Research Article

A New Augmented Lagrangian Method for Equality Constrained Optimization with Simple Unconstrained Subproblem

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We propose a new method for equality constrained optimization based on augmented Lagrangian method. We construct an unconstrained subproblem by adding an adaptive quadratic term to the quadratic model of augmented Lagrangian function. In each iteration, we solve this unconstrained subproblem to obtain the trial step. The main feature of this work is that the subproblem can be more easily solved. Numerical results show that this method is effective.

1. Introduction

In this paper, we consider the following equality constrained optimization:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & c_i(x) = 0, \quad i = 1, \dots, m, \end{aligned} \quad (1)$$

where $x \in R^n$, $f(x) : R^n \rightarrow R$, and $c_i(x) : R^n \rightarrow R$ ($i = 1, \dots, m$) are twice continuously differentiable.

The method presented in this paper is a variant of the augmented Lagrangian method (denoted by AL). In the late 1960s, AL method was proposed by Hestenes [1] and Powell [2]. Later Conn et al. [3, 4] presented a practical AL method and proved the global convergence under the LICQ condition. Since then, AL method attracted the attentions of many scholars and many variants were presented (see [5–11]). Up to now, there are many computer packages based on AL method, such as LANCELOT [4] and ALGENCAN [5, 6]. In the past decades, AL method was fully developed. Attracted by its well performance, there are still many scholars devoted to research AL method and its applications in recent years (see [7, 8, 11–15]).

For (1), we define the Lagrangian function

$$l(x, \lambda) = f(x) - \lambda^T c(x) \quad (2)$$

and the augmented Lagrangian function

$$L(x, \lambda, \sigma) = l(x, \lambda) + \frac{\sigma}{2} \|c(x)\|^2, \quad (3)$$

where λ is called the Lagrangian multiplier and σ is called the penalty parameter. In this paper, $\|\cdot\|$ refers to the Euclidean norm.

In a typical AL method, at the k th step, for given multiplier λ_k and penalty parameter σ_k , an unconstrained subproblem

$$\min_{x \in R^n} L(x, \lambda_k, \sigma_k) \quad (4)$$

is solved to find the next iteration point. Then, the multiplier and penalty parameter are updated by some rules. For convenience, for given λ_k and σ_k , we define

$$\Phi_k(x) = L(x, \lambda_k, \sigma_k) = l(x, \lambda_k) + \frac{\sigma_k}{2} \|c(x)\|^2. \quad (5)$$

Motivated by the regularized Newton method for unconstrained optimization (see [16–19]), we construct a new

subproblem of (1). At the k th iteration point x_k , $\Phi_k(x_k + s)$ is approximated by the following quadratic model:

$$\begin{aligned} \bar{m}_k(s) &= l(x_k, \lambda_k) + (\nabla_x l(x_k, \lambda_k))^T s + \frac{1}{2} s^T B_k s \\ &\quad + \frac{\sigma_k}{2} \|c(x_k) + A(x_k)s\|^2 \\ &= l(x_k, \lambda_k) + (\nabla_x f(x_k) - A(x_k)^T \lambda_k)^T s \\ &\quad + \frac{1}{2} s^T B_k s + \frac{\sigma_k}{2} \|c(x_k) + A(x_k)s\|^2 \\ &= \Phi_k(x_k) \\ &\quad + [\nabla_x f(x_k) - A(x_k)^T \lambda_k + \sigma_k A(x_k)^T c(x_k)]^T s \\ &\quad + \frac{1}{2} s^T [B_k + \sigma_k A(x_k)^T A(x_k)] s, \end{aligned} \quad (6)$$

where $A(x_k) = [\nabla_x c_1(x_k), \dots, \nabla_x c_m(x_k)]^T$ and B_k is a positive semidefinite approximation of $\nabla_{xx}^2 l(x_k, \lambda_k)$. Let

$$\begin{aligned} \tilde{g}_k &= \nabla \Phi_k(x_k) \\ &= \nabla_x f(x_k) - A(x_k)^T \lambda_k + \sigma_k A(x_k)^T c(x_k); \quad (7) \\ \tilde{B}_k &= B_k + \sigma_k A(x_k)^T A(x_k), \end{aligned}$$

thus we have

$$\bar{m}_k(s) = \Phi_k(x_k) + \tilde{g}_k^T s + \frac{1}{2} s^T \tilde{B}_k s. \quad (8)$$

In [14, 15], $\bar{m}_k(s)$ is minimized within a trust region to find the next iteration point. Motivated by the regularized Newton method, we add a regularization term to the quadratic model $\bar{m}_k(s)$ and define

$$\begin{aligned} m_k(s) &= \bar{m}_k(s) + \frac{1}{2} \mu_k \|s\|^2 \\ &= \Phi_k(x_k) + \tilde{g}_k^T s + \frac{1}{2} s^T \tilde{B}_k s + \frac{1}{2} \mu_k \|s\|^2, \end{aligned} \quad (9)$$

where μ_k is called regularized parameter. At the k th step of our algorithm, we solve the following convex unconstrained quadratic subproblem:

$$\min_{s \in \mathbb{R}^n} m_k(s) = \Phi_k(x_k) + \tilde{g}_k^T s + \frac{1}{2} s^T \tilde{B}_k s + \frac{1}{2} \mu_k \|s\|^2 \quad (10)$$

for finding the trial step s_k . Then, we compute the ratio between the actual reduction and predicted reduction

$$\rho_k = \frac{\text{Ared}_k}{\text{Pred}_k} = \frac{\Phi_k(x_k) - \Phi_k(x_k + s_k)}{\Phi_k(x_k) - m_k(x_k + s_k)}. \quad (11)$$

