quad (17)$$

with a positive number μ , where $\mu = \mu_1/3$ and μ_1 is determined by

$$\begin{aligned} (c - B) \lambda - D \lambda^2 - \partial_1 F(0, 0) - \mu_1 \\ - \partial_2 F(0, 0) f'(0) e^{\mu_1 r} M(r, \lambda) > 0, \\ \lambda \in [\lambda_1(c), \lambda_*], \end{aligned} \quad (18)$$

$$M(r, \lambda) = \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) e^{-\lambda(y+c\tau)} dy d\tau,$$

$$\begin{aligned} - \partial_1 F(K, f(K)) - \mu_1 \\ - \partial_2 F(K, f(K)) f'(K) e^{\mu_1 r} > 0. \end{aligned}$$

Remark 3. The authors in [24] investigated the exponential stability of traveling waves of (1) when

$$c > \max \left\{ c_* + B, \frac{2D}{c_*} [(2 + L) L_0 L_2 + 2L_1] + B \right\}, \quad (19)$$

where

$$\begin{aligned} L_0 &= \max_{u \in [0, K]} f'(u), \\ L_i &= \max_{(u, v) \in [0, K] \times [0, f(K)]} \partial_i F(u, v), \quad i = 1, 2, \\ L &= \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) e^{-(c_*/D)(y+c\tau)} dy d\tau. \end{aligned} \quad (20)$$

In addition, [24] needs the following assumption:

$$\begin{aligned} 2 [\partial_1 F(K, f(K)) + \partial_2 F(K, f(K)) f'(K)] \\ < -L \partial_2 F(K, f(K)) f'(K). \end{aligned} \quad (21)$$

But condition (A_5) is not required in [24]. In Section 4, we can demonstrate the advantage of our result by the same example.

3. Proof of the Stability

For the stability result, we need the following boundedness and the comparison principle for (1). The proofs are similar to those of [16, 24], so we omit them here.

Lemma 4 (boundedness). Assume (A_1) – (A_4) hold and the initial data satisfy

$$\begin{aligned} 0 = u_- \leq u_0(s, x) \leq u_+ = K, \\ \text{for } (s, x) \in [-r, 0] \times \mathbb{R}; \end{aligned} \quad (22)$$

thus the solution $u(t, x)$ of the Cauchy problem (1) and (2) exists uniquely and satisfies

$$\begin{aligned} 0 = u_- \leq u(t, x) \leq u_+ = K, \\ \text{for } (t, x) \in [0, +\infty) \times \mathbb{R}. \end{aligned} \quad (23)$$

Lemma 5 (comparison principle). Set $\bar{u}(t, x)$ and $\underline{u}(t, x)$ as the solutions of (1) and (2) with the initial data $\bar{u}_0(s, x)$ and $\underline{u}_0(s, x)$, respectively. When

$$\begin{aligned} 0 = u_- \leq \underline{u}_0(s, x) \leq \bar{u}_0(s, x) \leq u_+ = K, \\ \text{for } (s, x) \in [-r, 0] \times \mathbb{R}, \end{aligned} \quad (24)$$

thus

$$\begin{aligned} 0 = u_- \leq \underline{u}(t, x) \leq \bar{u}(t, x) \leq u_+ = K, \\ \text{for } (t, x) \in [0, +\infty) \times \mathbb{R}. \end{aligned} \quad (25)$$

If $u_0(s, x)$ satisfies

$$\begin{aligned} 0 = u_- \leq u_0(s, x) \leq u_+ = K, \\ \text{for } (s, x) \in [-r, 0] \times \mathbb{R}, \end{aligned} \quad (26)$$

letting

$$\begin{aligned} U_0^+(s, x) &= \max \{u_0(s, x), \phi(x + cs)\}, \\ &\text{for } (s, x) \in [-r, 0] \times \mathbb{R}, \\ U_0^-(s, x) &= \min \{u_0(s, x), \phi(x + cs)\}, \\ &\text{for } (s, x) \in [-r, 0] \times \mathbb{R}, \end{aligned} \quad (27)$$

then

$$\begin{aligned} u_- \leq U_0^-(s, x) \leq u_0(s, x) \leq U_0^+(s, x) \leq u_+, \\ \text{for } (s, x) \in [-r, 0] \times \mathbb{R}, \\ u_- \leq U_0^-(s, x) \leq \phi(x + cs) \leq U_0^+(s, x) \leq u_+, \\ \text{for } (s, x) \in [-r, 0] \times \mathbb{R}. \end{aligned} \quad (28)$$

Set $U^+(t, x)$ and $U^-(t, x)$ as the corresponding solutions of (1) and (2) with initial data $U_0^+(s, x)$ and $U_0^-(s, x)$ determined in (27); that is,

$$\begin{aligned} U_t^\pm(t, x) &= D \Delta U^\pm(t, x) + F(U^\pm(t, x)), \\ &\int_0^r \int_{-\infty}^{\infty} \mathcal{K}(s, y) f(U^\pm(t - s, x - y)) dy ds, \\ &t \geq 0, \quad x \in \mathbb{R}, \end{aligned} \quad (29)$$

$$U^\pm(s, x) = U_0^\pm(s, x), \quad (s, x) \in [-r, 0] \times \mathbb{R}.$$

Thus by Lemma 5 we get

$$\begin{aligned} u_- \leq U^-(t, x) \leq u(t, x) \leq U^+(t, x) \leq u_+, \\ \text{for } (t, x) \in [0, +\infty) \times \mathbb{R}, \\ u_- \leq U^-(t, x) \leq \phi(x + ct) \leq U^+(t, x) \leq u_+, \\ \text{for } (t, x) \in [0, +\infty) \times \mathbb{R}. \end{aligned} \quad (30)$$

Now we need the following three steps to get the stability result.

Step 1. The convergence of $U^+(t, x)$ to $\phi(x + ct)$ is as follows.

For every $c > c_* + B$, take $\xi = x + ct$ and

$$\begin{aligned} v(t, \xi) &= U^+(t, x) - \phi(x + ct), \\ v_0(s, \xi) &= U_0^+(s, x) - \phi(x + cs). \end{aligned} \quad (31)$$

From (28) and (30) we get

$$\begin{aligned} v(t, \xi) &\geq 0, \\ v_0(s, \xi) &\geq 0. \end{aligned} \quad (32)$$

Then it can be checked that $v(t, \xi)$ defined in (31) satisfies

$$\begin{aligned} v_t(t, \xi) + (c - B)v_\xi(t, \xi) - Dv_{\xi\xi}(t, \xi) - \partial_1 F(\xi)v(t, \xi) \\ = \partial_2 F(\xi)\tilde{\Phi}(\xi) + Q \end{aligned} \quad (33)$$

with the initial data

$$v(s, \xi) = v_0(s, \xi), \quad (s, \xi) \in [-r, 0] \times \mathbb{R}, \quad (34)$$

where

$$\begin{aligned} \widehat{\Phi}(\xi) &= \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(s, y) f(\phi(\xi - y - cs)) dy ds, \\ \widetilde{\Phi}(\xi) &= \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(s, y) f'(\phi(\xi - y - cs)) v(t - s, \xi \\ &\quad - y - cs) dy ds, \\ \overline{\Phi}(\xi) &= \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(s, y) f(v(t - s, \xi - y - cs) \\ &\quad + \phi(\xi - y - cs)) dy ds, \\ F(\xi) &= F(\phi(\xi), \widehat{\Phi}(\xi)), \end{aligned} \quad (35)$$

$$\begin{aligned} Q &= F(v(t, \xi) + \phi(\xi), \overline{\Phi}(\xi)) - F(\xi) - \partial_1 F(\xi)v(t, \xi) \\ &\quad - \partial_2 F(\xi)\tilde{\Phi}(\xi). \end{aligned} \quad (36)$$

