Research Article

The Threshold of a Stochastic SIRS Model with Vertical Transmission and Saturated Incidence

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The threshold of a stochastic SIRS model with vertical transmission and saturated incidence is investigated. If the noise is small, it is shown that the threshold of the stochastic system determines the extinction and persistence of the epidemic. In addition, we find that if the noise is large, the epidemic still prevails. Finally, numerical simulations are given to illustrate the results.

1. Introduction

Mathematical models can describe the progress of a disease and predict the trend of the disease and they can provide the theoretical basis for people to undertake prevention strategies. At present, researches have constructed a series of mathematical models [1–10], including SIRS model [5, 9, 10]. They divided people into susceptible, S , infective, I , and removed, R , categories and one of the most famous SIRS epidemic models is the following:

$$\begin{aligned} S'(t) &= \Lambda + b(S(t) + I(t) + R(t)) - \beta S(t)I(t) \\ &\quad - \mu S(t) + \varepsilon R(t), \\ I'(t) &= \beta S(t)I(t) - (\mu + a + \gamma)I(t), \\ R'(t) &= \gamma I(t) - (\mu + \varepsilon)R(t). \end{aligned} \quad (1)$$

Here, Λ is the birth rate, β is the transmission rate, b and μ are the birth rate and natural death rate, respectively, of S , I , and R individuals, a is the death rate of the disease, and γ is the recovery rate of the infective individuals. Assume $\mu > b$.

Medical research has shown that the herpes virus will be in the form of mother-to-child transmission (vertical transmission) to the baby. In addition, since susceptible individuals in contact with every infective individual are limited, we see that when the number of the susceptible

individuals is large, the bilinear incidence βSI is unreasonable to consider. In this case, saturated incidence is more suitable than bilinear incidence [11].

In this paper, the transmission rate is chosen as the saturated incidence rate $\beta SI/(1 + \alpha I)$, and the SIRS model is described as the following:

$$\begin{aligned} S'(t) &= \Lambda + b(S(t) + R(t)) + (1 - q)bI - \frac{\beta S(t)I(t)}{1 + \alpha I(t)} \\ &\quad - \mu S(t) + \varepsilon R(t), \\ I'(t) &= \frac{\beta S(t)I(t)}{1 + \alpha I(t)} + qbI(t) - (\gamma + a + \mu)I(t), \\ R'(t) &= \gamma I(t) - (\mu + \varepsilon)R(t). \end{aligned} \quad (2)$$

In system (2), q is vertical transmission rate, $R_0 = \beta\Lambda/(\mu - b)(\mu + \gamma + a - qb)$ is the threshold which determines whether the disease will die out or persist, and there always is a disease-free equilibrium $P^0 = (\Lambda/(\mu - b), 0, 0)$; see [5, 9, 10] and the references therein. When $R_0 < 1$, the disease-free equilibrium P^0 is globally asymptotically stable; when $R_0 > 1$, P^0 is unstable and there is an endemic equilibrium $P^* = (S^*, I^*, R^*)$ where $I^* = ((\mu + \varepsilon)(\mu - b)(\mu + \gamma + a - qb)/(\beta(\mu - b)\gamma + [\beta(\mu - b) + \beta + \alpha(\mu - b)(\mu + \gamma + a - qb)](\mu + \varepsilon)))(R_0 - 1)$, $R^* = (\gamma/(\mu + \varepsilon))I^*$, $S^* = (\mu + \gamma + a - qb)(1 + \alpha I^*)/\beta$, which is globally stable under a sufficient condition.

However, all parameters in system (2) are affected by environmental noise, so it is of benefit to use a stochastic model. Stochastic models are more realistic compared to deterministic models. Many stochastic models for epidemic populations have been studied [12–19]. Tornatore et al. [18] studied an stochastic SIR model. They showed that under the condition $0 < \beta < \min\{\gamma + \mu - \sigma^2/2, 2\mu\}$, the disease-free equilibrium is locally stable, but the authors do not discuss under which condition the disease will prevail. Concerning the transmission coefficient β , Gray et al. [20] considered the stochastic SIS (susceptible-infective-susceptible) epidemic model with fluctuation. They proved threshold R_0 which determines the extinction and persistence of $I(t)$ according to the fluctuation. Here, R_0 is the threshold of the deterministic model; however, it is more difficult to get the threshold of the stochastic model.