When ρ_k is close to 1, we accept $x_k + s_k$ as the next iteration point. At the same time, we think the quadratic model $m_k(s)$ is

a sufficiently “good” approximation of $\Phi_k(x_k + s)$ and reduce the value of μ_k . Conversely, when ρ_k is close to zero, we set $x_{k+1} = x_k$ and increase the value of μ_k , by which we wish to reduce the length of the next trial step. This technique is similar to the update rule of trust region radius. Actually, sufficiently large μ_k indeed reduces the length of the trial step s_k . However, the regularized parameter is different from trust region radius. In [14, 15], the authors construct a trust region subproblem

$$\min_{s \in \Delta_k} \bar{m}_k(s) = \Phi(x_k) + \tilde{g}_k^T s + \frac{1}{2} s^T \tilde{B}_k s. \quad (12)$$

The exact solution s_k of (12) satisfies the first-order critical conditions if there exists some $\bar{\mu}_k \geq 0$ such that $\tilde{B}_k + \bar{\mu}_k I$ is positive semidefinite and

$$\begin{aligned} \tilde{g}_k + (\tilde{B}_k + \bar{\mu}_k I) s_k &= 0 \\ \|s_k\| &\leq \Delta_k \\ \bar{\mu}_k (\|s_k\| - \Delta_k) &= 0, \end{aligned} \quad (13)$$

while the first-order critical condition of (10) is

$$\tilde{g}_k + (\tilde{B}_k + \mu_k I) s_k = 0. \quad (14)$$

Equations (13) and (15) can show the similarities and differences between regularized subproblem (10) and trust region subproblem (12). It seems that the parameter μ_k plays a role similar to the multiplier $\bar{\mu}_k$ in the trust region subproblem. But, actually, the update rule of μ_k (see (26)) shows that μ_k is not the approximation of $\bar{\mu}_k$. The update of μ_k depends on the quality of last trial step s_{k-1} and has no direct relation with system (13).

To establish the global convergence of an algorithm, some kind of constraint qualification is required. There are many well-known constraint qualifications, such as LICQ, MFCQ, CRCQ, RCR, CPLD, and RCPLD. In case there are only equality constraints, LICQ is equivalent to MFCQ in which $\{\nabla c_i(x) \mid i = 1, \dots, m\}$ has full rank; CRCQ is equivalent to CPLD in which any subset of $\{\nabla c_i(x) \mid i = 1, \dots, m\}$ maintains constant rank in a neighborhood of x ; RCR is equivalent to RCPLD in which $\{\nabla c_i(x) \mid i = 1, \dots, m\}$ maintains constant rank in a neighborhood of x . RCPLD is weaker than CRCQ, and CRCQ is weaker than LICQ. In this paper, we use RCPLD which is defined in the following.

Definition 1. One says that RCPLD holds at a feasible point x^* of (1), if there exists a neighborhood $U(x^*)$ of x^* such that $\{\nabla c_i(x) \mid i = 1, \dots, m\}$ maintains constant rank for all $x \in U(x^*)$.

The rest of this paper is organized as follows. In Section 2, we give a detailed description of the presented algorithm. The global convergence is proved in Section 3. In Section 4, we present the numerical experiments. Some conclusions are given in Section 5.

Notations. For convenience, we abbreviate $\nabla_x f(x_k)$ to g_k , $f(x_k)$ to f_k , $c(x_k)$ to c_k , and $A(x_k)$ to A_k . In this paper, $\lambda^{(i)}$ denotes the i th component of the vector λ .

2. Algorithm

In this section, we give a detailed description of the proposed algorithm.

As mentioned in Section 1, we solve the unconstrained subproblem (10) to obtain the trial step s_k . Since B_k is at least positive semidefinite and $\tilde{B}_k = B_k + \sigma_k A_k^T A_k$, $\tilde{B}_k + \mu_k I$ is positive definite as $\mu_k > 0$. Therefore, (10) is a strictly convex quadratic unconstrained optimization. s_k solves (10) if and only if

$$\tilde{g}_k + (\tilde{B}_k + \mu_k I) s_k = 0 \quad (15)$$

holds. Global convergence does not depend on the exact solution of (15), although the linear system (15) is easy to be solved. For minimizer of (10) along the direction $-\tilde{g}_k$, specifically, we consider the following subproblem:

$$\begin{aligned} \min_{\alpha \geq 0} \quad m_k(-\alpha \tilde{g}_k) &= \Phi_k(x_k) - \alpha \|\tilde{g}_k\|^2 + \frac{1}{2} \alpha^2 \tilde{g}_k^T \tilde{B}_k \tilde{g}_k \\ &\quad + \frac{1}{2} \alpha^2 \mu_k \|\tilde{g}_k\|^2. \end{aligned} \quad (16)$$