Lemma 6. For $\omega_1 = e^{-\lambda(\xi - \xi_0)}$ ($\lambda \in (\lambda_1(c), \lambda^*]$), by choosing $\mu_1 > 0$ such that

$$\begin{aligned} (c - B)\lambda - D\lambda^2 - \partial_1 F(0, 0) - \mu_1 \\ - \partial_2 F(0, 0)f'(0)e^{\mu_1 r}M(r, \lambda) = C_1 > 0, \end{aligned} \quad (37)$$

where

$$M(r, \lambda) = \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) e^{-\lambda(y + c\tau)} dy d\tau, \quad (38)$$

then, for $c > c_* + B$, the following holds:

$$\begin{aligned} \|v(t)\|_{L^1_{\omega_1}(\mathbb{R})} + C_1 \int_0^t \int_{-\infty}^{\infty} e^{-\mu_1(t-s)} \omega_1(\xi) v(s, \xi) d\xi ds \\ \leq Ce^{-\mu_1 t}. \end{aligned} \quad (39)$$

Proof. Assume the solution of (33)-(34) is mollifying enough. Multiplying (33) by $\omega_1(\xi)e^{\mu_1 t}$, where μ_1 is defined in (37), we get

$$\begin{aligned} (e^{\mu_1 t} \omega_1 v)_t + e^{\mu_1 t} [(c - B)\omega_1 v - D\omega_1 v_\xi + D\omega_1' v]_\xi \\ - \mu_1 e^{\mu_1 t} \omega_1 v + e^{\mu_1 t} [-(c - B)\omega_1' - D\omega_1'' - \partial_1 F(\xi) \\ \cdot \omega_1] v = e^{\mu_1 t} \omega_1 \left[\partial_2 F(\xi) \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) \right. \\ \cdot f'(\phi(\xi - y - cs)) \\ \cdot v(s - \tau, \xi - y - c\tau) dy d\tau + Q \Big]. \end{aligned} \quad (40)$$

Integrating both sides of (40) over $\mathbb{R} \times [0, t]$ with respect to ξ and t , thus

$$\begin{aligned} e^{\mu_1 t} \int_{-\infty}^{\infty} \omega_1(\xi) v(t, \xi) d\xi + \int_0^t \int_{-\infty}^{\infty} e^{\mu_1 s} [-(c - B)\omega_1' - D\omega_1'' - \partial_1 F(\xi)\omega_1 - \mu_1 \omega_1] v d\xi ds = \|v_0(0)\|_{L^1_{\omega_1}(\mathbb{R})} \\ + \int_0^t \int_{-\infty}^{\infty} e^{\mu_1 s} \omega_1(\xi) Q(s, \xi) d\xi ds \\ + \int_0^t \int_{-\infty}^{\infty} e^{\mu_1 s} \omega_1(\xi) \left[\partial_2 F(\xi) \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) f'(\phi(\xi - y - cs)) v(s - \tau, \xi - y - c\tau) dy d\tau \right] d\xi ds. \end{aligned} \quad (41)$$

For Q , applying Taylor's formula to (36) and noting (A_3) -(A_4); then

$$\begin{aligned} Q &\leq \frac{1}{2} \partial_{11} F(\bar{\phi}, \bar{\phi}) v^2 + \partial_{12} F(\bar{\phi}, \bar{\phi}) [\overline{\Phi} - \widehat{\Phi}] v \\ &\quad + \frac{1}{2} \partial_{22} F(\bar{\phi}, \bar{\phi}) [\overline{\Phi} - \widehat{\Phi}]^2 \leq 0, \end{aligned} \quad (42)$$

where $\bar{\phi}$ is some function between ϕ and $\phi + v$ and $\bar{\phi}$ is some function between $\overline{\Phi}$ and $\widehat{\Phi}$.

For the third term on the right-hand side of (41) by Fubini's theorem and making the change of variables $\xi - y - c\tau \rightarrow \xi, s - \tau \rightarrow s, y \rightarrow y, \tau \rightarrow \tau$, it follows that

$$\begin{aligned}
& \int_0^t \int_{-\infty}^{\infty} e^{\mu_1 s} \omega_1(\xi) \left[\partial_2 F(\xi) \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) f'(\phi(\xi - y - c\tau)) v(s - \tau, \xi - y - c\tau) dy d\tau \right] d\xi ds \\
&= \int_0^r \int_{-\infty}^{\infty} \int_0^t \int_{-\infty}^{\infty} e^{\mu_1 s} \omega_1(\xi) \partial_2 F(\xi) \mathcal{K}(\tau, y) f'(\phi(\xi - y - c\tau)) v(s - \tau, \xi - y - c\tau) d\xi ds dy d\tau \\
&= \int_0^r \int_{-\infty}^{\infty} \int_{-\tau}^t \int_{-\infty}^{\infty} e^{\mu_1(s+\tau)} \omega_1(\xi + y + c\tau) \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) f'(\phi(\xi)) v(s, \xi) d\xi ds dy d\tau \\
&\leq e^{\mu_1 r} \int_0^t \int_{-\infty}^{\infty} \int_0^r \int_{-\infty}^{\infty} e^{\mu_1 s} \omega_1(\xi + y + c\tau) \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) f'(\phi(\xi)) v(s, \xi) dy d\tau d\xi ds \\
&+ e^{\mu_1 r} \int_{-r}^0 \int_{-\infty}^{\infty} \int_0^r \int_{-\infty}^{\infty} e^{\mu_1 s} \omega_1(\xi + y + c\tau) \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) f'(\phi(\xi)) v_0(s, \xi) dy d\tau d\xi ds.
\end{aligned} \tag{43}$$

Substituting (42) and (43) into (41), it follows that

$$\begin{aligned}
& e^{\mu_1 t} \int_{-\infty}^{\infty} \omega_1(\xi) v(t, \xi) d\xi + \int_0^t \int_{-\infty}^{\infty} e^{\mu_1 s} \omega_1(\xi) v(s, \xi) A_{c, \mu_1, \omega_1}(\xi) d\xi ds \leq \|v_0(0)\|_{L^1_{\omega_1}(\mathbb{R})} \\
& + e^{\mu_1 r} \int_{-r}^0 \int_{-\infty}^{\infty} \int_0^r \int_{-\infty}^{\infty} e^{\mu_1 s} \omega_1(\xi + y + c\tau) \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) f'(\phi(\xi)) v_0(s, \xi) dy d\tau d\xi ds,
\end{aligned} \tag{44}$$

where

$$\begin{aligned}
A_{c, \mu_1, \omega_1}(\xi) &= -(c - B) \frac{\omega_1'(\xi)}{\omega_1(\xi)} - D \frac{\omega_1''(\xi)}{\omega_1(\xi)} - \partial_1 F(\xi) \\
&- \mu_1 \\
&- e^{\mu_1 r} \int_0^r \int_{-\infty}^{\infty} \frac{\omega_1(\xi + y + c\tau)}{\omega_1(\xi)} \partial_2 F(\xi + y + c\tau) \\
&\cdot \mathcal{K}(\tau, y) f'(\phi(\xi)) dy d\tau.
\end{aligned} \tag{45}$$

