The vertical distribution of suspended sediment concentration in steady turbulent flow is an important measure when evaluating the suspended flux in natural rivers and canals [1–3]. The physical mechanism of a steady sediment suspension distribution is a dynamic equilibrium of vertical fluxes between downward sediment settling and upward turbulent diffusion. Nonequilibrium of the two opposing processes may result in erosion or deposition, which consequently brings environmental issues such as water and soil erosion and reservoirs deposition.

The theories of gravity, diffusion, mixing, energy dissipation, similarity, and stochastic models have been applied to study the vertical diffusion of sediment [4–7]. The theories of gravity and diffusion, among others, are the most frequently used. Yet, the gravity theory has been questioned due to the ambiguous definition of “suspension energy” [8]. The diffusion theory has been more widely applied in recent decades since it explains many observations in laboratory and field settings [9–11]. In addition, the two-phase flow system, which has been applied to investigate a variety of problems of interest in fluvial hydraulics, may be described in terms of either macroscopic or microscopic methods [12–18].

The diffusion theory is derived from the traditional advection-diffusion equation (ADE) model by assuming Fick’s first law for sediment diffusion in turbulence:

$$\frac{\partial S}{\partial t} = \omega \frac{\partial S}{\partial y} + \varepsilon y \frac{\partial^2 S}{\partial y^2},$$

(1)
where $S$ [ML$^3$] is the sediment volumetric concentration, $y$ [L] is the vertical coordinate, $\omega$ [L/T] is the sediment settling velocity, and $\varepsilon_{sy}$ [L$^2$/T] is the sediment turbulent diffusion coefficient along the $y$ direction. When downward sediment settling and upward turbulent diffusion reach the balanced state, it results in

$$\omega S + \varepsilon_{sy} \frac{\partial S}{\partial y} = 0. \quad (2)$$

Here $\varepsilon_{sy} = \beta \varepsilon_{sw}$, and $\varepsilon_{sw}$ is the fluid eddy viscosity and $\varepsilon_{sw} = \kappa u_*(1-y/h) y$ based on the Karman-Prandtl logarithmic velocity profile. The analytical solution of (2) yields the well-known Rouse vertical concentration profile [19]:

$$s = \frac{S}{S_0} = \left[ \frac{h/y - 1}{h/a - 1} \right]^{\omega/\kappa u_*}, \quad (3)$$

where $s$ is the relative sediment concentration, $S_0$ is a reference concentration at a given height $a$ above the riverbed, $h$ [L] is the water depth, $a$ [L] is the reference height, $\kappa$ [dimensionless] is the Von Karman constant, $u_*$ [L/T] is the shear velocity, and $\beta$ [dimensionless] is a proportionality coefficient related to diffusion of sediment particles.

As a milestone in the history of sediment transport, the Rouse formula (3) has been widely used for decades [4, 8, 28]. However, limitations of this theory are also obvious: the sediment concentration is calculated as zero at the water surface and infinity at the riverbed. Vanoni also stated, in his classical manual [29], that the Rouse formula can only represent the shape of the distribution not the actual values in a predictive sense. Therefore, many researchers have put forward modifications or improvements on the Rouse formula [8, 16, 20–23]. Besides the formula investigated in the next section, there are many valuable works on this topic. For example, Greimann et al. [24, 30] offered both numerical and analytical expressions for concentration profiles, based on two-phase flow analysis. Bombardelli and Jha [31] and Jha and Bombardelli [32–34] established a framework composed of the complete two-fluid model, a partial two-fluid model, and a standard sediment-transport model and further discussed different models in describing three datasets. Toorman [35] derived Eulerian equations for the vertical flux and momentum of suspended particles in dilute sediment-laden open-channel flow in equilibrium using the two-fluid approach.

Recently, the fractional advection-dispersion equation (fADE) model has been developed to describe anomalous diffusion of sediment [21, 36–38]. As an extension of the traditional advection-dispersion equation, the fADE model can describe the anomalous diffusive characteristics of suspended sediment in turbulence, for example, nonlocal displacement or superdiffusion in certain circumstances such as turbulence bursting. However, the advantages and parameter determination method of the fADE model in describing suspended sediment transport are still unclear [36–43]. In this study, we investigate the Rouse formula and six improved models (including the fADE model) with explicit expressions. A comparison of these models in describing previously published experimental data has been presented for illustrative purposes. Furthermore, this study develops empirical formulas for accessing the fractional derivative order in terms of particle sizes or sediment settling velocities. The aim is to help river engineers in estimating the vertical distribution of suspended sediment concentration via the fADE model in real-world applications.

