Research Article

Hartman-Wintner-Type Inequality for a Fractional Boundary Value Problem via a Fractional Derivative with respect to Another Function

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We consider a fractional boundary value problem involving a fractional derivative with respect to a certain function g . A Hartman-Wintner-type inequality is obtained for such problem. Next, several Lyapunov-type inequalities are deduced for different choices of the function g . Moreover, some applications to eigenvalue problems are presented.

1. Introduction

In this work, we are concerned with the following fractional boundary value problem:

$$\begin{aligned} (D_{a,g}^\alpha u)(t) + q(t)u(t) &= 0, \quad a < t < b, \\ u(a) &= u(b) = 0, \end{aligned} \quad (1)$$

where $(a, b) \in \mathbb{R}^2$, $a < b$, $\alpha \in (1, 2)$, $q : [a, b] \rightarrow \mathbb{R}$ is a continuous function, and $D_{a,g}^\alpha$ is the fractional derivative operator of order α with respect to a certain nondecreasing function $g \in C^1([a, b]; \mathbb{R})$ with $g'(x) > 0$, for all $x \in (a, b)$. A Hartman-Wintner-type inequality is derived for problem (1). As a consequence, several Lyapunov-type inequalities are deduced for different types of fractional derivatives. Next, we end the paper with some applications to eigenvalue problems.

Let us start by describing some historical backgrounds about Lyapunov inequality and some related works. In the late XIX century, the mathematician A. M. Lyapunov established the following result (see [1]).

Theorem 1. *If the boundary value problem*

$$\begin{aligned} u''(t) + q(t)u(t) &= 0, \quad a < t < b, \\ u(a) &= u(b) = 0 \end{aligned} \quad (2)$$

has a nontrivial solution, where $q : [a, b] \rightarrow \mathbb{R}$ is a continuous function, then

$$\int_a^b |q(s)| ds > \frac{4}{b-a}. \quad (3)$$

Inequality (3) is known as Lyapunov inequality. It is proved to be very useful in various problems in connection with differential equations, including oscillation theory, asymptotic theory, eigenvalue problems, and disconjugacy. For more details, we refer the reader to [2–12] and references therein.

In [8], Hartman and Wintner proved that if boundary value problem (2) has a nontrivial solution, then

$$\int_a^b (s-a)(b-s)q^+(s) ds > b-a, \quad (4)$$

where

$$q^+(s) = \max\{q(s), 0\}, \quad s \in [a, b]. \quad (5)$$

Using the fact that

$$\max_{a \leq s \leq b} (s-a)(b-s) = \frac{(b-a)^2}{4}, \quad (6)$$

Lyapunov inequality (3) follows immediately from inequality (4). Many other generalizations and extensions of inequality (3) exist in the literature; see, for instance, [7, 13–22] and references therein.

Due to the positive impact of fractional calculus on several applied sciences (see, for instance, [23]), several authors investigated Lyapunov-type inequalities for various classes of fractional boundary value problems. The first work in this direction is due to Ferreira [24], where he considered the fractional boundary value problem

$$\begin{aligned}({}^{\text{RL}}D_a^\alpha u)(t) + q(t)u(t) &= 0, \quad a < t < b, \\ u(a) &= u(b) = 0,\end{aligned}\quad (7)$$

where $(a, b) \in \mathbb{R}^2$, $a < b$, $\alpha \in (1, 2)$, $q : [a, b] \rightarrow \mathbb{R}$ is a continuous function, and ${}^{\text{RL}}D_a^\alpha$ is the Riemann-Liouville fractional derivative operator of order α . The main result obtained in [24] is the following fractional version of Theorem 1.

Theorem 2. *If fractional boundary value problem (7) has a nontrivial solution, then*

$$\int_a^b |q(s)| ds > \Gamma(\alpha) \left(\frac{4}{b-a} \right)^{\alpha-1}, \quad (8)$$

where Γ is the Gamma function.

Observe that (3) can be deduced from Theorem 2 by passing to the limit as $\alpha \rightarrow 2$ in (8). For other related works, we refer the reader to Ferreira [25, 26], Jleli and Samet [27, 28], Jleli et al. [29, 30], O'Regan and Samet [31], Al Arifi et al. [32], Rong and Bai [33], Chidouh and Torres [34], Agarwal and Özbekler [35], Ma [36], and the references therein.

