Research Article

Dynamics of Endomorphism of Finite Abelian Groups

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We study the dynamics of endomorphisms on a finite abelian group. We obtain the automorphism group for these dynamical systems. We also give criteria and algorithms to determine whether it is a fixed point system.

1. Introduction

A discrete dynamical system is consisting of a pair (X, f) , where X is a nonempty set and $f : X \rightarrow X$ is a function. The functional graph of (X, f) is a directed graph with vertex set X and an arrow from x to y if and only if $f(x) = y$. We denote the functional graph of (X, f) by $G(X, f)$.

Finite discrete dynamical systems appeared in engineering, control systems computer science, and genetic networks; see [1, 2]. One of the most important examples of discrete dynamical systems is given by $(\mathbb{F}_q^n, f)$, where $\mathbb{F}_q^n$ is the vector space over a finite field $\mathbb{F}_q$ and f is a linear transformation. The structure these dynamical systems can be described by the elementary divisors of the corresponding matrix; see [3, 4]. By using the same method, Brown and Vaughan [5] studied the cycle structure of linear dynamical system over $\mathbb{Z}_n$. Bach and Bridy [6] gave an upper and a lower bound for the distinct number of functional graph defined by affine mapping over finite dimensional linear space over finite fields. But the study of linear dynamical systems over a general finite commutative ring R has difficulties since the factorization of polynomials in $R[x]$ is not unique. Brown and Vaughan [5] showed that R^n can be decompose into $R^n = \ker f^k \oplus \text{Im } f^k$ by Fitting's Lemma and (R^n, f) is the product of the dynamical systems induced on $\ker f^k$ and $\text{Im } f^k$. They also provided an algorithm to compute the minimal positive integer k in the above decomposition, but the algorithm is not efficient. Xu and Zou [7] obtained an efficient algorithm to compute k under a reasonable assumption on the size of

the ring R . Deng [8] gave a criterion to determine whether a linear dynamical system is over the ring of integer modulo m .

Let H be a finite abelian group written additively and $\text{End}(H)$ be the endomorphism ring of H . In this paper we are interested in the dynamical systems (H, f) defined by $f \in \text{End}(H)$. These dynamical systems have nice structure and were studied in many special cases; see [9]. In this paper we are interested in the case in which X is a finite abelian group H written additive, and $f : H \rightarrow H$ is an endomorphism of H .

The paper is organized as follows. In Section 2, we give some notations and basic results on the structure of functional graphs $G(H, f)$. In Section 3, we obtain necessary and sufficient conditions for $G(H_1, f_1)$ and $G(H_2, f_2)$. We also give an explicit form of the automorphism group of $G(H, f)$. In Section 4, we obtain a criterion to determine whether $G(H, f)$ is a fixed point system.

2. Elementary Properties of $G(H, f)$

A *component* of a digraph is a maximal connected subgraph of the associated nondirected graph. The *indegree* of a vertex a of $G(X, f)$, denoted by $\text{indeg}_X^f(a)$, is the number of directed edges coming into a . The *outdegree* $\text{outdeg}_X^f(a)$ of a is the number of directed edges leaving the vertex a . It is clear that the outdegree of any vertex in $G(X, f)$ is equal to 1, and the indegree of a vertex a is the number of vertices coming to a . A vertex x in $G(X, f)$ is called a cycle vertex if

$f^k(x) = x$ for some positive integers k . In particular, a cycle vertex x is called a fixed point of $G(X, f)$ if $f(x) = x$. Let $A_t(X, f)$ denote the number of t -cycles contained in $G(X, f)$. Let $\mathcal{A}(X, f)$ denote the set of positive integers which is the length of the cycle which appeared in $G(X, f)$. We need the following results about $A_t(X, f)$ and $\mathcal{A}(X, f)$. In particular, let $d(X, f) = |\{x \in X : f(x) = x\}|$ be the number of fixed point in (X, f) .

Definition 1. We define a height function on the vertices and components of $G(X, f)$. Let a be a vertex of $G(X, f)$. Then we define $h(a)$ to be the minimal nonnegative integer i such that $f^i(a)$ is a cycle vertex in $G(X, f)$. We set $h(D) = \sup_{a \in D} h(a)$ if D is a component of $G(X, f)$.

Definition 2. For any integer $i \geq 0$ and any vertex $a \in G(X, f)$, define

$$\begin{aligned} \mathcal{F}^0(a) &= \{a\}, \\ \mathcal{F}^i(a) &= \{b \mid f^i(b) = a, f^{i-1}(b) \text{ is not a cycle vertex if } i > 0\}. \end{aligned} \quad (1)$$

Let $T(a)$ be the subdigraph of $G(X, f)$ with the vertices set $\bigcup_{i=0}^{\infty} \mathcal{F}^i(a)$, and there is a directed arrow from x to y in $T(a)$ if $f(x) = y$ and $h(x) = h(y) + 1$. Hence, $T(a)$ is a directed tree with root a .

Let H be a finite abelian group and $f \in \text{End}(H)$. The basic properties of $G(H, f)$ in this section are proved in some special cases; see [10–14]. We give some basic properties of these digraphs. The proof is similar, but we still present it.