### 2. Improved Models Based on the Rouse Formula

#### 2.1. Model 1 (M1).

To obtain a sediment concentration distribution formula, which can be applied throughout the flow region along the depth in natural rivers, Zhang [20] developed a vertical distribution formula of suspended sediment by solving the diffusion equation accompanied by the velocity distribution formula in sediment-laden flow:

$$s = \frac{S}{S_0} = \exp \left[ \frac{\omega}{\kappa u_*} \left( \arctan \frac{h}{y} - 1 - 1.345 \right) \right]. \quad (4)$$

#### 2.2. fADE Model.

Numerous studies have shown that particle dynamics in turbulent flow exhibits anomalous diffusive characteristics and can be well described using the fractional Fick’s law [44]. Chen et al. [21] pointed out that turbulence bursting sometimes plays a key role in sediment diffusion and further proposed a fADE model using the fractional derivative to characterize nonlocal particle movement in steady turbulence:

$$\omega S + \varepsilon_{sy} \frac{\partial^\alpha S}{\partial y^\alpha} = 0, \quad (5)$$

where $\alpha$ is the fractional derivative order ($0 < \alpha \leq 1$). Considering that the anomalous diffusion occurs in an entire water body, they replaced the depth-dependent coefficient $\varepsilon_{sy}$ with a depth-averaged diffusivity $\varepsilon_{sy}$ in (5). New $\varepsilon_{sy}$ is obtained by integrating Rouse’s [29] expression of $\varepsilon_{sy}$ from the reference height $a$ ($a = 0.05h$) to the water surface ($y = h$):

$$\varepsilon_{sy} = \int_{0.05h}^{h} \kappa u_* (1-y/h) y \, dy = \frac{209\kappa u_* h}{1200}. \quad (6)$$

In model (5), the definition of fractional derivative is expressed as follows [45]:

$$\frac{\partial^\alpha S}{\partial y^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} S(\tau) \, d\tau, \quad 0 < \alpha \leq 1. \quad (7)$$

By employing the theoretical technique of fractional diffusion equation and property of Mittag-Leffler function, the analytical solution of (5) for sediment suspension can be written as follows [39]:

$$s = \frac{S}{S_0} = E_\alpha \left[ -\frac{\omega}{\varepsilon_{sy}} (y - y_a)^\alpha \right], \quad (8)$$

in which the Mittag-Leffler function $E_\alpha(*)$ is expressed as

$$E_\alpha (z) = \sum_{k=0}^{\infty} \frac{z^k}{(ak + 1)^\alpha}. \quad (9)$$
2.4. Power Law Model. This model was introduced from the turbulent energy theory approach, which yields a sediment distribution function for vertical sediment concentration, which considers the interaction between flow and sediment [46]. Zheng et al. [22] further developed a continuous sediment concentration distribution formula by modifying the Van Rijn formula. Then, the vertical distribution formula of suspended sediment can be written as

\[ s = \frac{S}{S_a} = \left( \frac{h - a}{a} \right)^{\frac{y}{h}} \left( 1 + e^4 \right)^{-Z}, \]  

where the suspension index \( Z = \omega/\kappa u_* \).

2.5. Wang Model. A typical theoretical model for the particle concentration distribution is given by Wang and coworkers using the kinetic theory for two-phase flow [14]:

\[ s = \frac{S}{S_a} = \left( \frac{h - y}{h - a} \right)^{Z} \left( h - a \right)^{1-\eta} \left( y \right)^{-\eta}, \]  

where \( \eta = y/h \), \( a \) is the reference height, \( Z = \omega/\kappa u_* \), and \( \gamma_c \) is the coefficient accounting for the crossing-trajectory and continuity effects.