Very recently, Ma et al. [37] investigated the fractional boundary value problem

$$\begin{aligned}({}^H D_1^\alpha u)(t) + q(t)u(t) &= 0, \quad 1 < t < e, \\ u(1) &= u(e) = 0,\end{aligned}\quad (9)$$

where $\alpha \in (1, 2)$, $q : [1, e] \rightarrow \mathbb{R}$ is a continuous function, and ${}^H D_1^\alpha$ is the Hadamard fractional derivative operator of order α . The main result in [37] is the following.

Theorem 3. *If fractional boundary value problem (9) has a nontrivial solution, then*

$$\int_1^e |q(s)| ds > \Gamma(\alpha) \lambda^{1-\alpha} (1-\lambda)^{1-\alpha} e^\lambda, \quad (10)$$

where $\lambda = (2\alpha - 1 - \sqrt{(2\alpha - 2)^2 + 1})/2$.

In the same paper [37], the authors formulated the following question: How to get the Lyapunov inequality for the following Hadamard fractional boundary value problem:

$$\begin{aligned}({}^H D_a^\alpha u)(t) + q(t)u(t) &= 0, \quad a < t < b, \\ u(a) &= u(b) = 0,\end{aligned}\quad (11)$$

where $(a, b) \in \mathbb{R}^2$, $1 \leq a < b$, $\alpha \in (1, 2)$, and $q : [a, b] \rightarrow \mathbb{R}$ is a continuous function. Note that one of our obtained results is an answer to the above question.

2. Preliminaries

Before stating and proving the main results in this work, some preliminaries are needed.

Let $I = [a, b]$ be a certain interval in $\mathbb{R}$, where $(a, b) \in \mathbb{R}^2$, $a < b$. We denote by $AC(I; \mathbb{R})$ the space of real valued and absolutely continuous functions on I . For $n = 1, 2, \dots$, we denote by $AC^n(I; \mathbb{R})$ the space of real valued functions $f(x)$ which have continuous derivatives up to order $n-1$ on I with $f^{(n-1)} \in AC(I; \mathbb{R})$; that is,

$$\begin{aligned}AC^n(I; \mathbb{R}) &= \left\{ f : I \rightarrow \mathbb{R} \text{ such that } D^{n-1}f \right. \\ &\left. \in AC(I; \mathbb{R}) \left(D = \frac{d}{dx} \right) \right\}.\end{aligned}\quad (12)$$

Clearly, we have $AC^1(I; \mathbb{R}) = AC(I; \mathbb{R})$.

Definition 4 (see [23]). Let $f \in L^1((a, b); \mathbb{R})$. The Riemann-Liouville fractional integral of order $\alpha > 0$ of f is defined by

$$({}^{\text{RL}}I_a^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad \text{a.e } t \in [a, b]. \quad (13)$$

Definition 5 (see [23]). Let $\alpha > 0$ and n be the smallest integer greater than or equal to α . Let $f : [a, b] \rightarrow \mathbb{R}$ be a function such that ${}^{\text{RL}}I_a^{n-\alpha} f \in AC^n([a, b]; \mathbb{R})$. Then the Riemann-Liouville fractional derivative of order α of a function f is defined by

$$\begin{aligned}({}^{\text{RL}}D_a^\alpha f)(t) &= \left(\frac{d}{dt} \right)^n {}^{\text{RL}}I_a^{n-\alpha} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,\end{aligned}\quad (14)$$

for a.e. $t \in [a, b]$.

Let $\alpha > 0$ and n be the smallest integer greater than or equal to α . By $AC^\alpha([a, b]; \mathbb{R})$ (see [38]), one denotes the set of all functions $f : [a, b] \rightarrow \mathbb{R}$ that have the representation:

$$f(t) = \sum_{i=0}^{n-1} \frac{c_i}{\Gamma(\alpha - n + 1 + i)} (t-a)^{\alpha-n+i} + {}^{\text{RL}}I_a^\alpha \varphi(t), \quad (15)$$

a.e. $t \in [a, b]$,

where $c_0, c_1, \dots, c_{n-1} \in \mathbb{R}$ and $\varphi \in L^1((a, b); \mathbb{R})$.

Lemma 6 (see [38]). Let $\alpha > 0$, n be the smallest integer greater than or equal to α , and $f \in L^1((a, b); \mathbb{R})$. Then ${}^{\text{RL}}D_a^\alpha f(t)$ exists almost everywhere on $[a, b]$ if and only if $f \in \text{AC}^\alpha([a, b]; \mathbb{R})$; that is, f has representation (15). In such a case, one has

$$({}^{\text{RL}}D_a^\alpha f)(t) = \varphi(t), \quad \text{a.e. } t \in [a, b]. \quad (16)$$

Let $g \in C^1([a, b]; \mathbb{R})$ be a nondecreasing function with $g'(x) > 0$, for all $x \in (a, b)$.