Lemma 3. Let $f \in \text{End}(H)$. Then

- (1) the indegree of any vertex in $G(H, f)$ is 0 or $|\ker f|$.
- (2) the number of the vertices with indegree $|\ker f|$ is $|H|/|\ker f|$.

Proof. Let x be a vertex with positive indegree. Then $\{y : f(y) = x\}$ is a coset of $\ker f$ and they have the same number of elements. The statement (2) follows immediately from (1). $\square$

We present two results on the tree structure attached to a vertex.

Lemma 4. Let c be a cycle vertex of $G(H, f)$. Then $T(c) \simeq T(0)$.

Proof. For any $h > 0$ there exists a unique cycle vertex c_h such that $f^h(c_h) = c$ since c is a cycle vertex. Now if $x \in \mathcal{F}^h(0)$, we define a map $\varphi_h : \mathcal{F}^h(0) \rightarrow \mathcal{F}^h(c)$ as follows:

$$\varphi_h(x) = x + c_h. \quad (2)$$

Then $f^h(x + c_h) = f^h(x) + f^h(c_h) = 0 + c = c$. We observe that $f^{h-1}(x)$ is not a cycle vertex, but $f^{h-1}(c_h)$ is a cycle vertex; then $f^{h-1}(x + c_h)$ is not a cycle vertex since the set of all cycle

vertices of $G(H, f)$ is a subgroup of H . The map φ_h is well-defined and clearly injective. If $b \in \mathcal{F}^h(c)$, then $b - c_h \in \mathcal{F}^h(0)$ and $\varphi_h(b - c_h) = b$. φ_h is also surjective.

Now if $a \in \mathcal{F}^h(0)$ and $b \in \mathcal{F}^{h-1}(0)$ with $f(a) = b$, then

$$\begin{aligned} f(\varphi_h(a)) &= f(a + c_h) = f(a) + f(c_h) = b + c_{h-1} \\ &= \varphi_{h-1}(b). \end{aligned} \quad (3)$$

Finally we set $\varphi : T(0) \rightarrow T(c)$:

$$\varphi(x) = \varphi_i(x), \quad (4)$$

if $x \in \mathcal{F}^i(0)$. By (3) φ is an isomorphism. $\square$

Corollary 5. Let D_1 and D_2 be two components of $G(H, f)$. Then $D_1 \simeq D_2$ if and only if they have the same cycle length.

Proof. It follows immediately from Lemma 4. $\square$

Lemma 6. Let H_1, H_2 be two finite abelian groups and $f_i \in \text{End}(H_i)$. Let $c_i \in G(H_i, f_i)$ with positive height. Then $T(c_1) \simeq T(c_2)$ if and only if $|\mathcal{F}^j(c_1)| = |\mathcal{F}^j(c_2)|$ for any $j \geq 0$.

Proof. Let $h_i = h(T(c_i))$. First we assume that $T(c_1) \simeq T(c_2)$ and let φ be an isomorphism. Since the length of the longest directed path in $T(c_i)$ is h_i and this path ends at c_i , it follows that $\varphi(c_1) = c_2$. Thus, $\varphi(\mathcal{F}^j(c_1)) \subseteq \mathcal{F}^j(c_2)$ and $|\mathcal{F}^j(c_1)| = |\mathcal{F}^j(c_2)|$.

Conversely, suppose that $|\mathcal{F}^j(c_1)| = |\mathcal{F}^j(c_2)| = n_j$ for any $j \geq 0$. Then $h = h_1 = h_2$. We induct on h . The case $h = 0, 1$ is trivial. Note that for any vertex $a \in G(H_i, f_i)$ with positive height, by Definition 2 $|\mathcal{F}^k(a)| = |\{x \in H_i : f_i^k(x) = a\}| = \text{indeg}_{H_i}^{f_i^k}(a)$. Since f_i^k is also a group homomorphism, by Lemma 3 we have $|\mathcal{F}^k(a)| = n_k$ or 0 for any $k = 0, 1, \dots, h$. Let $\mathcal{F}^1(c_1) = \{a_1, \dots, a_{n_1}\}$ and $\mathcal{F}^1(c_2) = \{b_1, \dots, b_{n_1}\}$. Let $x_i = |\{a \in \mathcal{F}^1(c_1) : h(T(a)) = i\}|$ and $y_i = |\{b \in \mathcal{F}^1(c_2) : h(T(b)) = i\}|$. Since $\mathcal{F}^1(c_1) = \bigcup_{a \in \mathcal{F}^1(c_1)} \mathcal{F}^{k-1}(a)$ is a disjoint union for any $k > 1$ and $|\mathcal{F}^k(a)| = |\mathcal{F}^k(c_1)| = n_k$ if and only if $h(T(a)) \geq k$, one has $|\mathcal{F}^k(c_1)| = n_k = \sum_{a \in \mathcal{F}^{k-1}(c_1)} |\mathcal{F}^{k-1}(a)| = n_{k-1}(x_1 + \dots + x_{k-1})$. It follows that

$$\begin{aligned} x_0 + x_1 + \dots + x_{h-1} &= n_1 \\ n_1(x_1 + \dots + x_{h-1}) &= n_2 \\ &\vdots \\ n_{h-1}x_{h-1} &= n_h. \end{aligned} \quad (5)$$