3. Comparison and Discussion

A summary of seven existing models including the Rouse formula and its extensions is shown in Table 1. Based on the observation of Table 1, it is clear that the six extended models improve the applicability of the Rouse formula and overcome the main description drawbacks at the riverbed and surface (except for the power law model and the two-phase flow model at the surface). As shown in Table 1, M1, M2, and Wang model have less parameters, while the fADE model, the power law model, and two-phase flow model contain one more parameter than the other models. M2 is obtained by modifying the Van Rijn model, among others. In the Wang model, a particle velocity distribution function is obtained in the equilibrium state or in a dilute steady state for a particle in two-phase flow, and then a theoretical model for the particle concentration distribution is derived from the kinetic theory. The power law model and M1 are obtained by a combination of the velocity distribution formula. In the two-phase flow model, a theoretical expression is obtained by assuming dilute concentrations and small particles. The fADE model is achieved by employing a fractional derivative to characterize anomalous diffusion in turbulence.

To further explore the feature of the above formulas in characterizing the vertical distribution of sediment suspension, here we investigate the vertical concentration profiles obtained using the above models. Figure 1 shows significant differences in the shape of the vertical profile for the different models. Generally speaking, the concentration calculated by the fADE model and Wang model under 0.05 \( h \) changes slowly, but the other models change dramatically, especially for the Rouse model. The M2, power law model, and Rouse model are coincident under 0.05 \( h \). M1, the power law model, and the two-phase flow model provide similar results as the Rouse formula near the water surface, in which the concentration tends to be zero. The Rouse model, M1, the fADE model, Wang model, M2, the power law model, and the two-phase flow model offer a similar description of the vertical distribution between heights 0.05 \( h \) and 0.9 \( h \).

When the settling velocity \( \omega \) increases from 0.01 to 0.04 (Figure 2), the concentration profiles of different models exhibit more nonuniform distribution features. The water surface concentration calculated by the fADE model is larger than the other models, while the concentration using the Rouse model, M1, Wang model, M2, the power law model, and the two-phase flow model tends to be zero. It should also be noted that all of the calculated concentration profiles of the existing models have a similar decay trend under the height of 0.05 \( h \) and the corresponding variation tendency
Table 1: Comparison of different vertical distribution formulas for sediment suspension.

<table>
<thead>
<tr>
<th>Name</th>
<th>Theory</th>
<th>Parameters</th>
<th>Reference concentration ($y = 0.05h$)</th>
<th>Water surface concentration ($y = h$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rouse [19]</td>
<td>Diffusion theory</td>
<td>$\omega, \kappa, u_*$</td>
<td>$s = \infty$</td>
<td>$s = 0$</td>
</tr>
<tr>
<td>M1 [20]</td>
<td>Velocity distribution formula</td>
<td>$\omega, \kappa, u_*$</td>
<td>$\exp(1.20\omega/k_u)$</td>
<td>$\exp(-7.17\omega/k_u)$</td>
</tr>
<tr>
<td>fADE [21]</td>
<td>Superdiffusion of turbulent bursting</td>
<td>$\omega, \kappa, u_*, \alpha$</td>
<td>$1$</td>
<td>$E_{\alpha} \left[ (-\omega/\varepsilon) (h - y_s)^{\alpha} \right]$</td>
</tr>
<tr>
<td>M2 [22]</td>
<td>Coupling effect of flow and sediment</td>
<td>$\omega, \kappa, u_*$</td>
<td>$\left[ ((h - a)/20a) (1 + e^{-1.8}) \right]^{\omega/k_u}$</td>
<td>$\left[ ((h - a)/a) e^{1} \right]^{\omega/k_u}$</td>
</tr>
<tr>
<td>Power law [23]</td>
<td>Logarithmic velocity distribution law</td>
<td>$\omega, \kappa, u_*, n$</td>
<td>$1$</td>
<td>$\exp[-6(\omega/k_u_a)(h - a)/h)]$</td>
</tr>
<tr>
<td>Wang [14]</td>
<td>Kinetic theory in two-phase flow</td>
<td>$\omega, \kappa, u_*$</td>
<td>$1$</td>
<td>$0$</td>
</tr>
<tr>
<td>Two-phase Flow [24]</td>
<td>Two-phase flow</td>
<td>$\omega, \kappa, u_*, \gamma'$</td>
<td>$[a/19 (1 - a)]^{\omega/k_u_a}$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
Figure 1: Numerical results of different models in describing the vertical distribution of sediment suspension. Von Karman constant $\kappa = 0.4$, shear velocity $u_* = 0.10$, sediment settling velocity $\omega = 0.01$, water depth $h = 1.0$, $a = 0.05h$, and $\alpha = 0.9$ in the fADE model. For description simplicity, units are not used for parameters in this figure.