Since $\omega_1'(\xi) = -\lambda e^{-\lambda(\xi - \xi_0)}$, $\omega_1''(\xi) = \lambda^2 e^{-\lambda(\xi - \xi_0)}$, $\omega_1(\xi + y + c\tau)/\omega_1(\xi) = e^{-\lambda(y + c\tau)}$ and noting (A₅), the following holds:

$$\begin{aligned}
A_{c, \mu_1, \omega_1}(\xi) &\geq (c - B) \lambda - D \lambda^2 - \partial_1 F(0, 0) - \mu_1 \\
&- e^{\mu_1 r} \partial_2 F(0, 0) f'(0) \\
&\cdot \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) e^{-\lambda(y + c\tau)} dy d\tau.
\end{aligned} \tag{46}$$

Using the fact that, for $c > c_* + B$, $\lambda_1 < \lambda \leq \lambda_*$,

$$\begin{aligned}
& (c - B) \lambda - D \lambda^2 - \partial_1 F(0, 0) \\
& - \partial_2 F(0, 0) f'(0) \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) e^{-\lambda(y + c\tau)} dy d\tau \\
& > 0,
\end{aligned} \tag{47}$$

then we can choose a small $\mu_1 > 0$ such that

$$\begin{aligned}
A_{c, \mu_1, \omega_1}(\xi) &\geq (c - B) \lambda - D \lambda^2 - \partial_1 F(0, 0) - \mu_1 \\
&- e^{\mu_1 r} \partial_2 F(0, 0) f'(0) \\
&\cdot \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) e^{-\lambda(y + c\tau)} dy d\tau = C_1 > 0.
\end{aligned} \tag{48}$$

For the second term on the right-hand side of (44), we have

$$\begin{aligned}
& e^{\mu_1 r} \int_{-r}^0 \int_{-\infty}^{\infty} \int_0^r \int_{-\infty}^{\infty} e^{\mu_1 s} \omega_1(\xi + y + c\tau) \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) f'(\phi(\xi)) \omega_1(\xi) v_0(s, \xi) dy d\tau d\xi ds \\
&\leq e^{\mu_1 r} \partial_2 F(0, 0) f'(0) \int_{-r}^0 \int_{-\infty}^{\infty} e^{\mu_1 s} \omega_1(\xi) v_0(s, \xi) \left[\int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) e^{-\lambda(y + c\tau)} dy d\tau \right] d\xi ds \leq C \int_{-r}^0 \|v_0(s)\|_{L^1_{\omega_1}(\mathbb{R})} ds.
\end{aligned} \tag{49}$$

Substituting (48)-(49) into (44) we can get

$$\begin{aligned} & e^{\mu_1 t} \int_{-\infty}^{\infty} \omega_1(\xi) v(t, \xi) d\xi \\ & + C_1 \int_0^t \int_{-\infty}^{\infty} e^{\mu_1 s} \omega_1(\xi) v(s, \xi) d\xi ds \leq C; \end{aligned} \quad (50)$$

that is,

$$\begin{aligned} & \|v(t)\|_{L^1_{\omega_1}(\mathbb{R})} + C_1 \int_0^t \int_{-\infty}^{\infty} e^{-\mu_1(t-s)} \omega_1(\xi) v(s, \xi) d\xi ds \\ & \leq C e^{-\mu_1 t}, \quad \text{for } c > c_* + B. \end{aligned} \quad (51)$$

This completes the proof. $\square$

Lemma 7. For every $c > c_* + B$, $t \geq 0$, there holds that

$$\|v(t)\|_{L^2(\mathbb{R})}^2 + \int_0^t \|v(s)\|_{H^1(\mathbb{R})}^2 ds \leq C. \quad (52)$$

Proof. In view of $\omega_1(\xi) = e^{-\lambda(\xi-\xi_0)} \geq 1$ for $\xi \in (-\infty, \xi_0]$, it can follow from (50) that, for any $c > c_* + B$,

$$\int_{-\infty}^{\xi_0} v(t, \xi) d\xi + C_1 \int_0^t \int_{-\infty}^{\xi_0} v(s, \xi) d\xi ds \leq C \quad \forall t \geq 0, \quad (53)$$

and letting $t \rightarrow \infty$, we obtain

$$\int_0^{\infty} \int_{-\infty}^{\xi_0} v(s, \xi) d\xi ds \leq C. \quad (54)$$

Since $0 \leq v(t, \xi) \leq u_+ = K$, for $(t, \xi) \in \mathbb{R}_+ \times \mathbb{R}$, we have

$$\begin{aligned} & \int_0^{\infty} \int_{-\infty}^{\xi_0} v^2(s, \xi) d\xi ds \leq u_+ \int_0^{\infty} \int_{-\infty}^{\xi_0} v(s, \xi) d\xi ds \\ & \leq C. \end{aligned} \quad (55)$$

Multiplying (33) by $v(t, \xi)$ and integrating it over $\mathbb{R} \times [0, t]$ with respect to ξ and t , it yields that

$$\begin{aligned} & \|v(t)\|_{L^2(\mathbb{R})}^2 + 2D \int_0^t \|v_\xi(s, \xi)\|_{L^2(\mathbb{R})}^2 ds - 2 \int_0^t \int_{-\infty}^{\infty} \partial_1 F(\xi) v^2(s, \xi) d\xi ds = \|v_0(0)\|_{L^2(\mathbb{R})}^2 + 2 \int_0^t \int_{-\infty}^{\infty} Q \cdot v(s, \xi) d\xi ds \\ & + 2 \int_0^t \int_{-\infty}^{\infty} \partial_2 F(\xi) \left[\int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) f'(\phi(\xi - y - c\tau)) v(s - \tau, \xi - y - c\tau) dy d\tau \right] v(s, \xi) d\xi ds. \end{aligned} \quad (56)$$

By employing the Cauchy inequality $2|xy| \leq \eta x^2 + 1/\eta \cdot y^2$ for $\eta > 0$, we get

$$\begin{aligned} & 2 \int_0^t \int_{-\infty}^{\infty} \partial_2 F(\xi) \left[\int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) f'(\phi(\xi - y - c\tau)) v(s - \tau, \xi - y - c\tau) dy d\tau \right] v(s, \xi) d\xi ds \\ & \leq \eta \int_0^t \int_{-\infty}^{\infty} \partial_2 F(\xi) \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) f'(\phi(\xi - y - c\tau)) v^2(s, \xi) dy d\tau d\xi ds + \frac{1}{\eta} \\ & \cdot \int_0^t \int_{-\infty}^{\infty} \partial_2 F(\xi) \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) f'(\phi(\xi - y - c\tau)) v^2(s - \tau, \xi - y - c\tau) dy d\tau d\xi ds. \end{aligned} \quad (57)$$