We derive that $x_i = n_{i+1}/n_i - n_{i+2}/n_{i+1}$ for $i \in \{0, 1, \dots, h-2\}$ and $x_{h-2} = n_h/n_{h-1}$. We obtain $y_i = x_i$ by replacing c_1 by c_2 in the above arguments. After reordering we may assume that $h(T(a_i)) = h(T(b_i))$ for any i . Then $|\mathcal{F}^j(a_i)| = |\mathcal{F}^j(b_i)|$ for any fixed i and j . By induction, there exists digraph isomorphism $\varphi_i : T(a_i) \rightarrow T(b_i)$. We set $\varphi : T(c_1) \rightarrow T(c_2)$, where $\varphi(c_1) = c_2$ and $\varphi(z) = \varphi_i(z)$ if $z \in T(a_i)$. Then φ is also a digraph isomorphism. This finishes the proof. $\square$

3. Automorphism Group of $G(H, f)$

We first show when $G(H, f_1)$ is isomorphic to $G(H, f_2)$ for $f_1, f_2 \in \text{End}(H)$.

Lemma 7. *Let H_1, H_2 be two finite abelian groups and $f_i \in \text{End}(H_i)$. Let D_i be the component of $G(H_i, f_i)$ containing the vertex 0. Then $G(H_1, f_1) \simeq G(H_2, f_2)$ if and only if*

- (1) $D_1 \simeq D_2$;
- (2) $A_t(H_1, f_1) = A_t(H_2, f_2)$ for any t .

Proof. It follows immediately from Lemma 4 and Corollary 5. $\square$

Theorem 8. *Let H_1, H_2 be two finite abelian groups and $f_i \in \text{End}(H_i)$. Then $G(H_1, f_1) \simeq G(H_2, f_2)$ if and only if for any positive integer k*

- (1) $|\ker f_1^k| = |\ker f_2^k|$;
- (2) $|\ker (id - f_1^k)| = |\ker (id - f_2^k)|$.

Proof. Let D_i be the component of $G(H_i, f_i)$ containing the vertex 0. Since $x \in \ker f_i^k$ if and only if $f_i^j(x) = 0$ for some $1 \leq j \leq k$, so $\ker f_i^k = \bigcup_{j=0}^k \mathcal{F}^j(D_i)$ is a disjoint union. It follows that $|\ker f_i^k| = \sum_{j=0}^k |\mathcal{F}^j(D_i)|$ for any $k \geq 1$. Thus, $|\ker f_1^k| = |\ker f_2^k|$ for any $k \geq 1$ if and only if $|\mathcal{F}^k(D_1)| = |\mathcal{F}^k(D_2)|$ for any $k \geq 1$. By Lemma 6, condition (1) is equivalent to (1) of Lemma 7. Note that $|\ker (id - f_i^k)| = |\{x \in H_i : f_i^k(x) = x\}| = \sum_{t|k} t A_t(H_i, f_i)$. Thus, condition (2) is equivalent to (2) of Lemma 7. The proof is finished by Lemma 7. $\square$

Let H be a finite abelian group. By the structure of finite abelian group, we have

$$H \simeq \bigoplus_p H^p, \quad (6)$$

where p runs over the set of prime divisor of $|H|$ and H^p is the unique Sylow p -subgroup of H . Moreover, if there is another factorization of $H \simeq \bigoplus F^p$ in this way, it must have $H^p \simeq F^p$ for each p .

We also recall the product of graphs. Let G_1 and G_2 be two digraphs. The digraph product $G_1 \times G_2$ is the digraph whose vertices are the set of order pairs (a_1, a_2) , where a_i is a vertex of G_i . There is a directed edge from (a_1, a_2) to (b_1, b_2) in $G_1 \times G_2$ if and only if there is a directed edge from a_1 to b_1 in G_1 and there is a directed edge from a_2 to b_2 in G_2 .

Lemma 9. *Let H be a finite abelian group and $f \in \text{End}(H)$. Suppose that $|H| = n = \prod_{i=1}^r p_i^{e_i}$ is prime factorization. Then*

$$G(H, f) \simeq G(H^{p_1}, f) \times \cdots \times G(H^{p_r}, f). \quad (7)$$

Proof. It follows immediately from the fact that $H = H^{p_1} \oplus H^{p_2} \oplus \cdots \oplus H^{p_r}$ and $f(H^p) \subseteq H^p$. $\square$

Theorem 10. *Let H_1, H_2 be two abelian groups and $f_i \in \text{End}(H_i)$. Then $G(H_1, f_1) \simeq G(H_2, f_2)$ if and only if $G(H_1^p, f_1) \simeq G(H_2^p, f_2)$ for any prime p .*

Proof. First we assume that $G(H_1^p, f_1) \simeq G(H_2^p, f_2)$. By Lemma 9, we have

$$G(H_i, f_i) \simeq G(H_i^{p_1}, f_i) \times G(H_i^{p_2}, f_i) \times \cdots \times G(H_i^{p_r}, f_i). \quad (8)$$

So we obtain $G(H_1, f_1) \simeq G(H_2, f_2)$.