Definition 7 (see [23]). Let $f \in L^1((a, b); \mathbb{R})$. The fractional integral of order $\alpha > 0$ of f with respect to the function g is defined by

$$(I_{a,g}^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g'(s) f(s)}{(g(t) - g(s))^{1-\alpha}} ds, \quad (17)$$

a.e. $t \in [a, b]$.

Definition 8 (see [23]). Let $\alpha > 0$ and n be the smallest integer greater than or equal to α . Let $f : [a, b] \rightarrow \mathbb{R}$ be a function such that $((1/g'(t))(d/dt))^n I_{a,g}^{n-\alpha} f$ exists almost everywhere on $[a, b]$. In this case, the fractional derivative of order α of f with respect to the function g is defined by

$$\begin{aligned} D_{a,g}^\alpha f(t) &= \left(\frac{1}{g'(t)} \frac{d}{dt} \right)^n I_{a,g}^{n-\alpha} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{g'(t)} \frac{d}{dt} \right)^n \int_a^t \frac{g'(s) f(s)}{(g(t) - g(s))^{\alpha-n+1}} ds, \end{aligned} \quad (18)$$

for a.e. $t \in [a, b]$.

The following lemma is crucial for the proof of our main result.

Lemma 9. Let $\alpha > 0$ and n be the smallest integer greater than or equal to α . Suppose that the function $y \mapsto f(g^{-1}(y))$ belongs to the space $\text{AC}^\alpha([g(a), g(b)]; \mathbb{R})$. Then

$$\begin{aligned} D_{a,g}^\alpha f(g^{-1}(y)) &= {}^{\text{RL}}D_{g(a)}^\alpha (f \circ g^{-1})(y), \\ &\text{a.e. } y \in [g(a), g(b)]. \end{aligned} \quad (19)$$

Proof. At first, observe that, from Lemma 6, ${}^{\text{RL}}D_{g(a)}^\alpha (f \circ g^{-1})(y)$ exists for a.e. $y \in [g(a), g(b)]$. Now, using the change of variable $x = g^{-1}(y)$, $y \in (g(a), g(b))$, the chain rule yields

$$\frac{d}{dy} = \frac{d}{dx} \frac{dx}{dy} = \frac{d}{dx} \frac{1}{g'(g^{-1}(y))} = \frac{1}{g'(x)} \frac{d}{dx}. \quad (20)$$

Therefore, we obtain

$$\begin{aligned} {}^{\text{RL}}D_{g(a)}^\alpha (f \circ g^{-1})(y) &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dy} \right)^n \int_{g(a)}^y \frac{f(g^{-1}(s))}{(y-s)^{\alpha-n+1}} ds \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{g'(x)} \frac{d}{dx} \right)^n \int_{g(a)}^{g(x)} \frac{f(g^{-1}(s))}{(g(x)-s)^{\alpha-n+1}} ds. \end{aligned} \quad (21)$$

Next, using the change of variable $t = g^{-1}(s)$, we obtain

$$\begin{aligned} {}^{\text{RL}}D_{g(a)}^\alpha (f \circ g^{-1})(y) &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{g'(x)} \frac{d}{dx} \right)^n \int_a^x \frac{g'(t) f(t)}{(g(x) - g(t))^{\alpha-n+1}} dt \\ &= D_{a,g}^\alpha f(x) = D_{a,g}^\alpha f(g^{-1}(y)), \end{aligned} \quad (22)$$

which proves the desired result. $\square$

In the sequel, we denote by $\Xi_g([a, b]; \mathbb{R})$ the functional space defined by

$$\begin{aligned} \Xi_g([a, b]; \mathbb{R}) &= \{f : [a, b] \rightarrow \mathbb{R} : f \circ g^{-1} \\ &\in \text{AC}^\alpha([g(a), g(b)]; \mathbb{R})\}. \end{aligned} \quad (23)$$

Definition 10 (see [23]). Let $\alpha > 0$ and n be the smallest integer greater than or equal to α . Let $g(x) = \ln x$, where $x \in [a, b]$ and $0 < a < b$. The Hadamard fractional derivative of order α of $f \in \Xi_g([a, b]; \mathbb{R})$ is defined by

$${}^H D_a^\alpha f(t) = D_{a,g}^\alpha f(t), \quad \text{a.e. } t \in [a, b]. \quad (24)$$

We refer the reader to Ferreira [24] for the proofs of the following results.