For the second term on the right-hand side of (57), by making the change of variables $\xi - y - c\tau \rightarrow \xi$, $s - \tau \rightarrow s$, $y \rightarrow y$, $\tau \rightarrow \tau$ and using (55), we have

$$\begin{aligned} & \frac{1}{\eta} \int_0^t \int_{-\infty}^{\infty} \partial_2 F(\xi) \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) f'(\phi(\xi - y - c\tau)) v^2(s - \tau, \xi - y - c\tau) dy d\tau d\xi ds = \frac{1}{\eta} \\ & \cdot \int_0^r \int_{-\infty}^{\infty} \int_0^{t-\tau} \int_{-\infty}^{\infty} \partial_2 F(\xi) \mathcal{K}(\tau, y) f'(\phi(\xi - y - c\tau)) v^2(s - \tau, \xi - y - c\tau) d\xi ds dy d\tau = \frac{1}{\eta} \\ & \cdot \int_0^r \int_{-\infty}^{\infty} \int_{-\tau}^{t-\tau} \int_{-\infty}^{\infty} \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) f'(\phi(\xi)) v^2(s, \xi) d\xi ds dy d\tau \leq \frac{1}{\eta} \\ & \cdot \int_{-r}^0 \int_{-\infty}^{\infty} \int_0^r \int_{-\infty}^{\infty} \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) f'(\phi(\xi)) v^2(s, \xi) dy d\tau d\xi ds + \frac{1}{\eta} \end{aligned}$$

$$\begin{aligned}
& \cdot \int_0^t \int_{-\infty}^{\xi_0} \int_0^r \int_{-\infty}^{\infty} \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) f'(\phi(\xi)) v^2(s, \xi) dy d\tau d\xi ds + \frac{1}{\eta} \\
& \cdot \int_0^t \int_{\xi_0}^{\infty} \int_0^r \int_{-\infty}^{\infty} \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) f'(\phi(\xi)) v^2(s, \xi) dy d\tau d\xi ds \leq \frac{1}{\eta} \partial_2 F(0, 0) f'(0) \int_{-r}^0 \int_{-\infty}^{\infty} v_0^2(s, \xi) d\xi ds \\
& + \frac{1}{\eta} \partial_2 F(0, 0) f'(\phi(\xi_0)) \int_0^t \int_{\xi_0}^{\infty} v^2(s, \xi) d\xi ds + C = \frac{1}{\eta} \partial_2 F(0, 0) f'(\phi(\xi_0)) \int_0^t \int_{\xi_0}^{\infty} v^2(s, \xi) d\xi ds + C.
\end{aligned} \tag{58}$$

For the first term on the right-hand side of (57), by using (55), we have

Substituting (58), (59) into (57), we obtain

$$\begin{aligned}
& \eta \int_0^t \int_{-\infty}^{\infty} \partial_2 F(\xi) \\
& \cdot \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) f'(\phi(\xi - y - c\tau)) v^2(s, \xi) dy d\tau d\xi ds \quad (59) \\
& \leq \eta f'(0) \partial_2 F(\phi(\xi_0), 0) \int_0^t \int_{\xi_0}^{\infty} v^2(s, \xi) d\xi ds + C.
\end{aligned}$$

$$\begin{aligned}
& 2 \int_0^t \int_{-\infty}^{\infty} \partial_2 F(\xi) \left[\int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) f'(\phi(\xi - y - c\tau)) v(s - \tau, \xi - y - c\tau) dy d\tau \right] v(s, \xi) d\xi ds \leq \left[\eta f'(0) \right. \\
& \cdot \partial_2 F(\phi(\xi_0), 0) + \frac{1}{\eta} \partial_2 F(0, 0) f'(\phi(\xi_0)) \left. \right] \int_0^t \int_{\xi_0}^{\infty} v^2(s, \xi) d\xi ds + C.
\end{aligned} \tag{60}$$

Noticing that $v(t, \xi) \geq 0$, $Q \leq 0$ and using (60) to (56), it yields that

$$\begin{aligned}
& \|v(t)\|_{L^2(\mathbb{R})}^2 + 2D \int_0^t \|v_\xi(s, \xi)\|_{L^2(\mathbb{R})}^2 ds \\
& - 2 \int_0^t \int_{-\infty}^{\xi_0} \partial_1 F(\xi) v^2(s, \xi) d\xi ds
\end{aligned}$$

$$\begin{aligned}
& - \int_0^t \int_{\xi_0}^{\infty} \left[2\partial_1 F(\xi) + \eta f'(0) \partial_2 F(\phi(\xi_0), 0) \right. \\
& \left. + \frac{1}{\eta} \partial_2 F(0, 0) f'(\phi(\xi_0)) \right] v^2(s, \xi) d\xi ds \leq C.
\end{aligned} \tag{61}$$

Because of (A_5) and (A_6) , we can select $\eta > 0$ as follows:

$$\begin{aligned}
0 & < \frac{-\partial_1 F(K, f(K)) - \sqrt{\partial_1^2 F(K, f(K)) - f'(0) \partial_2 F(K, 0) \partial_2 F(0, 0) f'(K)}}{f'(0) \partial_2 F(K, 0)} < \eta \\
& < \frac{-\partial_1 F(K, f(K)) + \sqrt{\partial_1^2 F(K, f(K)) - f'(0) \partial_2 F(K, 0) \partial_2 F(0, 0) f'(K)}}{f'(0) \partial_2 F(K, 0)}.
\end{aligned} \tag{62}$$

Then we have

Choose ξ_0 large enough such that if $|\phi(\xi_0) - u_+| \ll 1$, we can obtain

$$\begin{aligned}
& 2\partial_1 F(K, f(K)) + \eta f'(0) \partial_2 F(K, 0) \\
& + \frac{1}{\eta} \partial_2 F(0, 0) f'(K) < 0.
\end{aligned} \tag{63}$$

$$\begin{aligned}
C_2 & := - \left[2\partial_1 F(\xi_0) + \eta f'(0) \partial_2 F(\phi(\xi_0), 0) \right. \\
& \left. + \frac{1}{\eta} \partial_2 F(0, 0) f'(\phi(\xi_0)) \right] > 0.
\end{aligned} \tag{64}$$

Because of $\partial_{ij}F(x, y) \leq 0$, for $\xi \in [\xi_0, \infty)$, we have

$$\begin{aligned} & 2\partial_1 F(\phi(\xi), \widehat{\Phi}(\xi)) + \eta f'(0) \partial_2 F(\phi(\xi_0), 0) \\ & + \frac{1}{\eta} \partial_2 F(0, 0) f'(\phi(\xi_0)) \leq 2\partial_1 F(\phi(\xi_0), \widehat{\Phi}(\xi_0)) \\ & + \eta f'(0) \partial_2 F(\phi(\xi_0), 0) + \frac{1}{\eta} \partial_2 F(0, 0) f'(\phi(\xi_0)) \\ & = -C_2 < 0. \end{aligned} \quad (65)$$

Thus substituting (65) into (61) and setting $C_3 = \min\{C_2, -2\partial_1 F(0, 0)\}$ we can obtain

$$\begin{aligned} & \|v(t)\|_{L^2(\mathbb{R})}^2 + 2D \int_0^t \|v_\xi(s, \xi)\|_{L^2(\mathbb{R})}^2 ds \\ & + C_3 \int_0^t \|v(s)\|_{L^2(\mathbb{R})}^2 ds \leq C. \end{aligned} \quad (66)$$

This completes the proof. $\square$

Next, we differentiate (33) with respect to ξ , multiply it by $v_\xi(t, \xi)$, and integrate it over $[0, t] \times \mathbb{R}$ with respect to t and ξ . Combining with Lemma 7, we can get the following energy estimate. Here we omit the details of the proof.