If $\|\tilde{g}_k\| > 0$, then the minimizer of (16) is $\alpha_k^* = \|\tilde{g}_k\|^2 / (\tilde{g}_k^T \tilde{B}_k \tilde{g}_k + \mu_k \|\tilde{g}_k\|^2)$. Therefore, at the k th step, it follows that

$$\begin{aligned} \text{Pred}_k &\geq \Phi_k(x_k) - m_k(-\alpha_k^* \tilde{g}_k) \\ &= \frac{\|\tilde{g}_k\|^4}{2(\tilde{g}_k^T \tilde{B}_k \tilde{g}_k + \mu_k \|\tilde{g}_k\|^2)}. \end{aligned} \quad (17)$$

By direct calculation, we have that

$$\text{Pred}_k \geq \frac{\|\tilde{g}_k\|^2}{4} \min \left\{ \frac{1}{\|\tilde{B}_k\|}, \frac{1}{\mu_k} \right\}. \quad (18)$$

In Section 3, we always suppose that (18) holds.

In a typical AL algorithm, the update rule of σ_k depends on the improvement of constraint violation. A commonly used update rule is that if $\|c_{k+1}\| < \tau_k \|c_k\|$, where $0 < \tau_k < 1$, one may think that the constraint violation is reduced sufficiently and thus $\sigma_{k+1} = \sigma_k$ is a good choice. Otherwise, if $\|c_{k+1}\| \geq \tau_k \|c_k\|$, one thinks that current penalty parameter can not sufficiently reduce the constraint violation and increase it in the next iteration. In [20], Yuan proposed a different update rule of σ_k for trust region algorithm. Specifically, if

$$\text{Pred}_k < \delta_k \sigma_k \min \{ \Delta_k \|c_k\|, \|c_k\|^2 \}, \quad (19)$$

σ_k is increased. In (19), δ_k is an auxiliary parameter such that $\delta_k \sigma_k$ tends to zero. We slightly modify (19) in our algorithm. Specifically, if

$$\text{Pred}_k < \delta_k \sigma_k \min \left\{ \frac{\|c_k\| \max \{1, \|\tilde{g}_k\|\}}{\mu_k}, \|c_k\|^2 \right\}, \quad (20)$$

σ_k is increased.

In typical AL method, next iteration point x_{k+1} is obtained by minimizing $L(x, \lambda_k, \sigma_k)$. In most AL methods, x_{k+1} satisfies that $\|\nabla_x L(x_{k+1}, \lambda_k, \sigma_k)\| < \varepsilon_k$, where ε_k is controlling parameter which tends to zero. As

$$\nabla_x L(x_{k+1}, \lambda_k, \sigma_k) = g_{k+1} - A_{k+1}^T (\lambda_k - \sigma_k c_k), \quad (21)$$

when ε_k is sufficiently small, $\lambda_k - \sigma_k c_k$ is a good estimate of the next multiplier λ_{k+1} . As we obtain x_{k+1} by minimizing $m_k(s)$, the critical point of $m_k(s)$ has no direct relation to $\|\nabla_x L(x_{k+1}, \lambda_k, \sigma_k)\|$. Therefore, the update rule $\lambda_{k+1} = \lambda_k - \sigma_k c_k$ does not suit our algorithm. We obtain λ_{k+1} by approximately solving the following least squares problem:

$$\min_{\lambda \in R^m} \frac{1}{2} \|g_{k+1} - A_{k+1}^T \lambda\|^2. \quad (22)$$

Most AL algorithms require that $\{\lambda_k\}$ is bounded to ensure the global convergence. Hence, all components of λ_k are restricted to certain interval $[\bar{\lambda}, \underline{\lambda}]$. This technique is also used in our algorithm.

Now, we give the detailed algorithm in the following.

Algorithm 2.

Step 0 (initialization). Choose the parameters $0 < \gamma_1 < \gamma_2 < 1$, $0 < \theta_1 < 1 \leq \theta_2 < \theta_3$, $\bar{\lambda} > \underline{\lambda}$. Determine $x_0 \in R^n$, $\lambda_0 \in R^m$, $\sigma_0 > 0$, $\delta_0 > 0$, $B_0 \in R^{n \times n}$. Let $R_0 = \max\{\|c(x_0)\|, 1\}$. Set $k := 0$.

Step 1 (termination test). If $\|\tilde{g}_k\| = 0$ and $\|c_k\| = 0$, return x_k as a KKT point. If $\|\tilde{g}_k\| = 0$, $\|c_k\| > 0$, and $\|A_k^T c_k\| = 0$, return x_k as an infeasible KKT point.