Lemma 11. Let $h \in C([A, B]; \mathbb{R})$, $(A, B) \in \mathbb{R}^2$, $A < B$, and $1 < \alpha < 2$. Then $F \in \text{AC}^\alpha([a, b]; \mathbb{R}) \cap C([a, b]; \mathbb{R})$ is a solution of the boundary value problem

$$\begin{aligned} ({}^{\text{RL}}D_A^\alpha F)(t) + h(t) &= 0, \quad A < t < B, \\ F(A) &= F(B) = 0 \end{aligned} \quad (25)$$

if, and only if, F satisfies the integral equation

$$F(t) = \int_A^B G(t, s) h(s) ds, \quad A \leq t \leq B, \quad (26)$$

where

$$\begin{aligned} & \Gamma(\alpha) G(t, s) \\ &= \begin{cases} \frac{(t-A)^{\alpha-1}}{(B-A)^{\alpha-1}} (B-s)^{\alpha-1} - (t-s)^{\alpha-1}, & A \leq s \leq t \leq B, \\ \frac{(t-A)^{\alpha-1}}{(B-A)^{\alpha-1}} (B-s)^{\alpha-1}, & A \leq t \leq s \leq B. \end{cases} \end{aligned} \quad (27)$$

Lemma 12. *The Green function G defined by (27) satisfies the following properties:*

(i) $G(t, s) \geq 0$ for all $A \leq t, s \leq B$.

(ii) For all $s \in [A, B]$, one has

$$\max_{t \in [A, B]} G(t, s) = G(s, s). \quad (28)$$

3. A Hartman-Wintner-Type Inequality for Boundary Value Problem (1)

In this section, a Hartman-Wintner-type inequality is established for fractional boundary value problem (1).

Problem (1) is investigated under the following assumptions:

(A1) $\alpha \in (1, 2)$.

(A2) $q \in C([a, b]; \mathbb{R})$.

(A3) $g \in C^1([a, b]; \mathbb{R})$.

(A4) g is a nondecreasing function with $g'(x) > 0$, for all $x \in (a, b)$.

We have the following result.

Theorem 13. *Under assumptions (A1)–(A4), if fractional boundary value problem (1) has a nontrivial solution $u \in \Xi_g([a, b]; \mathbb{R}) \cap C([a, b]; \mathbb{R})$, then*

$$\begin{aligned} & \int_a^b [(g(s) - g(a))(g(b) - g(s))]^{\alpha-1} g'(s) |q(s)| ds \\ & \geq \Gamma(\alpha) (g(b) - g(a))^{\alpha-1}. \end{aligned} \quad (29)$$

Proof. Suppose that u is a nontrivial solution of (1). Let us define the function $v : [g(a), g(b)] \rightarrow \mathbb{R}$ by

$$v(y) = u(g^{-1}(y)), \quad y \in [g(a), g(b)]. \quad (30)$$

Using Lemma 9, for all $y \in (g(a), g(b))$, we have

$$\begin{aligned} D_{a,g}^\alpha u(g^{-1}(y)) &= {}^{\text{RL}}D_{g(a)}^\alpha (u \circ g^{-1})(y) \\ &= {}^{\text{RL}}D_{g(a)}^\alpha v(y). \end{aligned} \quad (31)$$

On the other hand, since u is a solution of (1), we have

$$D_{a,g}^\alpha u(g^{-1}(y)) = -q(g^{-1}(y)) u(g^{-1}(y)), \quad y \in (g(a), g(b)), \quad (32)$$

$$u(g^{-1}(g(a))) = u(g^{-1}(g(b))) = 0.$$

Therefore, v is a nontrivial solution of the Riemann-Liouville fractional boundary value problem

$$\begin{aligned} & ({}^{\text{RL}}D_A^\alpha v)(y) + Q(y) v(y) = 0, \quad A < y < B, \\ & v(A) = v(B) = 0, \end{aligned} \quad (33)$$

where $A = g(a)$, $B = g(b)$, and $Q : [A, B] \rightarrow \mathbb{R}$ is the function defined by

$$Q(y) = q(g^{-1}(y)), \quad y \in [A, B]. \quad (34)$$

Now, by Lemma 11, we obtain

$$v(y) = \int_A^B G(y, s) Q(s) v(s) ds, \quad A \leq y \leq B, \quad (35)$$

where G is the Green function defined by (27). Next, let us consider the Banach space $C([A, B]; \mathbb{R})$ equipped with the standard norm

$$\|v\|_\infty = \max\{|v(y)| : A \leq y \leq B\}. \quad (36)$$

Clearly, since v is nontrivial, we have $\|v\|_\infty > 0$. Further, using (35) and Lemma 12, we have