Figure 2: Numerical results of different models in describing the vertical distribution of sediment suspension. Von Karman constant $\kappa = 0.4$, shear velocity $u_* = 0.10$, sediment settling velocity $\omega = 0.04$, water depth $h = 1.0$, $a = 0.05h$, and $\alpha = 0.9$ in the fADE model. For description simplicity, units are not used for parameters in this figure.
keeps consistent with an increasing settling velocity. The comparison results (Figures 1–3) also show that shear velocity is not a critical parameter in controlling the decay pattern of the concentration profiles. Hence, it can be concluded that shear velocity plays a noncritical role in determining the vertical concentration distribution when compared with the other factors.

To test the efficiency of the above models in describing the vertical distribution of suspended sediment, here we use two groups of experimental data from Einstein and Chien [27] and Lyn [26]. The experiments were carried out with natural sands in two-dimensional, fully developed, and steady open-channel flows. In the experiments, coarse sand (with a diameter of $d = 1.3$ mm) was used for runs S-2 and S-3, and medium sand ($d = 0.94$ mm) for runs S-8 and S-9 in Einstein and Chien [27], while 1965EQ ($d = 0.19$ mm) and 2565EQ ($d = 0.24$ mm) were in Lyn [26]. Details of the experiments can be found in the related references [17, 18, 47]. The comparison results of different vertical distribution formulas in fitting the experimental data are drawn in Figures 4–6. As shown in Figures 4 and 5, it is clear that most extensions better describe the vertical distribution of sediment suspension than the Rouse formula. Figures 4 and 5 show the concentration distributions of runs S-2 and S-3 for coarse sand ($d = 1.30$ mm) and S-8 and S-9 for medium sand ($d = 0.94$ mm), respectively. The fractional derivative order $\alpha$ is numerically calculated using the experimental data and was eventually determined as 0.98 for S-2, 0.94 for S-3, and 0.88 for S-8 and S-9, respectively. Generally speaking, the fADE model and the Wang model are better than the other models in fitting the experimental data. Moreover, the fADE model offers a better description of the vertical distribution near the riverbed ($0.05h < y < 0.10h$). Similar results are also found in Figure 5 for medium sand ($d = 0.94$ mm).

Figure 6 indicates that the fADE model is the best one in the seven models to fit the experimental data with natural sand conducted by Lyn [26]. Moreover, the fADE model and Wang model offer a better description of the vertical distribution near the riverbed ($0.05h < y < 0.10h$) than the other models.

It is noteworthy that the Rouse formula with parameter $\beta \neq 1$ has also been used to describe the concentration profile. Figure 7 provides the experimental data fitting result of the Rouse formula with $\beta \neq 1$. Clearly, it gives a better agreement with experimental data than the basic Rouse formula, especially at the bottom region. However, since it is necessary to adjust the parameter $\beta$ frequently to coordinate the suspended sediment concentration profiles with the measured data, this formula is not easy to use in real-world applications.

4. Fractional Index of the fADE Model and Hydraulic Parameters

The fractional index is a key parameter in the fADE model to characterize the impact of turbulent bursting on sediment suspension. Therefore, the determination of the fractional index is a critical issue in real-world applications of the fADE model. Here we investigate the relationship between
Figure 4: Results of the different models in describing the experimental data of Einstein and Chien [27]. (a) S-2 ($d = 1.3$ mm), the fractional derivative order $\alpha = 0.98$, and the reference height $a = 0.1h$. (b) S-3 ($d = 1.3$ mm), the fractional derivative order $\alpha = 0.94$, and the reference height $a = 0.1h$.

Figure 5: Results of the different models in describing the experimental data of Einstein and Chien [27]. (a) S-8 ($d = 0.94$ mm), the fractional derivative order $\alpha = 0.88$, and the reference height $a = 0.1h$. (b) S-9 ($d = 0.94$ mm), the fractional derivative order $\alpha = 0.88$, and the reference height $a = 0.1h$. 
Figure 6: Results of the different models in describing the experimental data of Lyn [26]. (a) 1965EQ ($d = 0.19$ mm), the fractional derivative order $\alpha = 0.93$, and the reference height $a = 0.1h$. (b) 2565EQ ($d = 0.24$ mm), the fractional derivative order $\alpha = 0.97$, and the reference height $a = 0.1h$. 