Step 2 (determine the trial step). Evaluate the trial step s_k by solving

$$\min_{s \in R^n} m_k(s) = \Phi_k(x_k) + \tilde{g}_k^T s + \frac{1}{2} s^T \tilde{B}_k s + \frac{1}{2} \mu_k \|s\|^2 \quad (23)$$

such that (18) holds. Compute the ratio between the actual reduction to the predicted reduction

$$\rho_k = \frac{\text{Ared}_k}{\text{Pred}_k}, \quad (24)$$

where $\text{Ared}_k = \Phi_k(x_k) - \Phi_k(x_k + s_k)$, $\text{Pred}_k = \Phi_k(x_k) - m_k(x_k + s_k)$. Set

$$x_{k+1} = \begin{cases} x_k + s_k, & \rho_k \geq \gamma_1, \\ x_k, & \rho_k < \gamma_1, \end{cases} \quad (25)$$

$$\mu_{k+1} = \begin{cases} \theta_1 \mu_k, & \rho_k \geq \gamma_2, \\ \theta_2 \mu_k, & \gamma_1 \leq \rho_k < \gamma_2, \\ \theta_3 \mu_k, & \rho_k < \gamma_1. \end{cases} \quad (26)$$

Step 3 (update the penalty parameter). If

$$\text{Pred}_k < \delta_k \sigma_k \min \left\{ \frac{\|c_k\| \max \{1, \|\tilde{g}_k\|\}}{\mu_k}, \|c_k\|^2 \right\}, \quad (27)$$

set

$$\begin{aligned}\sigma_{k+1} &= 2\sigma_k, \\ \delta_{k+1} &= \frac{1}{4}\delta_k.\end{aligned}\quad (28)$$

Otherwise, set

$$\begin{aligned}\sigma_{k+1} &= \sigma_k, \\ \delta_{k+1} &= \delta_k.\end{aligned}\quad (29)$$

Step 4 (update the multiplier). If $\|c_{k+1}\| \leq R_k$, set $R_{k+1} = (1/2)R_k$. Evaluate $\hat{\lambda}_{k+1}$ by

$$\min_{\lambda \in R^m} \|g_k - A_k^T \lambda\|, \quad (30)$$

and let

$$\lambda_{k+1}^{(i)} = \min \left\{ \max \left\{ \hat{\lambda}_{k+1}^{(i)}, \underline{\lambda} \right\}, \bar{\lambda} \right\}, \quad i = 1, \dots, m. \quad (31)$$

If $\|c_{k+1}\| > R_k$, set $R_{k+1} = R_k$ and $\lambda_{k+1} = \lambda_k$.
Set $k := k + 1$ and go to Step 1.

Remark 3. In practical calculation, it is not required to solve (30) exactly to find $\hat{\lambda}_{k+1}$. In our implementation of Algorithm 2, we use the Matlab subroutine *minres* to find an approximate solution of the linear system $A_k A_k^T \lambda = A_k g_k$ and take it as an approximation of $\hat{\lambda}_{k+1}$.

3. Global Convergence

In this section, we discuss the global convergence of Algorithm 2. We assume that Algorithm 2 can find an infinite set $\{x_k\}$ and give some assumptions in the following.

Assumptions 1. (A1) $f(x)$ and $c(x)$ are twice continuously differentiable.

(A2) $\{x_k\}$ and $\{B_k\}$ are bounded, where B_k is positive semidefinite approximation of $\nabla_{xx}^2 l(x_k, \lambda_k)$.

Firstly, we give a result on the upper bound of the trial step.

Lemma 4. *If s_k solves subproblem (23), then one has*

$$\|s_k\| \leq 4 \frac{\|\tilde{g}_k\|}{\mu_k} \quad (32)$$

$$\text{or } \mu_k \leq 2 \|\tilde{B}_k\| \quad \text{hold } \forall k \geq 0.$$

Proof. Any approximate solution s_k of (23) satisfies $\text{Pred}_k = \Phi_k(x_k) - m_k(s_k) \geq 0$. Clearly,

$$\begin{aligned}0 &\leq \Phi_k(x_k) - m_k(s_k) \\ &= -\tilde{g}_k^T s_k - \frac{1}{2} s_k^T \tilde{B}_k s_k - \frac{1}{2} \mu_k \|s_k\|^2 \\ &\leq \|\tilde{g}_k\| \|s_k\| + \frac{1}{2} \|\tilde{B}_k\| \|s_k\|^2 - \frac{1}{2} \mu_k \|s_k\|^2 \\ &= \left(\|\tilde{g}_k\| - \frac{1}{4} \mu_k \|s_k\| \right) \|s_k\| + \frac{1}{2} \left(\|\tilde{B}_k\| - \frac{1}{2} \mu_k \right) \|s_k\|^2.\end{aligned}\quad (33)$$

If $\|s_k\| = 0$, then (32) holds. If $\|s_k\| > 0$, $(\|\tilde{g}_k\| - (1/4)\mu_k\|s_k\|)\|s_k\| + (1/2)(\|\tilde{B}_k\| - (1/2)\mu_k)\|s_k\|^2 \geq 0$ implies that $\|\tilde{g}_k\| - (1/4)\mu_k\|s_k\| \geq 0$ or $\|\tilde{B}_k\| - (1/2)\mu_k \geq 0$. Thus we can obtain (32). $\square$

Now, we discuss convergence properties in two cases. One is that the penalty parameter σ_k tends to ∞ and the other is that $\{\sigma_k\}$ is bounded.