$$|v(y)| \leq \|v\|_\infty \int_A^B G(s, s) |Q(s)| ds, \quad y \in [A, B], \quad (37)$$

which yields

$$\|v\|_\infty \leq \|v\|_\infty \int_A^B G(s, s) |Q(s)| ds. \quad (38)$$

Therefore, we obtain

$$\int_A^B G(s, s) |Q(s)| ds \geq 1; \quad (39)$$

that is,

$$\int_{g(a)}^{g(b)} G(s, s) |q(g^{-1}(s))| ds \geq 1. \quad (40)$$

Using the change of variable $s = g(t)$, we get

$$\int_a^b G(g(t), g(t)) |q(t)| g'(t) dt \geq 1. \quad (41)$$

Note that by (27) we have

$$\begin{aligned} G(g(t), g(t)) &= \frac{(g(t) - g(a))^{\alpha-1} (g(b) - g(t))^{\alpha-1}}{\Gamma(\alpha) (g(b) - g(a))^{\alpha-1}}, \\ & \quad t \in [a, b]. \end{aligned} \quad (42)$$

Therefore,

$$\int_a^b [(g(t) - g(a))(g(b) - g(t))]^{\alpha-1} g'(t) |q(t)| dt \quad (43)$$

$$\geq \Gamma(\alpha) (g(b) - g(a))^{\alpha-1},$$

which is desired inequality (29). $\square$

4. Lyapunov-Type Inequalities for Different Choices of the Function g

In this section, using Theorem 13, several Lyapunov-type inequalities are deduced for different choices of the function g .

4.1. The Case $g(x) = x^\beta$, $\beta > 0$. Taking $g(x) = x^\beta$, $\beta > 0$, in Theorem 13, we deduce the following Hartman-Wintner-type inequality.

Corollary 14. *If fractional boundary value problem (1) has a nontrivial solution $u \in \Xi_g([a, b]; \mathbb{R}) \cap C([a, b]; \mathbb{R})$, where $g(x) = x^\beta$, $x \in [a, b]$, $0 < a < b$, then*

$$\begin{aligned} & \int_a^b [(s^\beta - a^\beta)(b^\beta - s^\beta)]^{\alpha-1} s^{\beta-1} |q(s)| ds \\ & \geq \frac{\Gamma(\alpha) (b^\beta - a^\beta)^{\alpha-1}}{\beta}. \end{aligned} \quad (44)$$

Next, let us define the function $\varphi_{\alpha, \beta} : [a, b] \rightarrow [0, \infty)$ by

$$\varphi_{\alpha, \beta}(s) = [(s^\beta - a^\beta)(b^\beta - s^\beta)]^{\alpha-1} s^{\beta-1}, \quad s \in [a, b]. \quad (45)$$

Since $\varphi_{\alpha, \beta}$ is continuous on $[a, b]$ and $\varphi_{\alpha, \beta}(a) = \varphi_{\alpha, \beta}(b) = 0$, there exists some $s^*(\alpha, \beta) \in (a, b)$ such that

$$\varphi_{\alpha, \beta}(s^*(\alpha, \beta)) = \max \{\varphi_{\alpha, \beta}(s) : s \in [a, b]\} > 0. \quad (46)$$

Therefore, from inequality (44), we obtain the following Lyapunov-type inequality.

Corollary 15. *If fractional boundary value problem (1) has a nontrivial solution $u \in \Xi_g([a, b]; \mathbb{R}) \cap C([a, b]; \mathbb{R})$, where $g(x) = x^\beta$, $x \in [a, b]$, $0 < a < b$, then*

$$\int_a^b |q(s)| ds \geq \frac{\Gamma(\alpha) (b^\beta - a^\beta)^{\alpha-1}}{\beta \varphi_{\alpha, \beta}(s^*(\alpha, \beta))}. \quad (47)$$

In order to compute the value of $s^*(\alpha, \beta)$ for $\alpha \in (1, 2)$ and $\beta > 0$, we have to study the variations of the function $\varphi_{\alpha, \beta}$ defined by (45). Observe that

$$\varphi_{\alpha, \beta}(s) = \phi_{\alpha, \beta}(s^\beta), \quad s \in [a, b], \quad (48)$$

where $\phi_{\alpha, \beta} : [M, N] \rightarrow [0, \infty)$ is the function defined by

$$\begin{aligned} \phi_{\alpha, \beta}(x) &= [(x - M)(N - x)]^{\alpha-1} x^{(\beta-1)/\beta}, \\ x &\in [M, N] \end{aligned} \quad (49)$$