Lemma 8. *If $c > c_* + B$ and $t \geq 0$, the following holds:*

$$\|v_\xi(t)\|_{L^2(\mathbb{R})}^2 + \int_0^t \|v_\xi(s)\|_{H^1(\mathbb{R})}^2 ds \leq C. \quad (67)$$

Lemma 9. *If $c > c_* + B$ and $t \geq 0$, the following holds:*

$$\|v(t)\|_{L^\infty(-\infty, \xi_0]} \leq Ce^{-\mu_1 t/3}. \quad (68)$$

Proof. Take $I = (-\infty, \xi_0]$. Thus

$$\begin{aligned} \|v(t)\|_{L^2(I)}^2 &= \int_{-\infty}^{\xi_0} |v(t, \xi)|^2 d\xi \\ &\leq \|v(t)\|_{L^\infty(I)} \|v(t)\|_{L^1(I)}, \end{aligned} \quad (69)$$

$$\begin{aligned} v^2(t, \xi) &= \int_{-\infty}^{\xi} \partial_\xi (v^2) d\xi \\ &= 2 \int_{-\infty}^{\xi} v(t, \xi) v_\xi(t, \xi) d\xi. \end{aligned} \quad (70)$$

$$\begin{aligned} & e^{\mu_1 t} \|v(t)\|_{L^1(\mathbb{R})} - \int_0^t \int_{-\infty}^{\infty} e^{\mu_1 s} [\partial_1 F(\xi) + \mu_1] v(s, \xi) d\xi ds \\ & - \int_0^t \int_{-\infty}^{\infty} e^{\mu_1 s} \partial_2 F(\xi) \left[\int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) f'(\phi(\xi - y - c\tau)) v(s - \tau, \xi - y - c\tau) dy d\tau \right] d\xi ds \leq \|v_0(0)\|_{L^1(\mathbb{R})}. \end{aligned} \quad (78)$$

Similar to (43), we have

$$\begin{aligned} & \int_0^t \int_{-\infty}^{\infty} e^{\mu_1 s} \partial_2 F(\xi) \left[\int_0^r \int_{-\infty}^{\infty} \mathcal{K}(\tau, y) f'(\phi(\xi - y - c\tau)) v(s - \tau, \xi - y - c\tau) dy d\tau \right] d\xi ds \\ & \leq e^{\mu_1 r} \int_0^t \int_{-\infty}^{\infty} \int_0^r \int_{-\infty}^{\infty} e^{\mu_1 s} \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) f'(\phi(\xi)) v(s, \xi) dy d\tau d\xi ds \\ & + e^{\mu_1 r} \int_{-r}^0 \int_{-\infty}^{\infty} \int_0^r \int_{-\infty}^{\infty} e^{\mu_1 s} \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) f'(\phi(\xi)) v_0(s, \xi) dy d\tau d\xi ds. \end{aligned} \quad (79)$$

By the Hölder inequality, we can get

$$\begin{aligned} & \|v(t)\|_{L^\infty(I)}^2 \\ & \leq 2 \left(\int_{-\infty}^{\xi_0} v^2(t, \xi) d\xi \right)^{1/2} \left(\int_{-\infty}^{\xi_0} v_\xi^2(t, \xi) d\xi \right)^{1/2} \\ & = 2 \|v(t)\|_{L^2(I)} \|v_\xi(t)\|_{L^2(I)}. \end{aligned} \quad (71)$$

It follows from (69) and (71) that

$$\|v(t)\|_{L^\infty(I)}^2 \leq 2 \|v(t)\|_{L^\infty(I)}^{1/2} \|v(t)\|_{L^1(I)}^{1/2} \|v_\xi(t)\|_{L^2(I)}; \quad (72)$$

that is,

$$\|v(t)\|_{L^\infty(I)} \leq \sqrt[3]{4} \|v(t)\|_{L^1(I)}^{1/3} \|v_\xi(t)\|_{L^2(I)}^{2/3}. \quad (73)$$

From (67) we have

$$\|v_\xi(t)\|_{L^2(\mathbb{R})} \leq C. \quad (74)$$

Noting that $\omega_1(\xi) = e^{-\lambda(\xi - \xi_0)} \geq 1$ if $\xi \in I$ and combining with (39), we can get

$$\|v(t)\|_{L^1(I)} \leq \|v(t)\|_{L_{\omega_1}^1(I)} \leq Ce^{-\mu_1 t}, \quad t \geq 0. \quad (75)$$

Then it is obvious from (73)–(75) that

$$\begin{aligned} \|v(t)\|_{L^\infty(I)} &\leq Ce^{-\mu_1 t/3}, \quad t \geq 0, \\ v(t, \xi_0) &\leq Ce^{-\mu_1 t/3}, \quad t \geq 0. \end{aligned} \quad (76)$$

This completes the proof. $\square$

Next, we need to prove the convergence of the traveling wave solutions in another interval $[\xi_0, +\infty)$.

Lemma 10. *If $c > c_* + B$ and $t \geq 0$, we have*

$$\|v(t)\|_{L^\infty[\xi_0, \infty)} \leq Ce^{-\mu_1 t/3}. \quad (77)$$

Proof. Similar to front, we multiply (33) by $e^{\mu_1 t}$ and integrate it over $[0, t] \times \mathbb{R}$ with respect to (t, ξ) . Since $Q \leq 0$, we can get

Substituting (79) into (78), it follows that

$$\begin{aligned}
& e^{\mu_1 t} \|v(t)\|_{L^1(\mathbb{R})} - \int_0^t \int_{-\infty}^{\infty} e^{\mu_1 s} [\partial_1 F(\xi) + \mu_1] v(s, \xi) d\xi ds \\
& - e^{\mu_1 r} \int_0^t \int_{-\infty}^{\infty} \int_0^r \int_{-\infty}^{\infty} e^{\mu_1 s} \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) f'(\phi(\xi)) v(s, \xi) dy d\tau d\xi ds \\
& - e^{\mu_1 r} \int_{-r}^0 \int_{-\infty}^{\infty} \int_0^r \int_{-\infty}^{\infty} e^{\mu_1 s} \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) f'(\phi(\xi)) v_0(s, \xi) dy d\tau d\xi ds \leq \|v_0(0)\|_{L^1(\mathbb{R})}.
\end{aligned} \tag{80}$$

Since $\mathbb{R} = (-\infty, \xi_0] \cup [\xi_0, \infty)$, we obtain

$$\begin{aligned}
& e^{\mu_1 t} \int_{\xi_0}^{\infty} v(t, \xi) d\xi - \int_0^t \int_{\xi_0}^{\infty} e^{\mu_1 s} [\partial_1 F(\xi) + \mu_1] v(s, \xi) d\xi ds \\
& - e^{\mu_1 r} \int_0^t \int_{\xi_0}^{\infty} e^{\mu_1 s} f'(\phi(\xi)) v(s, \xi) \\
& \cdot \left[\int_0^r \int_{-\infty}^{\infty} \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) dy d\tau \right] d\xi ds \\
& \leq \|v_0(0)\|_{L^1(\mathbb{R})} - I(t),
\end{aligned} \tag{81}$$

where

$$\begin{aligned}
I(t) &= e^{\mu_1 t} \int_{-\infty}^{\xi_0} v(t, \xi) d\xi - \int_0^t \int_{-\infty}^{\xi_0} e^{\mu_1 s} [\partial_1 F(\xi) + \mu_1] v(s, \xi) d\xi ds \\
& - e^{\mu_1 r} \int_0^t \int_{-\infty}^{\xi_0} e^{\mu_1 s} f'(\phi(\xi)) v(s, \xi) \left[\int_0^r \int_{-\infty}^{\infty} \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) dy d\tau \right] d\xi ds \\
& + e^{\mu_1 r} \int_{-r}^0 \int_{-\infty}^{\infty} \int_0^r \int_{-\infty}^{\infty} e^{\mu_1 s} \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) f'(\phi(\xi)) v_0(s, \xi) dy d\tau d\xi ds.
\end{aligned} \tag{82}$$