We consider certain stochastic environmental factors and assume that fluctuations in the environment will manifest themselves mainly as fluctuations in the parameter β , as in [20],

$$\beta \longrightarrow \beta + \dot{B}(t), \quad (3)$$

where $B(t)$ is standard Brownian motion with $B(0) = 0$ and σ is the intensity. The stochastic version corresponding to the deterministic model (2) is the following:

$$\begin{aligned} dS(t) &= \left[\Lambda + b(S(t) + R(t)) + (1 - q)bI(t) \right. \\ &\quad \left. - \frac{\beta S(t)I(t)}{1 + \alpha I(t)} - \mu S(t) + \varepsilon R(t) \right] dt - \sigma \\ &\quad \cdot \frac{S(t)I(t)}{1 + \alpha I(t)} dB(t), \\ dI(t) &= \left[\frac{\beta S(t)I(t)}{1 + \alpha I(t)} + qbI(t) - (\gamma + a + \mu)I(t) \right] dt \\ &\quad + \sigma \frac{S(t)I(t)}{1 + \alpha I(t)} dB(t), \\ dR(t) &= (\gamma I(t) - (\mu + \varepsilon)R(t)) dt. \end{aligned} \quad (4)$$

Throughout this paper, unless otherwise specified, let $(\Omega, \{\mathbf{F}_t\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{\mathbf{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\{\mathbf{F}_t\}_{t \geq 0}$ contains all P -null sets) and let $B(t)$ be the Brownian motion defined on the probability space.

For simplicity, define

$$\langle x(t) \rangle = \frac{1}{t} \int_0^t x(r) dr. \quad (5)$$

2. Existence and Uniqueness of the Nonnegative Solution

In this section, we will show that there is a unique positive solution of system (4).

Theorem 1. *For any initial value $(S(0), I(0), R(0))$ in R_+^3 , there is a unique solution $(S(t), I(t), R(t))$ of system (4) on $t \geq 0$, and the solution will remain in R_+^3 with probability 1.*

Proof. Since the coefficients of system (4) are locally Lipschitz continuous, for any initial value $(S(0), I(0), R(0)) \in R_+^3$, there is a unique local solution on $[0, \tau_e)$, where τ_e is the explosion time. To show that this solution is global, we need to have $\tau_e = \infty$ a.s. To show that this solution is global, we need to have $\tau_e = \infty$ a.s. Let $m_0 \geq 0$ be sufficiently large so that $(S(0), I(0), R(0))$ all lie in the interval $[1/m_0, m_0]$. For each integer $m \geq m_0$, define the stopping time

$$\begin{aligned} \tau_m &= \inf \left\{ t \in [0, \tau_e) : \min \{S(t), I(t), R(t)\} \right. \\ &\quad \left. \leq \frac{1}{m} \text{ or } \max \{S(t), I(t), R(t)\} \geq 1 \right\}, \end{aligned} \quad (6)$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_m is increasing as $m \rightarrow \infty$. Set $\tau_\infty = \lim_{m \rightarrow \infty} \tau_m$, whence $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_\infty = \infty$ and $(S(0), I(0), R(0)) \in R_+^3$ a.s. for all $t \geq 0$. In other words, to complete the proof all we need to show is that $\tau_\infty = \infty$ a.s. If this statement is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \epsilon$.

Hence there is an integer $m_1 \geq m_0$ such that

$$P\{\tau_\infty \leq T\} > \epsilon, \quad \forall m \geq m_1. \quad (7)$$

Besides, the total biomass $N(t) = S(t) + I(t) + R(t)$ of model (4) satisfies the following equation:

$$dN(t) = [\Lambda - (\mu - b)N - aI] dt. \quad (8)$$

It is easy to know that, for all $t < \tau_\infty$,

$$N(t) \leq \max \left\{ S_0, I_0, R_0, \frac{\Lambda}{\mu - b} \right\} := K_1. \quad (9)$$