with $M = a^\beta$ and $N = b^\beta$. A simple computation yields

$$\begin{aligned} \phi'_{\alpha, \beta}(x) &= \frac{\phi_{\alpha, \beta}(x)}{x(x - M)(N - x)} ((\gamma - 2\delta)x^2 \\ &+ (M + N)(\delta - \gamma)x + \gamma MN), \end{aligned} \quad (50)$$

for all $x \in (M, N)$, where $\gamma = (1 - \beta)/\beta$ and $\delta = \alpha - 1$. Next, we put

$$\begin{aligned} P(x) &= (\gamma - 2\delta)x^2 + (M + N)(\delta - \gamma)x + \gamma MN, \\ x &\in [M, N]. \end{aligned} \quad (51)$$

We consider three cases.

Case 1 (if $\beta = 1/(2\alpha - 1)$). In this case, we have $\gamma = 2\delta$ and $P(x) = 0$ if and only if $x = 2MN/(M + N)$. Moreover, we have $P(x) \geq 0$ for $x \in [M, 2MN/(M + N)]$ and $P(x) \leq 0$ for $x \in [2MN/(M + N), N]$. Therefore,

$$\begin{aligned} \phi_{\alpha, \beta}\left(\frac{2MN}{M + N}\right) &= \max \{\phi_{\alpha, \beta}(x) : x \in [M, N]\}, \\ s^*(\alpha, \beta) &= \left(\frac{2MN}{M + N}\right)^{1/\beta} = \left(\frac{2}{a^\beta + b^\beta}\right)^{1/\beta} ab. \end{aligned} \quad (52)$$

Thus, in this case we obtain

$$\varphi_{\alpha, \beta}(s^*(\alpha, \beta)) = \left[\frac{(b^\beta - a^\beta)^2}{4(ab)^\beta} \right]^{\alpha-1}. \quad (53)$$

Next, using (53), we deduce from Corollary 15 the following Lyapunov-type inequality in the case $\beta(2\alpha - 1) = 1$.

Corollary 16 (the case $\beta(2\alpha - 1) = 1$). *If fractional boundary value problem (1) has a nontrivial solution $u \in \Xi_g([a, b]; \mathbb{R}) \cap C([a, b]; \mathbb{R})$, where $g(x) = x^\beta$, $x \in [a, b]$, $0 < a < b$, then*

$$\int_a^b |q(s)| ds \geq \frac{\Gamma(\alpha)}{\beta} \left[\frac{4(ab)^\beta}{b^\beta - a^\beta} \right]^{\alpha-1}. \quad (54)$$

Case 2 (if $0 < \beta < 1/(2\alpha - 1)$). In this case, we have $\gamma > 2\delta > 0$ and $P(x)$ has two distinct zeros at

$$\begin{aligned} x_1 &= \frac{(\gamma - \delta)(M + N) - \sqrt{\Delta}}{2(\gamma - 2\delta)}, \\ x_2 &= \frac{(\gamma - \delta)(M + N) + \sqrt{\Delta}}{2(\gamma - 2\delta)}, \end{aligned} \quad (55)$$

where

$$\Delta = (M - N)^2 (\delta - \gamma)^2 + 4MN\delta^2. \quad (56)$$

It can be easily seen that

$$0 < M < x_1 < N < x_2. \quad (57)$$

Moreover, we have $P(x) \geq 0$ for $x \in [M, x_1]$ and $P(x) \leq 0$ for $x \in [x_1, N]$. Therefore,

$$\begin{aligned}\phi_{\alpha,\beta}(x_1) &= \max \{ \phi_{\alpha,\beta}(x) : x \in [M, N] \}, \\ s^*(\alpha, \beta) &= x_1^{1/\beta} = \left(\frac{(1-\alpha\beta)(a^\beta + b^\beta) - \sqrt{(a^\beta - b^\beta)^2(1-\alpha\beta)^2 + 4a^\beta b^\beta \beta^2(1-\alpha)^2}}{2[\beta(1-2\alpha) + 1]} \right)^{1/\beta}.\end{aligned}\quad (58)$$