Because $\omega_1(\xi) \geq 1$, $\xi \in (-\infty, \xi_0]$ and combining with (39), then we have

$$\begin{aligned}
& e^{\mu_1 t} \int_{-\infty}^{\xi_0} v(t, \xi) d\xi \leq C, \\
& \int_0^t \int_{-\infty}^{\xi_0} e^{\mu_1 s} v(s, \xi) d\xi ds \leq C.
\end{aligned} \tag{83}$$

Also in view of (A_1) , (A_3) , (A_4) and

$$\int_{-r}^0 \int_{-\infty}^{\infty} e^{\mu_1 s} v_0(s, \xi) d\xi ds \leq \int_{-r}^0 \int_{-\infty}^{\infty} v_0(s, \xi) d\xi ds, \tag{84}$$

then it yields that

$$|I(t)| \leq C, \quad \text{for } t \geq 0. \tag{85}$$

Substituting (85) into (81), we have

$$\begin{aligned}
& e^{\mu_1 t} \int_{\xi_0}^{\infty} v(t, \xi) d\xi - \int_0^t \int_{\xi_0}^{\infty} \left[\partial_1 F(\xi) + \mu_1 + f'(\phi(\xi)) \right. \\
& \cdot e^{\mu_1 r} \int_0^r \int_{-\infty}^{\infty} \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) dy d\tau \\
& \cdot e^{\mu_1 s} v(s, \xi) d\xi ds \leq C.
\end{aligned} \tag{86}$$

In view of the assumption $\partial_2 F(K, f(K))f'(K) + \partial_1 F(K, f(K)) < 0$, thus there is a constant $\mu_1 > 0$ small enough such that

$$\begin{aligned}
& -\partial_1 F(K, f(K)) - \mu_1 - \partial_2 F(K, f(K))f'(K)e^{\mu_1 r} \\
& > 0.
\end{aligned} \tag{87}$$

Since $\lim_{\xi \rightarrow +\infty} \phi(\xi) = K$, $\lim_{\xi \rightarrow +\infty} \widehat{\Phi}(\xi) = f(K)$, then when $\xi_0 \gg 1$, we can get

$$\begin{aligned}
& -\partial_1 F(\xi) - \mu_1 - f'(\phi(\xi)) \\
& \cdot e^{\mu_1 r} \int_0^r \int_{-\infty}^{\infty} \partial_2 F(\xi + y + c\tau) \mathcal{K}(\tau, y) dy d\tau > 0,
\end{aligned} \tag{88}$$

$$\xi \in [\xi_0, \infty).$$

Putting (88) to (86), it yields that

$$\int_{\xi_0}^{\infty} v(t, \xi) d\xi \leq C e^{-\mu_1 t}. \tag{89}$$

Similar to (68), it follows that

$$\begin{aligned}
\|v(t)\|_{L^\infty[\xi_0, \infty)} &\leq \sqrt[3]{4} \|v(t)\|_{L^1[\xi_0, \infty)}^{1/3} \|v_\xi(t)\|_{L^2[\xi_0, \infty)}^{2/3} \\
&\leq C e^{-\mu_1 t/3}.
\end{aligned} \tag{90}$$

This completes the proof. $\square$

From Lemmas 9 and 10, we can get the following result directly.

Lemma 11. *If $c > c_* + B$ and $t \geq 0$, it yields that*

$$\sup_{x \in \mathbb{R}} |U^+(t, x) - \phi(x + ct)| \leq Ce^{-\mu t}, \quad (91)$$

where $\mu = \mu_1/3$.

Step 2. The convergence of $U^-(t, x)$ to $\phi(x + ct)$ is as follows.
For every $c > c_* + B$, take $\xi = x + ct$ and

$$\begin{aligned} v(t, \xi) &= \phi(x + ct) - U^-(t, x), \\ v_0(s, \xi) &= \phi(x + cs) - U_0^-(s, x). \end{aligned} \quad (92)$$

Similar to process 1, we can get the following convergence of $U^-(t, x)$ to $\phi(x + ct)$.

Lemma 12. *If $c > c_* + B$ and $t \geq 0$, the following holds:*

$$\sup_{x \in \mathbb{R}} |U^-(t, x) - \phi(x + ct)| \leq Ce^{-\mu t}. \quad (93)$$

Step 3. The convergence of $u(t, x)$ to $\phi(x + ct)$ is as follows.

Lemma 13. *If $c > c_* + B$ and $t \geq 0$, the following holds:*

$$\sup_{x \in \mathbb{R}} |u(t, x) - \phi(x + ct)| \leq Ce^{-\mu t}. \quad (94)$$

Proof. Because of $U_0^-(s, x) \leq u_0(s, x) \leq U_0^+(s, x)$, by Lemma 5, we can get $U^-(t, x) \leq u(t, x) \leq U^+(t, x)$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Combining with Lemmas 11 and 12 and applying the squeeze theorem, it then follows that

$$\begin{aligned} \sup_{x \in \mathbb{R}} |u(t, x) - \phi(x + ct)| &\leq Ce^{-\mu t}, \\ &\text{for } t \geq 0, \quad c > c_* + B. \end{aligned} \quad (95)$$

This completes the proof. $\square$

4. Applications

In this section, we will apply the content of Theorem 2 to some realistic examples.

4.1. A Vector-Disease Model. In [3, 25], the authors considered the following equation:

$$\begin{aligned} u_t(t, x) &= D\Delta u(t, x) - au(t, x) + b[1 - u(t, x)] \\ &\quad \cdot \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(s, y) u(t - s, x - y) dy ds, \end{aligned} \quad (96)$$

where $b > a > 0$. This equation presented a host-vector model for a disease without immunity in which the current density of infectious vectors is related to the number of infectious hosts at earlier times. Here $u(t, x)$ is the density of infectious individual at time t and at position x , a denotes the recovery

rate of the infected individual, and b denotes the host-vector contact rate. D denotes the diffusion constant.

Clearly, (96) has two constant equilibria $u_0 \equiv 0$ and $u_+ \equiv 1 - a/b > 0$. The corresponding characteristic equation at $u_0 \equiv 0$ is

$$\begin{aligned} \mathcal{P}(\lambda, c) &= D\lambda^2 - c\lambda - a \\ &\quad + b \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(s, y) e^{-\lambda(y+cs)} dy ds. \end{aligned} \quad (97)$$

Set

$$\begin{aligned} c_* &= \inf \{c > 0 \mid \mathcal{P}(\lambda, c) \\ &= 0 \text{ has a positive real root}\}. \end{aligned} \quad (98)$$

There exists a corresponding number $\lambda_* > 0$ such that $\mathcal{P}(\lambda_*, c_*) = 0$. If $c > c_*$, $\mathcal{P}(\lambda, c)$ has two real roots $0 < \lambda_1(c) < \lambda_2(c)$.

Now set $F(x, y) = -ax + b(1 - x)y$, $B = 0$ and $f(x) = x$ in (1). It is easy to see that $\partial_1 F(x, y) = -a - by < 0$, $\partial_2 F(x, y) = b - bx > 0$, and $\partial_{ij} F(x, y) \leq 0$ for $(x, y) \in [0, u_+] \times [0, u_+]$. Also we have

$$\begin{aligned} f'(0) f'(K) \partial_2 F(0, 0) \partial_2 F(K, 0) - \partial_1^2 F[K, f(K)] \\ < 0, \end{aligned} \quad (99)$$

$$\partial_2 F(K, f(K)) f'(K) + \partial_1 F(K, f(K)) < 0,$$

where $K = 1 - a/b$. Thus we get the following result.