3.1. The Case of $\sigma_k \rightarrow \infty$

Lemma 5. *Suppose that (A1)-(A2) hold and $\sigma_k \rightarrow \infty$; then there exists a constant c^* such that $\|c_k\| \rightarrow c^*$.*

Proof. See Lemma 3.1 in Wang and Yuan [15]. $\square$

In Lemma 5, if $c^* > 0$, then any accumulation point of $\{x_k\}$ is infeasible. Sometimes (1) is naturally infeasible; in other words, the feasible set $\{x \mid c(x) = 0\}$ is empty. In this case, we wish to find a minimizer of constraint violation. Specifically, we wish to solve

$$\min_{x \in R^n} \|c(x)\|^2. \quad (34)$$

The solution of this problem is characterized by

$$A(x^*)^T c(x^*) = 0. \quad (35)$$

In the next theorem, we show that if $\{c_k\}$ is not convergent to zero, at least one of the accumulation points of $\{x_k\}$ satisfies (35).

Theorem 6. *Suppose that (A1)-(A2) hold and $\sigma_k \rightarrow \infty$. If $\|c_k\| \rightarrow c > 0$, then*

$$\liminf_{k \rightarrow \infty} \|A_k^T c_k\| = 0. \quad (36)$$

Proof. We prove this result by contradiction. Suppose that there exists some $\tau > 0$ such that

$$\|A_k^T c_k\| > 2\tau, \quad \forall k \geq 0. \quad (37)$$

By the definition of $\tilde{g}_k$ in (7), we know that

$$\|\tilde{g}_k\| \geq \sigma_k \|A_k^T c_k\| - \|g_k - A_k^T \lambda_k\|. \quad (38)$$

As $\{x_k\}$ and $\{\lambda_k\}$ are bounded, we can deduce the boundedness of $\|g_k - A_k^T \lambda_k\|$ by (A2); that is, there exists some $\bar{\tau} > 0$ such that

$$\|g_k - A_k^T \lambda_k\| < \bar{\tau}. \quad (39)$$

By (37), (38), and (39), we can conclude that

$$\|\tilde{g}_k\| \geq 2\tau\sigma_k - \bar{\tau} > \tau\sigma_k \quad (40)$$

holds for all sufficiently large k . By the boundedness of B_k and A_k , we can conclude that there exists $M > 0$, such that

$$\|\tilde{B}_k\| < M\sigma_k \quad (41)$$

holds for all sufficiently large k , where $\tilde{B}_k$ is defined by (7). By (18), (40), and (41),

$$\begin{aligned} \text{Pred}_k &\geq \frac{\|\tilde{g}_k\|^2}{4} \min \left\{ \frac{1}{\|\tilde{B}_k\|}, \frac{1}{\mu_k} \right\} \\ &= \frac{\|\tilde{g}_k\|}{4} \min \left\{ \frac{\|\tilde{g}_k\|}{\|\tilde{B}_k\|}, \frac{\|\tilde{g}_k\|}{\mu_k} \right\} \\ &\geq \frac{\tau \sigma_k}{4} \min \left\{ \frac{\tau}{M}, \frac{\|\tilde{g}_k\|}{\mu_k} \right\} \end{aligned} \quad (42)$$

holds for all sufficiently large k . By the update rule of σ_k and the fact that $\sigma_k \rightarrow \infty$, we have that

$$\text{Pred}_k < \delta_k \sigma_k \min \left\{ \frac{\|c_k\| \max \{1, \|\tilde{g}_k\|\}}{\mu_k}, \|c_k\|^2 \right\} \quad (43)$$

holds for infinitely many k . As $\|\tilde{g}_k\| > 1$ holds for all sufficiently k by (40), it is easy to see that (42) contradicts to (43) as $\delta_k \sigma_k \rightarrow 0$ and $\{\|c_k\|\}$ is convergent. Thus we can prove the desired result. $\square$

Lemma 7. Suppose that (A1)-(A2) hold, $\sigma_k \rightarrow \infty$, and $\|c_k\| \rightarrow 0$; then

$$\liminf_{k \rightarrow \infty} \|\tilde{g}_k\| = 0. \quad (44)$$

Proof. Assume that there exists $\tau > 0$ such that

$$\|\tilde{g}_k\| > \tau > 0, \quad \forall k \geq 0. \quad (45)$$

Then, by (18) and (41), we know that, for all sufficiently large k ,

$$\text{Pred}_k \geq \frac{\tau}{4} \min \left\{ \frac{\tau}{M \sigma_k}, \frac{\|\tilde{g}_k\|}{\mu_k} \right\}. \quad (46)$$

By the update rule of σ_k and $\sigma_k \rightarrow \infty$,

$$\text{Pred}_k < \delta_k \sigma_k \min \left\{ \frac{\|c_k\| \max \{1, \|\tilde{g}_k\|\}}{\mu_k}, \|c_k\|^2 \right\} \quad (47)$$

holds for infinitely many k . We will prove that (47) contradicts to (46). Let K be the index set containing all k such that (47) holds. Therefore, for all $k \in K$,

$$\text{Pred}_k < \delta_k \sigma_k \frac{\|c_k\| \max \{1, \|\tilde{g}_k\|\}}{\mu_k}, \quad (48)$$

$$\text{Pred}_k < \delta_k \sigma_k \|c_k\|^2. \quad (49)$$

If there exists an infinite subset $K_1 \subseteq K$ such that $\|\tilde{g}_k\| \leq 1$ holds for all $k \in K_1$, then, by (48), it holds that

$$\text{Pred}_k < \delta_k \sigma_k \frac{\|c_k\|}{\mu_k}, \quad \forall k \in K_1. \quad (50)$$