Define a C^2 -function $V : \mathbb{R}_+^3 \rightarrow \overline{\mathbb{R}}_+$ by

$$V(S, I, R) = -\log \frac{S}{K_1} - \log \frac{I}{K_1} - \log \frac{R}{K_1}. \quad (10)$$

By using Itô's formula, we get

$$dV = LV dt + \frac{\sigma(S - I)}{1 + \alpha I}, \quad (11)$$

where

$$\begin{aligned} LV &= -\frac{\Lambda}{S} - b \left(1 + \frac{R}{S} \right) - (1 - q)b \frac{I}{S} + \frac{\beta I}{1 + \alpha I} + \mu \\ &\quad - \varepsilon \frac{R}{S} - \frac{\beta S}{1 + \alpha I} + (\gamma + a + \mu - qb) - \frac{\gamma I}{R} \\ &\quad + (\mu + \varepsilon) + \frac{\sigma^2(I^2 + S^2)}{2(1 + \alpha I)^2} \end{aligned} \quad (12)$$

$$\begin{aligned} &\leq \frac{\beta I}{1 + \alpha I} + (3\mu + \gamma + a + \varepsilon - qb) + \frac{\sigma^2(I^2 + S^2)}{2(1 + \alpha I)^2} \\ &\leq \beta K_1 + (3\mu + \gamma + a + \varepsilon - qb) + K_1^2 \sigma^2 := K_2. \end{aligned}$$

The remainder of the proof follows that in Li and Mao [21, Theorem 2.1]. $\square$

Since there is a positive solution $(S(t), I(t), R(t)) \geq 0$ of system (4) for any given initial value $(S(0), I(0), R(0)) \in R_+^3$ and

$$\Gamma = \left\{ (S(t), I(t), R(t)) \in R_+^3 : S(t) + I(t) + R(t) \leq \frac{\Lambda}{\mu - b}, t \geq 0 \text{ a.s.} \right\} \quad (13)$$

is an invariant set [22], then from now on, we can assume that $(S(0), I(0), R(0)) \in \Gamma$.

3. Extinction

In this section, we discuss the conditions for the extinction of the disease.

Theorem 2. Let $(S(t), I(t), R(t))$ be the solution of system (4) with initial value $(S(0), I(0), R(0)) \in \Gamma$. If $\bar{R}_0 < 1$ and $\sigma^2 \leq \beta(\mu - b)/\Lambda$ then

$$\limsup_{t \rightarrow +\infty} \frac{\log I(t)}{t} \leq (\mu + a + \gamma - qb) (\bar{R}_0 - 1) < 0. \quad (14)$$

Namely, $I(t)$ tends to zero exponentially a.s. where

$$\bar{R}_0 = R_0 - \frac{\sigma^2 \Lambda^2}{2(\mu - b)^2(\mu + a + \gamma - qb)}. \quad (15)$$

Proof. Notice that $S/(1 + \alpha I) \in (0, \Lambda/(\mu - b))$; that is, $\sigma^2 \leq \beta(\mu - b)/\Lambda$ the quadratic function

$$f(z) = -\frac{\sigma^2}{2} z^2 + \beta z - (\mu + a + \gamma - qb) \quad (16)$$

gets its maximum value $f_{\max}$ on the interval $[0, \Lambda/(\mu - b)]$ at $z = \Lambda/(\mu - b)$, where

$$\begin{aligned} f_{\max} &= -\frac{\sigma^2}{2} \left(\frac{\Lambda}{\mu - b} \right)^2 + \beta \frac{\Lambda}{\mu - b} - (\mu + a + \gamma - qb) \\ &= (\mu + \gamma + a - qb) (\bar{R}_0 - 1). \end{aligned} \quad (17)$$

It follows from the monotonicity of the function $\beta S/(1 + \alpha I)$ on $[0, \Lambda/(\mu - b)]$ that when $\sigma^2 \leq \beta(\mu - b)/\Lambda$, we get

$$\begin{aligned} & -\frac{\sigma^2}{2} \left(\frac{S}{1 + \alpha I} \right)^2 + \beta \frac{S}{1 + \alpha I} - (\mu + a + \gamma - qb) \\ & \leq (\mu + \gamma + a - qb) (\bar{R}_0 - 1). \end{aligned} \quad (18)$$