Case 3 (if $\beta > 1/(2\alpha - 1)$). In this case, we have $\gamma < 2\delta$ and $P(x)$ has two distinct zeros at x_1 and x_2 . It can be easily seen that

$$x_2 < M < x_1 < N. \quad (59)$$

Moreover, we have $P(x) \geq 0$ for $x \in [M, x_1]$ and $P(x) \leq 0$ for $x \in [x_1, N]$. Therefore,

$$\begin{aligned}\phi_{\alpha,\beta}(x_1) &= \max \{ \phi_{\alpha,\beta}(x) : x \in [M, N] \}, \\ s^*(\alpha, \beta) &= x_1^{1/\beta} = \left(\frac{(1-\alpha\beta)(a^\beta + b^\beta) - \sqrt{(a^\beta - b^\beta)^2(1-\alpha\beta)^2 + 4a^\beta b^\beta \beta^2(1-\alpha)^2}}{2[\beta(1-2\alpha) + 1]} \right)^{1/\beta}.\end{aligned}\quad (60)$$

Observe that, for $\beta = 1$ ($g(x) = x$), problem (1) is equivalent to problem (7). Moreover, in this case we have

$$\begin{aligned}s^*(\alpha, 1) &= x_1 = \frac{a+b}{2}, \\ \varphi_{\alpha,1}(s^*(\alpha, 1)) &= \varphi_{\alpha,1}\left(\frac{a+b}{2}\right) = \left[\frac{(b-a)^2}{4}\right]^{\alpha-1}.\end{aligned}\quad (61)$$

Therefore, using (61) and Corollary 15, we obtain inequality (8), which is due to Ferreira [24].

4.2. A Lyapunov-Type Inequality via Hadamard Fractional Derivative. Taking $g(x) = \ln x$ in Theorem 13, we deduce the following Hartman-Wintner-type inequality for the following Hadamard fractional boundary value problem:

$$\begin{aligned}({}^H D_a^\alpha u)(t) + q(t)u(t) &= 0, \quad a < t < b, \\ u(a) &= u(b) = 0,\end{aligned}\quad (62)$$

where $(a, b) \in \mathbb{R}^2$, $0 < a < b$, $\alpha \in (1, 2)$, and $q : [a, b] \rightarrow \mathbb{R}$ is a continuous function.

Corollary 17. *If fractional boundary value problem (62) has a nontrivial solution $u \in \Xi_g([a, b]; \mathbb{R}) \cap C([a, b]; \mathbb{R})$, where $g(x) = \ln x$, $x \in [a, b]$, then*

$$\int_a^b \left[\left(\ln \frac{s}{a} \right) \left(\ln \frac{b}{s} \right) \right]^{\alpha-1} \frac{|q(s)|}{s} ds \geq \Gamma(\alpha) \left(\ln \frac{b}{a} \right)^{\alpha-1}. \quad (63)$$

Now, define the function $\psi : [a, b] \rightarrow [0, \infty)$ by

$$\psi(s) = \left[\left(\ln \frac{s}{a} \right) \left(\ln \frac{b}{s} \right) \right]^{\alpha-1} s^{-1}, \quad s \in [a, b]. \quad (64)$$

Observe that

$$\psi(s) = \mu(\ln s), \quad s \in [a, b], \quad (65)$$

where $\mu : [A, B] \rightarrow [0, \infty)$ is the function defined by

$$\mu(x) = [(x-A)(B-x)]^{\alpha-1} e^{-x}, \quad x \in [A, B] \quad (66)$$

with $A = \ln a$ and $B = \ln b$. A simple computation yields

$$\mu'(x) = [(x-A)(B-x)]^{\alpha-2} e^{-x} R(x), \quad x \in (A, B), \quad (67)$$

where

$$\begin{aligned}R(x) &= x^2 - (2(\alpha-1) + A+B)x + (\alpha-1)(A+B) \\ &\quad + AB, \quad x \in [A, B].\end{aligned}\quad (68)$$

Observe that $R(x)$ has two distinct zeros at

$$\begin{aligned}x_1 &= \frac{2(\alpha-1) + A+B - \sqrt{4(\alpha-1)^2 + (A-B)^2}}{2} \\ &:= \lambda(a, b),\end{aligned}\quad (69)$$

$$x_2 = \frac{2(\alpha-1) + A+B + \sqrt{4(\alpha-1)^2 + (A-B)^2}}{2}.$$

It can be easily seen that

$$A < x_1 < B < x_2. \quad (70)$$

Moreover, we have $R(x) \geq 0$ for $x \in [A, x_1]$ and $R(x) \leq 0$ for $x \in [x_1, B]$. Therefore, we deduce that

$$\mu(x_1) = \max \{ \mu(x) : x \in [A, B] \}, \quad (71)$$

$$\begin{aligned} \psi(e^{x_1}) &= \max \{ \psi(s) : s \in [a, b] \} \\ &= [(\lambda(a, b) - \ln a)(\ln b - \lambda(a, b))]^{\alpha-1} e^{-\lambda(a, b)}. \end{aligned} \quad (72)$$

Next, combining (63) with (72), we obtain the following Lyapunov-type inequality for fractional boundary value problem (62).