As $\delta_k \sigma_k \rightarrow 0$, $\|c_k\| \rightarrow 0$, and $\|\tilde{g}_k\| > \tau$, (50) implies that

$$\text{Pred}_k < \frac{\tau}{4} \frac{\|\tilde{g}_k\|}{\mu_k} \quad (51)$$

holds for all sufficiently large $k \in K_1$. If there exists an infinite subset $K_2 \subseteq K$ such that $\|\tilde{g}_k\| > 1$ holds for all $k \in K_2$, then by (48) we have that

$$\text{Pred}_k < \delta_k \sigma_k \frac{\|c_k\| \|\tilde{g}_k\|}{\mu_k} \quad (52)$$

for all $k \in K_2$. Equation (52) also implies (51) as $\delta_k \sigma_k \rightarrow 0$ and $\|c_k\| \rightarrow 0$. From (28) and (29), it follows that $\delta_k \sigma_k^2 = \delta_0 \sigma_0^2$ holds for all $k \geq 0$. Therefore, by (49) we know that, for all $k \in K$,

$$\text{Pred}_k < \delta_k \sigma_k \|c_k\|^2 = \delta_k \sigma_k^2 \|c_k\|^2 \frac{1}{\sigma_k} = \delta_0 \sigma_0^2 \|c_k\|^2 \frac{1}{\sigma_k}. \quad (53)$$

As $\|c_k\| \rightarrow 0$, (53) implies that

$$\text{Pred}_k < \frac{\tau}{4} \frac{\tau}{M \sigma_k} \quad (54)$$

holds for all sufficiently large $k \in K$. Thus from (47), we obtain (51) and (54) which contradict to (46). Thus we can complete the proof. $\square$

Theorem 8. Suppose that (A1)-(A2) hold. If $c_k \rightarrow 0$ and $\sigma_k \rightarrow \infty$, then there exists one cluster point x^* of $\{x_k\}$ such that x^* is a KKT point of (1) or the RCPLD condition does not hold at x^* .

Proof. Under the assumptions of this theorem, Lemma 7 implies that there exists an index set K such that $\{x_k \mid k \in K\}$ converge to some x^* ,

$$\lim_{k \rightarrow \infty, k \in K} c_k = 0, \quad (55)$$

$$\lim_{k \rightarrow \infty, k \in K} \tilde{g}_k = \lim_{k \rightarrow \infty, k \in K} [g_k - A_k^T (\lambda_k - \sigma_k c_k)] = 0,$$

where $\tilde{g}_k$ is defined in (7). With the help of Theorem 2 in Andreani et al. [21], (55) imply that x^* is a KKT point or the RCPLD condition does not hold at x^* . $\square$

3.2. The Case of $\{\sigma_k\}$ Being Bounded. In this subsection, without loss of generality, we assume that $\sigma_k = \sigma_0$ for all $k \geq 0$. Thus by the update rule (29), we have that $\delta_k = \delta_0$ and

$$\text{Pred}_k \geq \delta_0 \sigma_0 \min \left\{ \frac{\|c_k\| \max \{1, \|\tilde{g}_k\|\}}{\mu_k}, \|c_k\|^2 \right\} \quad (56)$$

holds for all $k \geq 0$. As σ_k remains constant, it follows from (A1) and (A2) that $\{\tilde{g}_k\}$ and $\{\tilde{B}_k\}$ are all bounded. If we define the index set

$$S \triangleq \{k \mid \rho_k \geq \gamma_1\}, \quad (57)$$

then $x_{k+1} = x_k$ for $k \notin S$.

Lemma 9. Suppose that (A1)-(A2) hold and $\sigma_k = \sigma_0$ for all $k \geq 0$. If $(1/\mu_k) \rightarrow 0$, as $k \rightarrow \infty$, and there exists some constant $c > 0$ such that

$$\text{Pred}_k > c \frac{\|\tilde{g}_k\|}{\mu_k} \quad \text{holds } \forall \text{ sufficiently large } k, \quad (58)$$

then $\sum_{k \in S} (\|\tilde{g}_k\|/\mu_k)$ is divergent.

Proof. We prove this lemma by contradiction. We will show that if $\sum_{k \in S} (\|\tilde{g}_k\|/\mu_k)$ is convergent, then $\mu_{k+1} < \mu_k$ holds for all sufficiently large k which contradicts to the fact that $1/\mu_k \rightarrow 0$, as $k \rightarrow \infty$.