Figure 7: Results of the Rouse model in describing the experimental data of Einstein and Chien [27] with $\beta = 1.75$, S-2, S-3 ($d = 1.3$ mm), and S-8, S-9 ($d = 0.94$ mm).
Table 2: The best fit fractional index $\alpha$ in the fADE model and corresponding hydraulic parameters, $d$-the particle size, and $\omega$-the settling velocity. Here, “A” denotes Chen et al. [21], “B” denotes Coleman [25], “C” denotes Lyn [26], and “D” denotes Einstein and Chien [27].

<table>
<thead>
<tr>
<th>Run number</th>
<th>Fractional index $\alpha$</th>
<th>Particle size $d$ (mm)</th>
<th>Settling velocity $\omega$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>0.095</td>
<td>0.035</td>
<td>0.000587</td>
</tr>
<tr>
<td>A2</td>
<td>0.12</td>
<td>0.075</td>
<td>0.002681</td>
</tr>
<tr>
<td>A3</td>
<td>0.20</td>
<td>0.175</td>
<td>0.013665</td>
</tr>
<tr>
<td>A4</td>
<td>0.41</td>
<td>0.375</td>
<td>0.044323</td>
</tr>
<tr>
<td>B1</td>
<td>0.55</td>
<td>0.105</td>
<td>0.00269</td>
</tr>
<tr>
<td>B2</td>
<td>0.90</td>
<td>0.210</td>
<td>0.0101</td>
</tr>
<tr>
<td>B3</td>
<td>0.75</td>
<td>0.420</td>
<td>0.03</td>
</tr>
<tr>
<td>C1</td>
<td>0.86</td>
<td>0.150</td>
<td>0.016</td>
</tr>
<tr>
<td>C2</td>
<td>0.93</td>
<td>0.190</td>
<td>0.023</td>
</tr>
<tr>
<td>C3</td>
<td>0.97</td>
<td>0.240</td>
<td>0.031</td>
</tr>
<tr>
<td>D1</td>
<td>0.80</td>
<td>0.274</td>
<td>0.035173</td>
</tr>
<tr>
<td>D2</td>
<td>0.88</td>
<td>0.940</td>
<td>0.114689</td>
</tr>
<tr>
<td>D3</td>
<td>0.94</td>
<td>1.300</td>
<td>0.140989</td>
</tr>
</tbody>
</table>

The fractional index in the fADE model and hydraulic parameters. The experimental data used in this section comes from Chen et al. [21], Einstein and Chien [27], Lyn [26], and Coleman [25].

Model parameters listed in Table 2 indicate that the fractional index $\alpha$ increases with particle size $d$ and settling velocity $\omega$. Small fractional indexes for fine particles mean that superdiffusion behavior dominates sediment particle movement, since the finer particles have more chances to jump long distances during a turbulent bursting event. Meanwhile, the coarse particles are not as easily influenced by the turbulent bursting event and hence tend to exhibit normal diffusion behavior, resulting in the fractional index $\alpha$ close to 1. Meanwhile, Table 2 also shows that the fractional index $\alpha$ is more sensitive to coarse or medium particles than fine particles. The best linear fitting functions, which can be used as empirical formulas, are offered in Figure 8. The figure depicts a linear relationship between the fractional index $\alpha$ and particle size $d$ for a given experimental condition. However, here we should point out that the fitting functions for different experiments are different, due to variations in the flow fields and the geometric structure of river sites.

The relationship between the fractional index $\alpha$ and settling velocity $\omega$ is similar to that of particle size. A particle’s settling velocity is determined by the sediment gravity and diffusion, and it can partially represent the turbulence bursting effect on sediment movement. Figure 9 shows that a higher settling velocity corresponds to a larger fractional index $\alpha$, implying that the sediment diffusion behavior changes from superdiffusion to normal diffusion with an increase in settling velocity $\omega$. In addition, it is clear that the particle size and settling velocity have similar influences on the fractional index. The reason may lie in the close relationship between settling velocity and particle size, where a large particle size usually yields a high settling velocity. Meanwhile the fractional index $\alpha$ exhibits a better linear relationship with the settling velocity from data fitting results.

5. Conclusions

This study evaluates seven existing formulas, including the one deduced by the fADE model, in describing the vertical distribution of sediment suspension under the steady-state flow condition. Numerical simulation results indicate that the fADE model provides a good agreement with the experimental data of Einstein and Chien [27] and Lyn [26], compared with the other models. Moreover, the fractional index in the fADE model can be estimated using a linear
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