Conversely, suppose that $G(H_1, f_1) \simeq G(H_2, f_2)$. Let $g_{i,k} = id - f_i^k$. By Theorem 8, $|\ker f_1^k| = |\ker f_2^k|$ and $|\ker g_{1,k}| = |\ker g_{2,k}|$ for any $k > 0$. It follows that $|\ker (f_1^k|_{H_1^p})| = |\ker (f_2^k|_{H_2^p})|$ and $|\ker (g_{1,k}|_{H_1^p})| = |\ker (g_{2,k}|_{H_2^p})|$. By Theorem 8 again, $G(H_1^p, f_1) \simeq G(H_2^p, f_2)$. The proof is finished. $\square$

Next we would determine the automorphism group of the digraph $G(H, f)$. We need the wreath product of groups; see [4, 15]. We rewrite in a simple way. Let G be a group and K be a permutation group on $\{1, \dots, n\}$; then the wreath product $G \wr K$ is generated by the direct product of n copies of G , together with the elements of K action on these n copies of G . The wreath product is associative, so we write $G_1 \wr G_2 \wr \cdots \wr G_k$ for $((G_1 \wr G_2) \wr G_3) \wr \cdots \wr G_{k-1}) \wr G_k$.

Theorem 11 (see [16]). *Let G be a graph. Suppose that the connected components of G consist of n_1 copies and $G_1, \dots, G_r$ copies of G_r , where $G_1, \dots, G_r$ are pairwise nonisomorphic. Then*

$$\text{Aut}(G) \simeq (\text{Aut}(G_1) \wr S_{n_1}) \oplus \cdots \oplus (\text{Aut}(G_r) \wr S_{n_r}), \quad (9)$$

where S_n is the symmetric group on a finite set of order n .

By the above Theorem and Corollary 5, we have

$$\text{Aut}(G(H, f)) \simeq \bigoplus_{i \in \mathcal{A}(H, f)} \text{Aut}(X_i) \wr S_{A_i(H, f)}, \quad (10)$$

where X_i is a component of $G(H, f)$ with cycle length i . Let T be the component containing 0 in $G(H, f)$. Sha [9] showed that $\text{Aut}(X_i) \simeq \text{Aut}(T) \wr C_i$, where C_i is the cyclic group of order i . Thus,

$$\text{Aut}(G(H, f)) \simeq \bigoplus_{i \in \mathcal{A}(H, f)} \text{Aut}(T) \wr C_i \wr S_{A_i(H, f)}. \quad (11)$$

But he did not obtain the structure of $\text{Aut}(T)$ in general. We would determine the automorphism group of T .

Lemma 12. *Let $c \in G(H, f)$ be a vertex with $h(c) > 0$. Suppose that $h(T(c)) = r$ and $|\ker f^i| = n_i$ for $i = 1, 2, \dots, r$. Then*

$$\begin{aligned} \text{Aut}(T(c)) &\simeq S_{n_1 - n_2/n_1 + 1} \\ &\oplus \left(\bigoplus_{1 < i_1 < i_2 < \cdots < i_r \leq r-1} S_{n_1 - n_2/n_1 + 1} \wr S_{x_{i_1}} \wr S_{x_{i_2}} \wr \cdots \wr S_{x_{i_r}} \right) \\ &\oplus \left(\bigoplus_{1 = i_1 < i_2 < \cdots < i_r \leq r-1} S_{n_1} \wr S_{x_{i_1}} \wr S_{x_{i_2}} \wr \cdots \wr S_{x_{i_r}} \right), \end{aligned} \quad (12)$$

where $x_i = n_{i+1}/n_i - n_{i+2}/n_{i+1}$.

Proof. By induction on r , let $y_i = |\{a \in \mathcal{F}^1(c) : h(T(a)) = i\}|$ for $i = 0, 1, \dots, r-1$. Then $n_j - n_{j-1} = |\mathcal{F}^j(c)| = \sum_{a \in \mathcal{F}^1(c), h(T(a)) \geq j-1} |\mathcal{F}^{j-1}(a)| = n_{j-1} \sum_{i=j-1}^{r-1} y_i$. Thus,

$$\begin{aligned} y_0 + y_1 + \dots + y_{r-1} &= n_1 \\ n_1 (y_1 + \dots + y_{r-1}) &= n_2 - n_1 \\ &\vdots \\ n_{r-1} y_{r-1} &= n_r - n_{r-1}. \end{aligned} \quad (13)$$

We derive that $y_0 = n_1 - n_2/n_1 + 1$ and $y_i = x_i$ for $i \geq 1$. Let $T_i = T(a_i)$ for $a_i \in \mathcal{F}^1(c)$ and $h(T(a_i)) = i$. By Lemma 6, we have $T(u) \simeq T(a_i)$ if $u \in \mathcal{F}^1(c)$ and $h(T(u)) = i$. By Theorem 11 and by induction we have