Corollary 18. *If fractional boundary value problem (62) has a nontrivial solution $u \in \Xi_g([a, b]; \mathbb{R}) \cap C([a, b]; \mathbb{R})$, where $g(x) = \ln x$, $x \in [a, b]$, then*

$$\begin{aligned} &\int_a^b |q(s)| ds \\ &\geq \Gamma(\alpha) \left[\frac{\ln b - \ln a}{(\lambda(a, b) - \ln a)(\ln b - \lambda(a, b))} \right]^{\alpha-1} e^{\lambda(a, b)}, \end{aligned} \quad (73)$$

where

$$\begin{aligned} &\lambda(a, b) \\ &= \frac{2(\alpha - 1) + \ln a + \ln b - \sqrt{4(\alpha - 1)^2 + (\ln a - \ln b)^2}}{2}. \end{aligned} \quad (74)$$

Observe that, in the particular case $(a, b) = (1, e)$, inequality (73) reduces to inequality (10) which is due to Ma et al. [37].

Remark 19. Corollary 18 is an answer to the open problem proposed in [37].

5. Applications to Eigenvalue Problems

Now, we present an application of the Hartman-Wintner-type inequality given by Theorem 13 to eigenvalue problems.

We say that the scalar λ is an eigenvalue of the fractional boundary value problem

$$\begin{aligned} (D_{a,g}^\alpha u)(t) + \lambda u(t) &= 0, \quad a < t < b, \\ u(a) &= u(b) = 0, \end{aligned} \quad (75)$$

where $(a, b) \in \mathbb{R}^2$, $a < b$, $\alpha \in (1, 2)$, and $g \in C^1([a, b]; \mathbb{R})$ with $g'(x) > 0$, for all $x \in (a, b)$, if problem (75) has at least a nontrivial solution $u_\lambda \in \Xi_g([a, b]; \mathbb{R}) \cap C([a, b]; \mathbb{R})$.

We have the following result which provides a lower bound of the eigenvalues of problem (75).

Corollary 20. *If λ is an eigenvalue of problem (75), then*

$$|\lambda| \geq \frac{\Gamma(\alpha)(B-A)^{\alpha-1}}{\int_A^B (x-A)^{\alpha-1}(B-x)^{\alpha-1} dx}, \quad (76)$$

where $A = g(a)$ and $B = g(b)$.

Proof. Suppose that λ is an eigenvalue of problem (75). Then problem (75) admits a nontrivial solution. Applying Theorem 13 with $q \equiv \lambda$, we obtain

$$\begin{aligned} |\lambda| \int_a^b (g(s) - g(a))^{\alpha-1} (g(b) - g(s))^{\alpha-1} g'(s) ds \\ \geq \Gamma(\alpha) (g(b) - g(a))^{\alpha-1}. \end{aligned} \quad (77)$$

Using the change of variable $x = g(s)$, we obtain

$$|\lambda| \int_A^B (x-A)^{\alpha-1} (B-x)^{\alpha-1} dx \geq \Gamma(\alpha) (B-A)^{\alpha-1}, \quad (78)$$

which proves the desired inequality. $\square$

Taking $g(x) = x^\beta$, $\beta > 0$, in Corollary 20, we obtain the following result.

Corollary 21. *If λ is an eigenvalue of problem (75) with $g(x) = x^\beta$, $\beta > 0$, $x \in [a, b]$, $0 < a < b$, then*

$$|\lambda| \geq \frac{\Gamma(\alpha) (b^\beta - a^\beta)^{\alpha-1}}{\int_{a^\beta}^{b^\beta} (x - a^\beta)^{\alpha-1} (b^\beta - x)^{\alpha-1} dx}. \quad (79)$$

Taking $g(x) = \ln x$ in Corollary 20, we obtain the following result.

Corollary 22. *If λ is an eigenvalue of the Hadamard fractional boundary value problem*

$$\begin{aligned} ({}^H D_a^\alpha u)(t) + \lambda u(t) &= 0, \quad a < t < b, \\ u(a) &= u(b) = 0, \end{aligned} \quad (80)$$

where $(a, b) \in \mathbb{R}^2$, $0 < a < b$, and $\alpha \in (1, 2)$, then

$$|\lambda| \geq \frac{\Gamma(\alpha) (\ln b - \ln a)^{\alpha-1}}{\int_{\ln a}^{\ln b} (x - \ln a)^{\alpha-1} (\ln b - x)^{\alpha-1} dx}. \quad (81)$$

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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