Applying Itô's formula to system (4) leads to

$$\begin{aligned} d(\log I) &= \left[-\frac{\sigma^2}{2} \left(\frac{S}{(1 + \alpha I)} \right)^2 + \beta \frac{S}{1 + \alpha I} \right. \\ & \quad \left. - (\mu + a + \gamma - qb) \right] dt + \frac{\sigma S}{1 + \alpha I} dB(t). \end{aligned} \quad (19)$$

Integrating this from 0 to t and diving t on both sides, we have

$$\begin{aligned} \frac{\log I(t) - \log I(0)}{t} &\leq (\mu + \gamma + a - qb) (\bar{R}_0 - 1) \\ &\quad + \frac{M(t)}{t}, \end{aligned} \quad (20)$$

where $M(t) := \int_0^t (\sigma S(r)/(1 + \alpha I(r))) dB(r)$, which is local martingale and $M(0) = 0$. Moreover,

$$\limsup_{t \rightarrow +\infty} \frac{\langle M, M \rangle_t}{t} \leq \frac{\sigma^2 \Lambda^2}{(\mu - b)^2} < \infty \text{ a.s.} \quad (21)$$

According to the large number theorem for martingales (see, e.g., [13]), we obtain

$$\lim_{t \rightarrow +\infty} \frac{M(t)}{t} = 0. \quad (22)$$

Then

$$\limsup_{t \rightarrow +\infty} \frac{\log I(t)}{t} \leq (\mu + \gamma + a - qb) (\bar{R}_0 - 1) < 0 \text{ a.s.} \quad (23)$$

The proof of Theorem 2 is completed. $\square$

Remark 3. We also notice that if $\beta/\sigma^2 \in (0, \Lambda/(\mu - b))$, that is, $\sigma^2 \geq \beta(\mu - b)/\Lambda$, the quadratic function $f(z)$ obtains the maximum value $\beta^2/2\sigma^2 - (\mu + \gamma + a - qb)$ at $z = \beta/\sigma^2$. The proof of Theorem 2 proceeded; we have

$$\limsup_{t \rightarrow +\infty} \frac{\log I(t)}{t} \leq \frac{\beta^2}{2\sigma^2} - (\mu + \gamma + a - qb) < 0 \text{ a.s.} \quad (24)$$

if $\sigma^2 > \beta^2/2(\mu + a + \gamma - qb)$. It means that the disease will always die out if the noise is large enough such that $\sigma^2 > \max\{\beta(\mu - b)/\Lambda, \beta^2/2(\mu + a + \gamma - qb)\}$.

Remark 4. From Theorem 2, we can get that the disease will die out if $\bar{R}_0 < 1$ and the white noise is not large such that $\sigma^2 \leq \beta(\mu - b)/\Lambda$. Meanwhile if white noise is large enough such that $\sigma^2 > \max\{\beta(\mu - b)/\Lambda, \beta^2/2(\mu + a + \gamma - qb)\}$ is satisfied, then the disease will also die out. Moreover, we notice $\bar{R}_0$ is smaller than the threshold of the corresponding deterministic model. The following examples illustrate this result more explicitly.

From Theorem 2, it is obvious that under some conditions

$$\limsup_{t \rightarrow +\infty} \frac{\log I(t)}{t} < 0 \text{ a.s.} \quad (25)$$

which implies

$$\lim_{t \rightarrow \infty} I(t) = 0 \text{ a.s.} \quad (26)$$

Next, according to system (4), we get

$$\begin{aligned} d(S(t) + I(t) + R(t)) \\ = [\Lambda - (\mu - b)(S(t) + I(t) + R(t)) - aI] dt \\ \geq [\Lambda - (\mu - b)(S(t) + I(t) + R(t))] dt \end{aligned} \quad (27)$$

which implies that

$$\liminf_{t \rightarrow +\infty} (S(t) + I(t) + R(t)) \geq \frac{\Lambda}{\mu - b} \text{ a.s.} \quad (28)$$

Together with (13), we have

$$\lim_{t \rightarrow +\infty} (S(t) + R(t)) = \frac{\Lambda}{\mu - b} \text{ a.s.} \quad (29)$$

According to (26) and the last equation of system (4), we obtain

$$\lim_{t \rightarrow +\infty} R(t) = 0 \text{ a.s.} \quad (30)$$

Therefore, by (29), we have

$$\lim_{t \rightarrow +\infty} S(t) = \frac{\Lambda}{\mu - b} \text{ a.s.} \quad (31)$$

We have the following theorem by combing these arguments.