$$\begin{aligned} \text{Aut}(T(c)) &\simeq S_{y_0} \oplus \left(\bigoplus_{j=1}^{r-1} \text{Aut}(T_j) \wr S_{x_j} \right) \\ &\simeq S_{y_0} \bigoplus_{j=1}^{r-1} \left(\bigoplus_{1 < i_1 < i_2 < \dots < i_t \leq j-1} S_{y_0} \wr S_{x_{i_1}} \wr S_{x_{i_2}} \wr \dots \wr S_{x_{i_t}} \right) \\ &\quad \wr S_{x_j} \\ &\quad \oplus \left(\bigoplus_{1=i_1 < i_2 < \dots < i_t \leq j-1} S_{n_1} \wr S_{x_{i_1}} \wr S_{x_{i_2}} \wr \dots \wr S_{x_{i_t}} \right) \\ &\quad \wr S_{x_j} \\ &\simeq S_{y_0} \oplus \left(\bigoplus_{1 < i_1 < i_2 < \dots < i_t \leq r-1} S_{y_0} \wr S_{x_{i_1}} \wr S_{x_{i_2}} \wr \dots \wr S_{x_{i_t}} \right) \\ &\quad \oplus \left(\bigoplus_{1=i_1 < i_2 < \dots < i_t \leq r-1} S_{n_1} \wr S_{x_{i_1}} \wr S_{x_{i_2}} \wr \dots \wr S_{x_{i_t}} \right). \end{aligned} \quad (14)$$

□

Theorem 13. Suppose that $|\ker f^i| = n_i$ and $x_i = n_{i+1}/n_i - n_{i+2}/n_{i+1}$. Then

$$\begin{aligned} \text{Aut}(T(0)) &\simeq S_{n_1 - n_2/n_1} \\ &\quad \oplus \left(\bigoplus_{1 < i_1 < i_2 < \dots < i_t \leq r-1} S_{n_1 - n_2/n_1 + 1} \wr S_{x_{i_1}} \wr S_{x_{i_2}} \wr \dots \wr S_{x_{i_t}} \right) \\ &\quad \oplus \left(\bigoplus_{1=i_1 < i_2 < \dots < i_t \leq r-1} S_{n_1} \wr S_{x_{i_1}} \wr S_{x_{i_2}} \wr \dots \wr S_{x_{i_t}} \right). \end{aligned} \quad (15)$$

Proof. The proof is similar to Lemma 12. We induct on $r = h(T(0))$. Let $z_i = |\{a \in \mathcal{F}^1(c) : h(T(a)) = i\}|$ for $i =$

$0, 1, \dots, r-1$. Note that $z_0 + z_1 + \dots + z_{r-1} = n_1 - n_0 = n_1 - 1$ since 0 is a fixed point. Thus,

$$\begin{aligned} z_0 + z_1 + \dots + z_{r-1} &= n_1 - 1 \\ n_1 (z_1 + \dots + z_{r-1}) &= n_2 - n_1 \\ &\vdots \\ n_{r-1} z_{r-1} &= n_r - n_{r-1}. \end{aligned} \quad (16)$$

It follows that $z_0 = n_1 - n_2/n_1$ and $z_i = x_i$ for $i \geq 1$. Let $T_i = T(a_i)$ for $a_i \in \mathcal{F}^1(0)$ and $h(T(a_i)) = i$. By induction,

$$\begin{aligned} \text{Aut}(T(c)) &\simeq S_{z_0} \oplus \left(\bigoplus_{j=1}^{r-1} \text{Aut}(T_j) \wr S_{x_j} \right) \simeq S_{z_0} \\ &\quad \cdot \left(\bigoplus_{j=1}^{r-1} \left(\bigoplus_{1 < i_1 < i_2 < \dots < i_t \leq j-1} S_{y_0} \wr S_{x_{i_1}} \wr S_{x_{i_2}} \wr \dots \wr S_{x_{i_t}} \right) \right. \\ &\quad \left. \wr S_{x_j} \oplus \left(\bigoplus_{1=i_1 < i_2 < \dots < i_t \leq j-1} S_{n_1} \wr S_{x_{i_1}} \wr S_{x_{i_2}} \wr \dots \wr S_{x_{i_t}} \right) \right) \\ &\quad \wr S_{x_j} \simeq S_{n_1 - n_2/n_1} \\ &\quad \oplus \left(\bigoplus_{1 < i_1 < i_2 < \dots < i_t \leq r-1} S_{n_1 - n_2/n_1 + 1} \wr S_{x_{i_1}} \wr S_{x_{i_2}} \wr \dots \wr S_{x_{i_t}} \right) \\ &\quad \oplus \left(\bigoplus_{1=i_1 < i_2 < \dots < i_t \leq r-1} S_{n_1} \wr S_{x_{i_1}} \wr S_{x_{i_2}} \wr \dots \wr S_{x_{i_t}} \right). \end{aligned} \quad (17)$$

□

4. Fixed Point Systems

A dynamical system is called a fixed point system if its cycle vertices are all fixed points. In this section we would describe when $G(H, f)$ is a fixed point system.

Definition 14. Let (X_1, f_1) and (X_2, f_2) be two dynamical systems. A morphism $\varphi : (X_1, f_1) \rightarrow (X_2, f_2)$ is a map $\varphi : X_1 \rightarrow X_2$ such that $f_2 \varphi = \varphi f_1$.

Lemma 15. Let $\varphi : (X_1, f_1) \rightarrow (X_2, f_2)$ be a morphism of dynamical systems. Suppose that (X_1, f_1) is a fixed point system and φ is surjective. Then (X_2, f_2) is also a fixed point system.