Suppose that

$$\sum_{k \in S} \frac{\|\tilde{g}_k\|}{\mu_k} = s. \quad (59)$$

$\lim_{k \rightarrow \infty} (1/\mu_k) = 0$ and (32) imply that

$$\|s_k\| \leq 4 \frac{\|\tilde{g}_k\|}{\mu_k} \quad (60)$$

holds for all sufficiently large k . By the definition of S ,

$$\sum_{k=0}^{\infty} \|x_{k+1} - x_k\| = \sum_{k \in S} \|s_k\|. \quad (61)$$

Equations (59)–(61) imply that $\{x_k\}$ is convergent. Let

$$\begin{aligned} r_k &= (\rho_k - \gamma_2) \text{Pred}_k \\ &= (1 - \gamma_2) \text{Pred}_k + (\text{Ared}_k - \text{Pred}_k). \end{aligned} \quad (62)$$

It is clear that $r_k > 0 \Leftrightarrow \rho_k > \gamma_2$. By Taylor's theorem, it holds that

$$\begin{aligned} \text{Ared}_k - \text{Pred}_k &= m_k(s_k) - \Phi_k(x_k + s_k) \\ &= -[\nabla \Phi_k(x_k + \xi_k) - \tilde{g}_k]^T s_k \\ &\quad + \frac{1}{2} s_k^T \tilde{B}_k s_k + \frac{1}{2} \mu_k \|s_k\|^2 \\ &\geq -\|\nabla \Phi_k(x_k + \xi_k) - \tilde{g}_k\| \|s_k\|, \end{aligned} \quad (63)$$

where ξ_k is a convex combination of x_k and $x_k + s_k$. According to (60), we have that

$$\text{Ared}_k - \text{Pred}_k \geq -4 \|\nabla \Phi_k(x_k + \xi_k) - \tilde{g}_k\| \frac{\|\tilde{g}_k\|}{\mu_k} \quad (64)$$

holds for all sufficiently large k and thus

$$r_k \geq [c(1 - \gamma_2) - 4 \|\nabla \Phi_k(x_k + \xi_k) - \tilde{g}_k\|] \frac{\|\tilde{g}_k\|}{\mu_k}. \quad (65)$$

The convergence of $\{x_k\}$ and the boundedness of $\{\lambda_k\}$ imply that $\|\nabla \Phi_k(x_k + \xi_k) - \tilde{g}_k\| \rightarrow 0$. Therefore, for all sufficiently large k , $r_k > 0$. This implies that $\rho_k > \gamma_2$ and $\mu_{k+1} < \mu_k$. $\square$

Lemma 10. Suppose that (A1)–(A2) hold and $\sigma_k = \sigma_0$ for all $k \geq 0$; then we have that

$$\lim_{k \rightarrow \infty} \|c_k\| = 0. \quad (66)$$

Proof. Firstly, we prove that the sum of Ared_k is bounded. Define the indices set

$$K = \{k \mid \|c_{k+1}\| \leq R_k\}, \quad (67)$$

where R_k is defined by Steps 0 and 4 in Algorithm 2. From Step 4 of Algorithm 2, we know that if $k \notin K$, then $\|c_{k+1}\| > R_k$ and $\lambda_{k+1} = \lambda_k$. Hence we have

$$\begin{aligned} \sum_{k=0}^{\infty} (-\lambda_k^T c_k + \lambda_k^T c_{k+1}) &= -\lambda_0^T c_0 + \sum_{k=0}^{\infty} (\lambda_k - \lambda_{k+1})^T c_{k+1} \\ &= -\lambda_0^T c_0 + \sum_{k \in K} (\lambda_k - \lambda_{k+1})^T c_{k+1} \\ &\leq -\lambda_0^T c_0 + 2 \|\lambda_{\max}\| \sum_{k \in K} \|c_{k+1}\| \\ &\leq -\lambda_0^T c_0 + 2 \|\lambda_{\max}\| \sum_{k \in K} R_k, \end{aligned} \quad (68)$$

where $\|\lambda_{\max}\|$ is the upper bound of $\{\lambda_k\}$. From Step 4 and (67), we have

$$\sum_{k \in K} R_k \leq R_0 \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) = 2R_0, \quad (69)$$

which implies

$$\sum_{k=0}^{\infty} (-\lambda_k^T c_k + \lambda_k^T c_{k+1}) \leq -\lambda_0^T c_0 + 4R_0 \|\lambda_{\max}\|. \quad (70)$$

Then, we have

$$\begin{aligned} \sum_{k=0}^{\infty} \text{Ared}_k &= \sum_{k=0}^{\infty} [\Phi_k(x_k) - \Phi_k(x_{k+1})] \\ &= \sum_{k=0}^{\infty} (f_k - f_{k+1}) + \sum_{k=0}^{\infty} (-\lambda_k^T c_k + \lambda_k^T c_{k+1}) \\ &\quad + \frac{\sigma_0}{2} \sum_{k=0}^{\infty} (\|c_k\|^2 - \|c_{k+1}\|^2) \\ &\leq f_0 + (-\lambda_0^T c_0 + 4R_0 \|\lambda_{\max}\|) + \frac{\sigma_0}{2} \|c_0\|^2 \\ &\triangleq \bar{A}, \end{aligned} \quad (71)$$

where $\Phi_k(x)$ is defined by (5).