Theorem 5. Let $(S(t), I(t), R(t))$ be the solution of system (4) with initial value $(S(0), I(0), R(0)) \in \Gamma$. If

- (i) $\widetilde{R}_0 < 1$ and $\sigma^2 \leq \beta(\mu - b)/\Lambda$ or
- (ii) $\sigma^2 > \max\{\beta(\mu - b)/\Lambda, \beta^2/2(\mu + a + \gamma - qb)\}$

then

$$\begin{aligned} \lim_{t \rightarrow +\infty} S(t) &= \frac{\Lambda}{\mu - b}, \\ \lim_{t \rightarrow +\infty} I(t) &= 0, \\ \lim_{t \rightarrow +\infty} R(t) &= 0 \text{ a.s.} \end{aligned} \quad (32)$$

Example 6. We assume that the unit of time is one day and the population sizes are measured in units of 1 million. The parameters in system (4) are chosen as follows:

$$\begin{aligned} \Lambda &= 0.5, \\ \beta &= 0.8, \\ \mu &= 0.3, \\ b &= 0.05, \\ \alpha &= 0.2, \\ q &= 0.2, \\ \gamma &= 0.39, \\ a &= 0.21, \\ \varepsilon &= 0.4, \\ \sigma &= 0.5. \end{aligned} \quad (33)$$

Note that

$$\widetilde{R}_0 = 0.6742 < 1, \quad (34)$$

$$\sigma^2 = 0.25 \leq \frac{\beta(\mu - b)}{\Lambda} = 0.4 \quad (35)$$

and then by Theorem 5(i), the solution $(S(t), I(t), R(t))$ of system (4) obeys

$$\begin{aligned} \limsup_{t \rightarrow +\infty} \frac{\log I(t)}{t} &\leq (\mu + \gamma + a - qb)(\widetilde{R}_0 - 1) = -0.41 \\ &< 0 \text{ a.s.} \end{aligned} \quad (36)$$

$$\begin{aligned} \lim_{t \rightarrow +\infty} S(t) &= \frac{\Lambda}{\mu - b} = 2, \\ \lim_{t \rightarrow +\infty} R(t) &= 0 \text{ a.s.} \end{aligned}$$

with any initial value $(S(0), I(0), R(0)) = (0.6, 0.4, 0.5) \in \Gamma$. Then the disease tends to zero exponentially with probability one.

On the other hand, for the responding deterministic SIRS model, $R_0 = 1.7978 > 1$; the disease will prevail. Using the method mentioned in [23], we provide the simulations shown in Figure 1 to support our results.

Example 7. We choose the same values of the parameters of (33) except for the value of σ . Because $\sigma^2 > \beta(\mu - b)/\Lambda = 0.4$, $\sigma^2 > \beta^2/2(\mu + \gamma + a - qb) = 0.3595$, we let $\sigma = 0.7$; then by Theorem 5(ii), the solution of system (4) obeys $\limsup_{t \rightarrow +\infty} (\log I(t)/t) \leq -(\mu + \gamma + a - qb) + \beta^2/2\sigma^2 = -0.2369 < 0$ a.s., in which case the disease will also die out. Note that $R_0 = 1.2584 > 1$, so that the disease for the corresponding system will prevail too (see Figure 2).

4. Persistence

Definition 8. System (4) is said to be persistent in the mean if

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t I(r) dr > 0 \text{ a.s.} \quad (37)$$

Theorem 9. If $\widetilde{R}_0 > 1$, then for any initial value $(S(0), I(0), R(0)) \in \Gamma$, the solution $(S(t), I(t), R(t))$ of system (4) has the following property:

FIGURE 1: Simulations of the path $(S(t), I(t), R(t))$ for the corresponding deterministic system (2) and stochastic system (4) with initial value $(S(0), I(0), R(0)) = (0.6, 0.4, 0.5)$, $\sigma = 0.5$.