Proof. It follows immediately from Lemma 3.1 in [8]. □

Lemma 16. Let H be a finite abelian group and $f \in \text{End}(H)$. Let $f_w : H \rightarrow H, f_w(x) = f(x) + w$ be the affine mapping for any $w \in H$. Then

- (1) $(H, f) \simeq (H, f_w)$ if and only if (H, f_w) has a fixed point;
- (2) (H, f_w) is a fixed point system if and only if it has a fixed point and (H, f) is a fixed point system.

Proof. It follows immediately from Lemma 3.3 in [8]. $\square$

The following lemma shows why we need to consider the affine system.

Lemma 17. *Let H_1, H_2 be two finite abelian groups and $f_i \in \text{End}(H_i)$. Suppose that $\varphi : H_1 \rightarrow H_2$ is a group homomorphism and is also a surjective morphism of dynamical systems. Let y_0 be a fixed point in (H_2, f_2) and $K = \varphi^{-1}(\{y_0\})$. Then $(K, f_1|_K) \simeq (\ker \varphi, g)$, where $g(x) = f_1(x) + f_1(x_0) - x_0$ for some $x_0 \in K$.*

Proof. It is clear that $K = x_0 + \ker \varphi$ is a coset of $\ker \varphi$. We define $\psi : \ker \varphi \rightarrow K, \psi(a) = a + x_0$. Then φ is bijective and $f_1\psi(a) = f_1(a + x_0) = f_1(a) + f_1(x_0) = \psi g(a)$. Thus, ψ is an isomorphism of dynamical systems. $\square$

Recall that $d(X, f)$ denotes the number of fixed points of (X, f) .

Lemma 18. *Let H_1, H_2 be two finite abelian groups and $f_i \in \text{End}(H_i)$. Suppose that $\varphi \in \text{Hom}(H_1, H_2)$ is a surjective morphism of dynamical systems. Let $F_i = \{t \in H_i : f_i(t) = t\}$. Then (H_1, f_1) is a fixed point system if and only if the following two conditions are satisfied:*

- (1) (H_2, f_2) is a fixed point system;
- (2) $d(H_1, f_1) = d(\ker \varphi, f_1)d(H_2, f_2)$.

Proof. We write $F_2 = \{y_i : 1 \leq i \leq |F_2|\}$ and $\varphi(x_i) = y_i$. Since φ maps cycle vertices into cycle vertices, so the cycle vertices of (H_1, f_1) are contained in $\bigcup_{i=1}^{|F_2|} (x_i + \ker \varphi, f_1)$, and $d(H_1, f_1) = \sum_{i=1}^{|F_2|} d(x_i + \ker \varphi, f_1)$. Thus, (H_1, f_1) is a fixed point system if and only if $(x_i + \ker \varphi, f_1)$ is a fixed point system for any $i = 1, \dots, |F_2|$. By Lemma 17, $(x_i + \ker \varphi, f_1)$ is a fixed point system if and only if $d(x_i + \ker \varphi, f_1) = d(\ker \varphi, f_1)$ and $(\ker \varphi, f_1)$ is a fixed point system. This finishes the proof. $\square$

By Theorem 10 we only need to deal with the case that H is a p -group. Let $H = \mathbb{Z}_{p^{a_1}} \oplus \dots \oplus \mathbb{Z}_{p^{a_n}}$ such that $a_1 \leq a_2 \leq \dots \leq a_n$. Recall that any endomorphism of f of H has a matrix representation $A = (a_{ij}) \in \mathbb{Z}^{n \times n}$, where $p^{a_i - a_j} \mid a_{ij}$ for $i > j$. If

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = A \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \quad (18)$$

then $u_i \equiv \sum_{k=1}^n a_{ik} v_k \pmod{p^{a_i}}$. We also write

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = A \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \left(\text{mod} \begin{pmatrix} p^{a_1} \\ p^{a_2} \\ \vdots \\ p^{a_n} \end{pmatrix} \right). \quad (19)$$

We recall the Smith normal form and invariant factor of matrices over a principal ideal ring.

Definition 19. Let R be a principal ideal ring and $A \in R^{n \times n}$. There exist invertible matrices $U, V \in R^{n \times n}$ such that

$$S = UAV = \begin{pmatrix} s_1 & & & \\ & \ddots & & \\ & & s_m & \\ & & & 0 \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}, \quad (20)$$

where $s_i \mid s_{i+1}$ for $1 \leq i \leq m-1$. S is called the Smith normal form of A . The diagonal entries s_i are called the invariant factors of A and are unique up to units.

Theorem 20 ([8], Theorem 5.2). *Let p be a prime and $H = \mathbb{Z}_{p^{a_n}}$ and $A \in \mathbb{Z}^{n \times n}$. Then (H, A) is a fixed point system if and only if the following conditions are satisfied.*

- (1) $A^n(A - I) \equiv 0 \pmod{p}$;
- (2) Each invariant factors of $A - I$ over $\mathbb{Z}_{p^{a_n}}$ is a unit or 0.