Corollary 14. *Suppose that (A_1) -(A_2) hold. For the given traveling wave $\phi(\xi)$ of (96) and $\phi(\pm\infty) = u_{\pm}$ with speed $c > c_*$, if the initial data satisfies*

$$\begin{aligned} 0 = u_- \leq u_0(s, x) \leq u_+ = K, \\ \text{for } (s, x) \in [-r, 0] \times \mathbb{R} \end{aligned} \quad (100)$$

and the initial perturbation is $u_0(s, x) - \phi(x + cs) \in C([-r, 0]; L_{\omega}^1(\mathbb{R}) \cap H^1(\mathbb{R}))$, then the solution of (96) satisfies

$$\begin{aligned} 0 = u_- \leq u(t, x) \leq u_+ = K, \\ \text{for } (t, x) \in [0, +\infty) \times \mathbb{R}, \end{aligned} \quad (101)$$

$$u(t, x) - \phi(x + ct)$$

$$\in C([0, +\infty); L_{\omega}^1(\mathbb{R}) \cap H^1(\mathbb{R})).$$

When $c > c_*$, the solution $u(t, x)$ converges to the traveling wave $\phi(x + ct)$ exponentially

$$\sup_{x \in \mathbb{R}} |u(t, x) - \phi(x + ct)| \leq Ce^{-\mu t}, \quad t \geq 0, \quad (102)$$

with a positive number $\mu = \mu_1/3$, where μ_1 is determined by

$$\begin{aligned} c\lambda - D\lambda^2 + a - \mu_1 \\ - be^{\mu_1 r} \int_0^r \int_{-\infty}^{\infty} \mathcal{K}(s, y) e^{-\lambda(y+cs)} dy ds > 0, \end{aligned} \quad (103)$$

$$\lambda \in (\lambda_1(c), \lambda_*],$$

$$b - \mu_1 - ae^{\mu_1 r} > 0.$$

Remark 15. In [24], under the additional condition $b > (1 + (1/2)L)a$, the exponential stability of traveling waves of (96) is obtained for

$$c > \max \left\{ c_*, \frac{2D}{c_*} [(2 + L)b - 2a] \right\}, \quad (104)$$

where $L = \int_0^r \int_{-\infty}^{\infty} e^{-(c_*/D)(y+cs)} \mathcal{K}(s, y) dy ds$.

Remark 16. Corollary 14 shows that if $b > a > 0$, the density of infectious individual tends to an endemic level $u_+ = 1 - a/b$ when time t goes on. Further, when $a \nearrow b$, the density of infectious individual tends to 0. That is, when the recovery rate of the infected individual is close to the host-vector contact rate, this disease will eventually die out. This shows that an effective way to control the epidemic is to reduce the host-vector contact rate. Therefore, the results will help predict the developing trend of infectious diseases, identify the key factors for the spread of infectious diseases, and seek the best strategies to prevent and control the spread of infectious diseases.

4.2. A Population Dynamic Model. In [17], the authors considered a single species in the spatial transport field and obtained the following general reaction advection diffusion equation with the nonlocal delayed effect if the mature mortality and diffusion rates are independent of age,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial x} \left(D_m \frac{\partial u}{\partial x} + Bu \right) - d_m u \\ &+ \varepsilon \int_{-\infty}^{\infty} \Phi_{\alpha}(x + Br - y) b(u(t - r, y)) dy, \end{aligned} \quad (105)$$

where

$$\Phi_{\alpha}(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{-x^2/4\alpha}. \quad (106)$$

Here $u(t, x)$ is the density of the population of the species at time t and location x , D_m is the diffusion rate and d_m is the mortality, B is the transport velocity of the field, $r \geq 0$ is the maturation time for the species, ε represents the impact of premature mortality, and α demonstrates the effect of immature population dispersal rate on mature population growth rate. Liang and Wu [17] obtained the existence of the traveling wavefronts by using the upper and lower solutions when $b(u)$ took three different birth functions which have been widely studied in the well-known Nicholson's blowflies equation. Finally, they gave the numerical stability of travelling waves by choosing some special data. Later, Wu et al. [24] showed the exponential stability of traveling waves with faster wave speed for (105). Now we will establish the exponential stability of all noncritical traveling waves for (105). First, we need to give the following hypotheses:

- (H₁) Let $K > 0$, $b(0) = 0$, $\varepsilon b(K) = d_m K$, $b(x) \in C^2[0, K]$, $b'(x) \geq 0$, $b''(x) \leq 0$ for $x \in [0, K]$, $\varepsilon b(x) > d_m x$ for $x \in (0, K)$.
 (H₂) $\varepsilon^2 b'(0) b'(K) < d_m^2$, $\varepsilon b'(K) < d_m$.

Clearly, $u_- = 0$ and $u_+ = K$ are two constant equilibria of (105). And the characteristic equation of (105) at $u_- = 0$ is

$$\begin{aligned} \mathcal{P}(\lambda, c) &= D_m \lambda^2 - (c - B) \lambda - d_m \\ &+ \varepsilon b'(0) e^{-\lambda cr} e^{(4B^2 + 4B\lambda + \lambda^2 - 4(Br)^2)/16\alpha}. \end{aligned} \quad (107)$$

There are $c_* > 0$ and $\lambda_* > 0$ which satisfy

$$\begin{aligned} \mathcal{P}(\lambda_*, c_* + B) &= 0, \\ \frac{\partial}{\partial \lambda} \mathcal{P}(\lambda, c_* + B)|_{\lambda_*} &= 0. \end{aligned} \quad (108)$$

If $c > c_* + B$, $\mathcal{P}(\lambda, c)$ has two positive real roots $0 < \lambda_1(c) < \lambda_2(c)$.

Taking $F(x, y) = -d_m x + \varepsilon y$, $f(x) = b(x)$, $\mathcal{K}(s, y) = \Phi_{\alpha}(y + Bs) \delta(s - r)$ in (1), then (1) turns to (105). Now we show the stability of traveling wave solutions of (105).

Corollary 17. Assume (H₁)-(H₂) hold. For the given traveling wave $\phi(\xi)$ of (105) and $\phi(\pm\infty) = u_{\pm}$ with speed $c > c_* + B$, if the initial data holds

$$\begin{aligned} 0 &= u_- \leq u_0(s, x) \leq u_+ = K, \\ &\text{for } (s, x) \in [-r, 0] \times \mathbb{R} \end{aligned} \quad (109)$$

and the initial perturbation is $u_0(s, x) - \phi(x + cs) \in C([-r, 0]; L^1_{\omega}(\mathbb{R}) \cap H^1(\mathbb{R}))$, then the solution of (105) satisfies

$$\begin{aligned} 0 &= u_- \leq u(t, x) \leq u_+ = K, \\ &\text{for } (t, x) \in [0, +\infty) \times \mathbb{R}, \\ u(t, x) - \phi(x + ct) &\in C([0, +\infty); L^1_{\omega}(\mathbb{R}) \cap H^1(\mathbb{R})). \end{aligned} \quad (110)$$