FIGURE 2: Simulations of the path $S(t), I(t), R(t)$ for the corresponding deterministic system (2) and stochastic system (4) with initial value $(S(0), I(0), R(0)) = (0.6, 0.4, 0.5)$, $\sigma = 0.7$.

$$\liminf_{t \rightarrow +\infty} \langle I(t) \rangle \geq \frac{(\mu + a + \gamma - qb)(\bar{R}_0 - 1)}{\beta(((\mu + \varepsilon)(\mu + a + \gamma - qb) - \gamma(b + \varepsilon)) / (\mu + \varepsilon)(\mu - b)) + (\mu + a + \gamma - qb)} := I_* \text{ a.s.} \quad (38)$$

Proof. An integration of system (4) yields

$$\begin{aligned} \frac{S(t) - S(0)}{t} &= \Lambda + b \langle S(t) \rangle + b \langle R(t) \rangle \\ &\quad + (1 - q)b \langle I(t) \rangle \\ &\quad - \beta \left\langle \frac{S(t)I(t)}{1 + \alpha I(t)} \right\rangle - \mu \langle S(t) \rangle \\ &\quad + \varepsilon \langle R(t) \rangle \end{aligned}$$

$$\begin{aligned} \frac{I(t) - I(0)}{t} &= \beta \left\langle \frac{S(t)I(t)}{1 + \alpha I(t)} \right\rangle \\ &\quad - (\mu + a + \gamma - qb) \langle I(t) \rangle \\ &\quad + \frac{\sigma}{t} \int_0^t \frac{S(r)I(r)}{1 + \alpha I(r)} dB(r), \\ \frac{R(t) - R(0)}{t} &= \gamma \langle I(t) \rangle - (\mu + \varepsilon) \langle R(t) \rangle. \end{aligned}$$

(39)

According to (39), we have

$$\begin{aligned} & \frac{1}{b+\varepsilon} \left(\frac{S(t)-S(0)}{t} + \frac{I(t)-I(0)}{t} \right) \\ & + \frac{1}{\mu+\varepsilon} \frac{R(t)-R(0)}{t} \\ & = \frac{\Lambda}{b+\varepsilon} - \frac{\mu-b}{b+\varepsilon} \langle S(t) \rangle \\ & + \left(\frac{\gamma}{\mu+\varepsilon} - \frac{\mu+\gamma+a-b}{b+\varepsilon} \right) \langle I(t) \rangle. \end{aligned} \quad (40)$$

After simple computing

$$\begin{aligned} & \langle S(t) \rangle \\ & = \frac{\Lambda}{\mu-b} \\ & - \frac{(\mu+\varepsilon)(\mu+a+\gamma-qb)-\gamma(b+\varepsilon)}{(\mu+\varepsilon)(\mu-b)} \langle I(t) \rangle \\ & + \varphi(t), \end{aligned} \quad (41)$$

where $\varphi(t) = -((b+\varepsilon)/(\mu-b))[(1/(b+\varepsilon))((S(t)-S(0))/t + (I(t)-I(0))/t) + (1/(\mu+\varepsilon))((R(t)-R(0))/t)]$ and $\lim_{t \rightarrow +\infty} \varphi(t) = 0$ a.s.

Let $X = ((\mu+\varepsilon)(\mu+a+\gamma-qb)-\gamma(b+\varepsilon))/(\mu+\varepsilon)(\mu-b)$. By Itô's formula, we obtain

$$\begin{aligned} d(\log I + \alpha I) & = \left[\beta S - (\mu+a+\gamma-qb) \right. \\ & \left. - \frac{\sigma^2 S^2}{2(1+\alpha I)^2} - (\mu+a+\gamma-qb) I \right] dt \\ & + \sigma S dB(t) \geq \left[\beta S - (\mu+a+\gamma-qb) - \frac{\sigma^2 \Lambda^2}{2(\mu-b)^2} \right. \\ & \left. - (\mu+a+\gamma-qb) I \right] dt + \sigma S dB(t) \end{aligned} \quad (42)$$