Corollary 21. *Let $H = \mathbb{Z}_{p^{a_1}} \oplus \dots \oplus \mathbb{Z}_{p^{a_n}}$ and $A = (a_{ij}) \in \mathbb{Z}^{n \times n}$, where $a_1 \leq a_2 \leq \dots \leq a_n$ and $p^{a_i - a_j} \mid a_{ij}$ for $i \geq j$. Suppose that $A^n(A - I) \equiv 0 \pmod{p}$ and each invariant factor of $A - I$ over $\mathbb{Z}_{p^{a_n}}$ is a unit or 0. Then (H, A) is a fixed point system.*

Proof. Since the canonical mapping $\varphi : (\mathbb{Z}_{p^{a_n}}, A) \rightarrow (H, A)$ is a surjective morphism of dynamical systems, it follows from Lemma 18 and Theorem 20 that (H, A) is a fixed point system. $\square$

However, the converse of Corollary 21 is not true in general. Consider the following example.

Example 22. Let $A = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}$ and $H_1 = \mathbb{Z}_3 \oplus \mathbb{Z}_3, H_2 = \mathbb{Z}_9 \oplus \mathbb{Z}_9$, and $H = \mathbb{Z}_3 \oplus \mathbb{Z}_9$. By a direct computation, (H_1, A) is a fixed point system with three fixed points $(0, 0)$, $(1, 0)$, and $(2, 0)$, and (H_2, A) has three fixed points $(0, 0)$, $(3, 0)$, and $(6, 0)$, and two 3-cycle $\{(1, 3), (4, 3), (7, 3)\}$ and $\{(2, 6), (8, 6), (5, 6)\}$, and (H, A) is a fixed point system with 3 fixed points $(0, 0)$, $(1, 3)$, and $(2, 6)$.

Let $A = (a_{ij})$ be an integer matrix and $n \in \mathbb{Z}$. We write $n \mid A$ if $n \mid a_{ij}$ for any i, j .

Theorem 23. *Let $H = \mathbb{Z}_{p^{a_1}}^{n_1} \oplus \mathbb{Z}_{p^{a_2}}^{n_2} \oplus \dots \oplus \mathbb{Z}_{p^{a_r}}^{n_r}$ with $a_1 < a_2 < \dots < a_r$. Let $T = (A_{ij})_{1 \leq i, j \leq r}$ be a matrix with partition $A_{ij} \in \mathbb{Z}^{n_j \times n_i}$ such that $p^{a_i - a_j} \mid A_{ij}$ for $i > j$. Let $B_{ij} = p^{a_j - a_i} A_{ij}$ and*

$$T_j = \begin{pmatrix} B_{jj} & B_{jj+1} & \dots & B_{jr} \\ B_{j+1j} & B_{j+1j+1} & \dots & B_{j+1r} \\ \vdots & \vdots & \ddots & \vdots \\ B_{rj} & B_{rj+1} & \dots & B_{rr} \end{pmatrix}, \quad (21)$$

for $j = 1, \dots, r$. Let $a_0 = 0$ and $H_j = \mathbb{Z}_{p^{a_j - a_{j-1}}}^{\sum_{i=j}^r n_i}$. Then (H, T) is a fixed point system if and only if

(1) (H_j, T_j) is a fixed point system for each j ;

(2) $d(H, T) = \prod_{j=1}^r d(H_j, T_j)$.

Proof. We induct on r . The case $r = 1$ is trivial. Let $F_1 = \mathbb{Z}_{p^{a_2 - a_1}}^{n_2} \oplus \mathbb{Z}_{p^{a_3 - a_1}}^{n_3} \oplus \dots \oplus \mathbb{Z}_{p^{a_r - a_1}}^{n_r}$ and

$$C_1 = \begin{pmatrix} A_{22} & A_{23} & \cdots & A_{2r} \\ A_{32} & A_{33} & \cdots & A_{3r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r2} & A_{r3} & \cdots & A_{rr} \end{pmatrix}. \quad (22)$$

Define

$$\begin{aligned} \varphi : H &\longrightarrow F_1, \\ \varphi \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{pmatrix} &\equiv \begin{pmatrix} X_2 \\ X_3 \\ \vdots \\ X_r \end{pmatrix} \\ &\quad \left(\text{mod} \begin{pmatrix} p^{a_2 - a_1} \\ p^{a_3 - a_1} \\ \vdots \\ p^{a_r - a_1} \end{pmatrix} \right), \end{aligned} \quad (23)$$

where $X_i \in \mathbb{Z}_{p^{a_i}}^{n_i}$. Since $p^{a_i - a_1} \mid A_{i1}$, one has

$$\begin{aligned} \varphi A \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{pmatrix} &= \begin{pmatrix} A_{21}X_1 + A_{22}X_2 + \cdots + A_{2r}X_r \\ A_{31}X_1 + A_{32}X_2 + \cdots + A_{3r}X_r \\ \vdots \\ A_{r1}X_1 + A_{r2}X_2 + \cdots + A_{rr}X_r \end{pmatrix} \\ &\equiv \begin{pmatrix} A_{22}X_2 + \cdots + A_{2r}X_r \\ A_{32}X_2 + \cdots + A_{3r}X_r \\ \vdots \\ A_{r2}X_2 + \cdots + A_{rr}X_r \end{pmatrix} \equiv C_1 \varphi \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{pmatrix} \end{aligned}$$

$$\left(\text{mod} \begin{pmatrix} p^{a_2 - a_1} \\ p^{a_3 - a_1} \\ \vdots \\ p^{a_r - a_1} \end{pmatrix} \right). \quad (24)$$