When $c > c_* + B$, the solution $u(t, x)$ converges to the traveling wave $\phi(x + ct)$ exponentially

$$\sup_{x \in \mathbb{R}} |u(t, x) - \phi(x + ct)| \leq C e^{-\mu t}, \quad t \geq 0, \quad (111)$$

with a positive constant $\mu = \mu_1/3$, where μ_1 is determined by

$$\begin{aligned} (c - B) \lambda - D_m \lambda^2 + d_m - \mu_1 \\ - \varepsilon b'(0) e^{-\lambda cr} e^{(4B^2 + 4B\lambda + \lambda^2 - 4(Br)^2)/16\alpha} > 0, \\ d_m - \mu_1 - \varepsilon b'(K) e^{\mu_1 r} > 0. \end{aligned} \quad (112)$$

Remark 18. The model can be used to study the behaviour of the mature population. For example, if we take birth function $b(u) = p u e^{-au^q}$, where p, a, q are positive constants and let $1 < \varepsilon p/d_m < e^{1/q}$, it is easy to verify that, for $c > c_* + B$, all noncritical traveling waves connecting 0 and $((1/a) \ln(\varepsilon p/d_m))^{1/q}$ are exponential stable. That is, the density of the population tends to a constant $((1/a) \ln(\varepsilon p/d_m))^{1/q}$ as time becomes large. The results show that the density of the population will eventually reach a stable state.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The authors declare that they share equal responsibility. All authors read and approved the final manuscript.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (no. 11471197) and the Natural Science Foundation of Shanxi Province (no. 2014011005-1).

References

- [1] S. A. Gourley and Y. Kuang, "Wavefronts and global stability in a time-delayed population model with stage structure," *Proceedings A*, vol. 459, no. 2034, pp. 1563–1579, 2003.
- [2] J. Al-Omari and S. A. Gourley, "Monotone travelling fronts in an age-structured reaction-diffusion model of a single species," *Journal of Mathematical Biology*, vol. 45, no. 4, pp. 294–312, 2002.
- [3] S. Ruan and D. Xiao, "Stability of steady states and existence of travelling waves in a vector-disease model," *Proceedings of the Royal Society of Edinburgh, Section: A Mathematics*, vol. 134, no. 5, pp. 991–1011, 2004.
- [4] J. W. So, J. Wu, and X. Zou, "A reaction-diffusion model for a single species with age structure. I. Travelling wavefronts on unbounded domains," *Proceedings A*, vol. 457, no. 2012, pp. 1841–1853, 2001.
- [5] P. C. Fife and J. B. McLeod, "A phase plane discussion of convergence to travelling fronts for nonlinear diffusion," *Archive for Rational Mechanics and Analysis*, vol. 75, no. 4, pp. 281–314, 1980/81.
- [6] T. Gallay, "Local stability of critical fronts in nonlinear parabolic partial differential equations," *Nonlinearity*, vol. 7, no. 3, pp. 741–764, 1994.
- [7] X. Hou and Y. Li, "Local stability of traveling-wave solutions of nonlinear reaction-diffusion equations," *Discrete and Continuous Dynamical Systems - Series A*, vol. 15, no. 2, pp. 681–701, 2006.
- [8] J.-C. Tsai and J. Sneyd, "Existence and stability of traveling waves in buffered systems," *SIAM Journal on Applied Mathematics*, vol. 66, no. 1, pp. 237–265, 2005.
- [9] Y. Wu and X. Xing, "Stability of traveling waves with critical speeds for p -degree Fisher-type equations," *Discrete and Continuous Dynamical Systems - Series A*, vol. 20, no. 4, pp. 1123–1139, 2008.
- [10] K. W. Schaaf, "Asymptotic behavior and traveling wave solutions for parabolic functional-differential equations," *Transactions of the American Mathematical Society*, vol. 302, no. 2, pp. 587–615, 1987.
- [11] H. L. Smith and X.-Q. Zhao, "Global asymptotic stability of traveling waves in delayed reaction-diffusion equations," *SIAM Journal on Mathematical Analysis*, vol. 31, no. 3, pp. 514–534, 2000.
- [12] Z.-C. Wang, W.-T. Li, and S. Ruan, "Existence and stability of traveling wave fronts in reaction advection diffusion equations with nonlocal delay," *Journal of Differential Equations*, vol. 238, no. 1, pp. 153–200, 2007.
- [13] M. Mei, C.-K. Lin, C.-T. Lin, and J. . So, "Traveling wavefronts for time-delayed reaction-diffusion equation. I. Local nonlinearity," *Journal of Differential Equations*, vol. 247, no. 2, pp. 495–510, 2009.
- [14] M. Mei, J. W. So, M. Y. Li, and S. S. Shen, "Asymptotic stability of travelling waves for Nicholson's blowflies equation with diffusion," *Proceedings of the Royal Society of Edinburgh, Section: A Mathematics*, vol. 134, no. 3, pp. 579–594, 2004.
- [15] M. Mei, C.-K. Lin, C.-T. Lin, and J. . So, "Traveling wavefronts for time-delayed reaction-diffusion equation: II. Nonlocal nonlinearity," *Journal of Differential Equations*, vol. 247, no. 2, pp. 511–529, 2009.
- [16] M. Mei and J. W. So, "Stability of strong travelling waves for a non-local time-delayed reaction-diffusion equation," *Proceedings of the Royal Society of Edinburgh, Section: A Mathematics*, vol. 138, no. 3, pp. 551–568, 2008.
- [17] D. Liang and J. Wu, "Travelling waves and numerical approximations in a reaction advection diffusion equation with nonlocal delayed effects," *Journal of Nonlinear Science*, vol. 13, no. 3, pp. 289–310, 2003.
- [18] M. Mei, C. Ou, and X.-Q. Zhao, "Global stability of monostable traveling waves for nonlocal time-delayed reaction-diffusion equations," *SIAM Journal on Mathematical Analysis*, vol. 42, no. 6, pp. 2762–2790, 2010.
- [19] M. Mei and Y. Wang, "Remark on stability of traveling waves for nonlocal Fisher-KPP equations," *International Journal of Numerical Analysis and Modeling. Series B*, vol. 2, no. 4, pp. 379–401, 2011.
- [20] Z.-C. Wang, W.-T. Li, and S. Ruan, "Traveling fronts in monostable equations with nonlocal delayed effects," *Journal of Dynamics and Differential Equations*, vol. 20, no. 3, pp. 573–607, 2008.
- [21] H. Wang, "On the existence of traveling waves for delayed reaction-diffusion equations," *Journal of Differential Equations*, vol. 247, no. 3, pp. 887–905, 2009.
- [22] X. H. Wang and G. Y. Lv, "Stability of traveling wave fronts for nonlocal reaction diffusion equations with delay," *Acta Mathematica Sinica*, vol. 58, no. 1, pp. 13–28, 2015.
- [23] M. Cencini, C. Lopez, and D. Vergni, "Reaction-diffusion systems: front propagation and spatial structures," in *The Kolmogorov legacy in physics*, vol. 636 of *Lecture Notes in Phys.*, pp. 187–210, Springer, Berlin, 2003.
- [24] S.-L. Wu, W.-T. Li, and S.-Y. Liu, "Exponential stability of traveling fronts in monostable reaction-advection-diffusion equations with non-local delay," *Discrete and Continuous Dynamical Systems - Series B*, vol. 17, no. 1, pp. 347–366, 2012.
- [25] X.-Q. Zhao and D. Xiao, "The asymptotic speed of spread and traveling waves for a vector disease model," *Journal of Dynamics and Differential Equations*, vol. 18, no. 4, pp. 1001–1019, 2006.