and then

$$\begin{aligned} & \frac{\log I(t) - \log I(0)}{t} \geq \left[\beta \langle S(t) \rangle - (\mu+a+\gamma-qb) \right. \\ & \left. - \frac{\sigma^2 \Lambda^2}{2(\mu-b)^2} - (\mu+a+\gamma-qb) \langle I(t) \rangle \right] dt + \frac{\sigma}{t} \\ & \cdot \int_0^t S(r) dB(r) - \alpha \frac{I(t) - I(0)}{t} \geq \beta \frac{\Lambda}{\mu-b} \end{aligned}$$

$$\begin{aligned} & + \beta \varphi(t) - (\mu+a+\gamma-qb) - [\beta X \\ & + (\mu+a+\gamma-qb)] \langle I(t) \rangle - \frac{\sigma^2 \Lambda^2}{2(\mu-b)^2} dt + \frac{\sigma}{t} \\ & \cdot \int_0^t S(r) dB(r) - \alpha \frac{I(t) - I(0)}{t} = (\mu+a \\ & + \gamma-qb)(\widetilde{R}_0 - 1) + \beta \varphi(t) - [\beta X \\ & + (\mu+a+\gamma-qb)] \langle I(t) \rangle dt + \frac{\sigma}{t} \int_0^t S(r) dB(r) \\ & - \alpha \frac{I(t) - I(0)}{t}. \end{aligned} \quad (43)$$

Then the inequality can be rewritten as

$$\begin{aligned} \langle I(t) \rangle & \geq \frac{1}{\beta X + (\mu+a+\gamma-qb)} \left[(\mu+a+\gamma-qb) \right. \\ & \cdot (\widetilde{R}_0 - 1) + \beta \varphi(t) - \frac{\log I(t) - \log I(0)}{t} - \alpha \\ & \cdot \left. \frac{I(t) - I(0)}{t} + \frac{M(t)}{t} \right]. \end{aligned} \quad (44)$$

By (29), then taking the limit inferior of both sides (44) leads to

$$\liminf_{t \rightarrow +\infty} \langle I(t) \rangle \geq \frac{(\mu+a+\gamma-qb)(\widetilde{R}_0 - 1)}{\beta X + (\mu+a+\gamma-qb)} := I_*. \quad (45)$$

This completes the proof of Theorem 9. $\square$

Example 10. Choose the parameters in system (4) as follows:

$$\begin{aligned} \Lambda & = 0.5, \\ \beta & = 0.8, \\ \mu & = 0.3, \\ b & = 0.05, \\ \alpha & = 0.2, \\ q & = 0.2, \\ \gamma & = 0.39, \\ a & = 0.21, \\ \varepsilon & = 0.4, \\ \sigma & = 0.2. \end{aligned} \quad (46)$$

Note that

$$\begin{aligned} \widetilde{R}_0 & = 1.7078 > 1, \\ \sigma^2 & = 0.04 \leq \frac{\beta(\mu-b)}{\Lambda} = 0.4. \end{aligned} \quad (47)$$

FIGURE 3: Simulations of the path $S(t)$, $I(t)$, $R(t)$ for the corresponding deterministic system (2) and stochastic system (4) with initial value $(S(0), I(0), R(0)) = (0.6, 0.4, 0.5)$, $\sigma = 0.2$.

FIGURE 4: Simulations of the path $S(t)$, $I(t)$, $R(t)$ for the corresponding deterministic system (2) and stochastic system (4) with initial value $(S(0), I(0), R(0)) = (0.6, 0.4, 0.5)$, $\sigma = 0.06$.

and then by Theorems 2, with any initial value $(S(0), I(0), R(0)) = (0.6, 0.4, 0.5) \in \Gamma$, we get that the solution of system (4) obeys

$$\liminf_{t \rightarrow +\infty} \langle I(t) \rangle \leq 0.2146 \text{ a.s.} \quad (48)$$

Using the method mentioned in [23], we give the simulations shown in Figure 3, and we find the disease is persistent.