It follows that $\varphi : (H, A) \rightarrow (F_1, C_1)$ is a surjective morphism of dynamical systems. By Lemma 18, (H, A) is a fixed point system if and only if (F_1, C_1) is a fixed point system and $d(H, A) = d(\ker \varphi, A)d(F_1, C_1)$. By induction, (F_1, C_1) is a fixed point system if and only if (H_j, T_j) is a fixed point system for each $j \geq 2$ and $d(F_1, C_1) = \prod_{j=2}^r d(H_j, T_j)$. It remains to show that $(\ker \varphi, A) \simeq (H_1, T_1)$, where $T_1 = (p^{a_j - a_1} A_{ij})_{1 \leq i, j \leq r}$.

We set $\psi : H_1 = \mathbb{Z}_{p^{a_1}}^{\sum_{i=1}^r n_i} \rightarrow H$.

$$\begin{aligned} \psi \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_r \end{pmatrix} &\equiv \begin{pmatrix} Y_1 \\ p^{a_2 - a_1} Y_2 \\ \vdots \\ p^{a_r - a_1} Y_r \end{pmatrix} \\ &\quad \left(\text{mod} \begin{pmatrix} p^{a_1} \\ p^{a_2} \\ \vdots \\ p^{a_r} \end{pmatrix} \right), \end{aligned} \quad (25)$$

where $Y_i \in \mathbb{Z}_{p^{a_i}}^{n_i}$. It is clear that ψ is injective and $\text{Im } \psi = \ker \varphi$. We have

$$\begin{aligned} \psi T_1 \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_r \end{pmatrix} &= \begin{pmatrix} A_{11}Y_1 + p^{a_2 - a_1} A_{12}Y_2 + \cdots + p^{a_r - a_1} A_{1r}Y_r \\ A_{21}Y_1 + p^{a_2 - a_1} A_{22}Y_2 + \cdots + p^{a_r - a_1} A_{2r}Y_r \\ \vdots \\ A_{r1}Y_1 + p^{a_2 - a_1} A_{r2}Y_2 + \cdots + p^{a_r - a_1} A_{rr}Y_r \end{pmatrix} \\ &\equiv A \psi \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_r \end{pmatrix} \left(\text{mod} \begin{pmatrix} p^{a_1} \\ p^{a_2} \\ \vdots \\ p^{a_r} \end{pmatrix} \right). \end{aligned} \quad (26)$$

Thus, $(H_1, \psi) \simeq (\text{Im } \psi, A) = (\ker \varphi, A)$. The proof is finished by induction. $\square$

Remark 24. By Theorem 23, we can determine whether (H_j, T_j) is a fixed point system and also can compute

$d(H_j, T_j)$ be using the Smith normal form of $T - I$. So the problem for determining whether (H, T) is a fixed point system is now reduced to solve the congruence equation

$$(T - I)X \equiv 0 \pmod{\begin{pmatrix} p^{a_1} \\ p^{a_2} \\ \vdots \\ p^{a_r} \end{pmatrix}}. \quad (27)$$

Example 25. Let $H = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}$ and $T = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix}$. By Theorem 23, let $T' = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 12 \\ 0 & 1 & 2 \end{pmatrix}$, and let

$$\begin{aligned} H_1 &= \mathbb{Z}_2^3, \\ T_1 &= \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 12 \\ 0 & 1 & 2 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \pmod{2}, \\ H_2 &= \mathbb{Z}_2^2, \\ T_2 &= \begin{pmatrix} 2 & 12 \\ 1 & 2 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \pmod{2}, \\ H_3 &= \mathbb{Z}_4, \\ T_3 &= (2) \pmod{4}. \end{aligned} \quad (28)$$

It is straightforward to check that (H_1, T_1) is a fixed point system with two fixed points $(0, 0, 0)^T, (1, 1, 1)^T$, (H_2, T_2) is a fixed point system with one fixed point $(0, 0)^T$, and (H_3, T_3) is a fixed point system with two fixed points 0. Now we compute the number of fixed points of (H, T) . Let $X = (x, y, z)^T$ be a fixed point in (H, T) . Then $TX = X$ and

$$\begin{aligned} y &\equiv 0 \pmod{2}, \\ 2x + y + 3z &\equiv 0 \pmod{4}, \\ 4y + z &\equiv 0 \pmod{8}. \end{aligned} \quad (29)$$

We derive that $y \equiv 0, 2 \pmod{4}$ and $z \equiv 0 \pmod{8}$ by the first and third equation. Hence, the second equation is equivalent to $2x + y \equiv 0 \pmod{4}$. It follows that $x \equiv 0 \pmod{2}$ if $y \equiv 0 \pmod{4}$, and $x \equiv 1 \pmod{2}$ if $y \equiv 2 \pmod{4}$. Thus, (H, T) has two fixed points $(0, 0, 0)^T$ and $(1, 2, 0)^T$. By Theorem 23 again, (H, T) is a fixed point system.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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