Example 11. We consider the same values of the parameters of (46) except σ . Change σ to 0.06 to further illustrate the effect of the noise intensity σ on this SIRS model. Notice that $\bar{R}_0 = 1.7816 > 1$ and $\sigma^2 = (0.06)^2 \leq \beta(\mu - b)/\Lambda = 0.4$. Using the method mentioned in [14], we give the simulations shown in Figure 4 to support our results. And we find the fluctuation of the solution of system (4) is becoming weaker compared with the picture in Figure 3.

5. Conclusion

In this paper, we study a stochastic SIRS model with vertical transmission and saturated incidence. We find that the

extinction or persistence of the disease is mainly determined by the value of $\bar{R}_0$, and it is obvious that $\bar{R}_0 < R_0$ when $\sigma \neq 0$. We find when the noise is so small (in fact $\sigma^2 < \beta(\mu - b)/\Lambda$), the extinction and persistence are completely determined by $\bar{R}_0$: if $\bar{R}_0 < 1$, the disease dies out; if $\bar{R}_0 > 1$, the disease prevails. So the noise is adverse to the survival of the disease. We also find that if the noise is so large that $\sigma^2 > \max\{\beta(\mu - b)/\Lambda, \beta^2/2(\mu + a + \gamma - qb)\}$, the disease prevails too.

To understand how the noise effects the survival of the disease better, the conditions in Theorem 2 can be written as

$$\begin{aligned} \sigma &> \frac{1}{\Lambda} \sqrt{2(\mu - b)[\beta\Lambda - (\mu - b)(\mu + a + \gamma - qb)]} \\ &:= \tilde{\sigma}, \\ \sigma &\leq \frac{1}{\Lambda} \sqrt{\beta\Lambda(\mu - b)}, \end{aligned} \quad (49)$$

and rewrite the condition $\bar{R}_0 > 1$ in Theorem 5 as follows: $\sigma < \tilde{\sigma}$.

We obtain the main results as follows:

- (a) If $R_0 < 1$, the disease will always die out without considering the existence of the noise according to (15).
- (b) If $R_0 > 1$, the disease will persist for the deterministic model (2) while for the stochastic model (4), the disease will persist when $\sigma < \tilde{\sigma}$ and die out when $\tilde{\sigma} < \sigma \leq (1/\Lambda)\sqrt{\beta\Lambda(\mu-b)}$. The existence of noise determines the extinction of the disease.
- (c) If $\tilde{\sigma} > \max\{(1/\Lambda)\sqrt{\beta\Lambda(\mu-b)}, (\beta/2(\mu+a+\gamma-qb))\sqrt{2(\mu+a+\gamma-qb)}\}$, the disease will always die out.

From (b) we get that when $R_0 > 1$, $\tilde{\sigma} = \sigma$ is the threshold which determines the extinction and persistence of the disease for stochastic system (4). Meanwhile, we can obtain from cases (b) and (c) that if

$$\max \left\{ \frac{1}{\Lambda} \sqrt{\beta\Lambda(\mu-b)}, \frac{\beta}{2(\mu+a+\gamma-qb)} \cdot \sqrt{2(\mu+a+\gamma-qb)} \right\} = \frac{1}{\Lambda} \sqrt{\beta\Lambda(\mu-b)}, \quad (50)$$

then disease will die out when $\sigma > \tilde{\sigma}$; meanwhile if

$$\max \left\{ \frac{1}{\Lambda} \sqrt{\beta\Lambda(\mu-b)}, \frac{\beta}{2(\mu+a+\gamma-qb)} \cdot \sqrt{2(\mu+a+\gamma-qb)} \right\} = \frac{\beta}{2(\mu+a+\gamma-qb)} \cdot \sqrt{2(\mu+a+\gamma-qb)}, \quad (51)$$

the disease will die out only for $\sigma \in (\tilde{\sigma}, (1/\Lambda)\sqrt{\beta\Lambda(\mu-b)}) \cup ((\beta/2(\mu+a+\gamma-qb))\sqrt{2(\mu+a+\gamma-qb)}, \infty)$.

Competing Interests

The authors declare that they have no competing interests.

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