Secondly, we prove

$$\liminf_{k \rightarrow \infty} \|c_k\| = 0 \quad (72)$$

by contradiction. Suppose that there exists some $\tau > 0$ such that

$$\|c_k\| > \tau > 0, \quad \forall k \geq 0. \quad (73)$$

Equations (56) and (73) imply that

$$\text{Pred}_k \geq \delta_0 \sigma_0 \min \left\{ \frac{\max \{1, \|\tilde{g}_k\|\}}{\mu_k} \tau, \tau^2 \right\}. \quad (74)$$

Considering the sum of Pred_k on the index set S (see (57)), we have by (71) that

$$\sum_{k \in S} \text{Pred}_k \leq \frac{1}{\gamma_1} \sum_{k \in S} \text{Ared}_k = \frac{1}{\gamma_1} \sum_{k=0}^{\infty} \text{Ared}_k \leq \frac{1}{\gamma_1} \bar{A}. \quad (75)$$

It can be deduced by (74) and (75) that

$$\sum_{k \in S} \frac{\max\{1, \|\tilde{g}_k\|\}}{\mu_k} < +\infty \quad (76)$$

and thus

$$\begin{aligned} \sum_{k \in S} \frac{\|\tilde{g}_k\|}{\mu_k} &< +\infty, \\ \sum_{k \in S} \frac{1}{\mu_k} &< +\infty. \end{aligned} \quad (77)$$

If S is a finite set, then it follows from (57) and Step 2 that $\rho_k \geq \gamma_1$ and $\mu_{k+1} = \theta_3 \mu_k$ ($\theta_3 > 1$) hold for all sufficiently large k . Therefore, $1/\mu_k \rightarrow 0$, as $k \rightarrow \infty$. If S is an infinite set, the second inequality in (77) implies that $1/\mu_k \rightarrow 0$, as $k \rightarrow \infty$ and $k \in S$. From Step 2, we know that if $k \notin S$, then $\mu_{k+1} \geq \mu_k$. Hence, we have $1/\mu_k \rightarrow 0$, as $k \rightarrow \infty$. The fact that $1/\mu_k \rightarrow 0$ and (74) imply that

$$\text{Pred}_k \geq \delta_0 \sigma_0 \tau \frac{\|\tilde{g}_k\|}{\mu_k} \quad (78)$$

holds for all sufficiently large k . Hence it can be deduced by Lemma 9 that $\sum_{k \in S} (\|\tilde{g}_k\|/\mu_k)$ is divergent which contradicts the first part in (77).

Finally, we prove (66). If S is a finite set, then $\{x_k\}$ is convergent. Thus, (72) implies (66). From now on we assume that S is an infinite set. Suppose that (66) does not hold; then there exist an infinite index set $\tilde{K} = \{k_i\}$ ($\tilde{K} \subset S$) and a constant $c > 0$ such that

$$\|c_{k_i}\| > 2c \quad \text{holds } \forall k_i \in \tilde{K}. \quad (79)$$

By (72), there also exists an infinite index set $\hat{K} = \{l_i\}$ ($\hat{K} \subset S$) such that $k_i < l_i$,

$$\|c_k\| \geq c \quad \text{holds, } \forall k \text{ such that } k_i \leq k < l_i, \quad (80)$$

$$\|c_{l_i}\| < c \quad \text{holds } \forall l_i \in \hat{K}. \quad (81)$$

Let $\bar{K} = \{k \mid k \in S, \|c_k\| \geq c\}$; then $\bar{K} \subset \tilde{K}$ and $\bar{K}$ is an infinite index set,

$$\sum_{k \in \bar{K}} \text{Pred}_k \leq \sum_{k \in S} \text{Pred}_k. \quad (82)$$

Therefore, by (75), we have that

$$\sum_{k \in \bar{K}} \text{Pred}_k \leq \frac{1}{\gamma_1} \bar{A}. \quad (83)$$

With the help of (56), (80), and (83), we obtain that

$$\sum_{k \in \bar{K}} \frac{\max\{1, \|\tilde{g}_k\|\}}{\mu_k} < +\infty. \quad (84)$$

A direct conclusion which can be drawn by (84) is

$$\frac{1}{\mu_k} \rightarrow 0, \quad \text{as } k \in \bar{K}, k \rightarrow \infty. \quad (85)$$

Thus by Lemma 4, we have that, for all sufficiently large $k \in \bar{K}$,

$$\|s_k\| \leq 4 \frac{\|\tilde{g}_k\|}{\mu_k}. \quad (86)$$

Therefore, for sufficiently large j ,

$$\|x_{l_i} - x_{k_i}\| \leq \sum_{k_i \leq k < l_i, k \in \bar{K}} \|s_k\| \leq 4 \sum_{k_i \leq k < l_i, k \in \bar{K}} \frac{\|\tilde{g}_k\|}{\mu_k}. \quad (87)$$

Equations (84) and (87) imply that $\|x_{l_i} - x_{k_i}\| \rightarrow 0$, as $i \rightarrow \infty$. Therefore (79) contradicts (81). Thus we complete the proof. $\square$

Lemma 11. Suppose that (A1)-(A2) hold and $\sigma_k = \sigma_0$ for all $k \geq 0$; then we have

$$\liminf_{k \rightarrow \infty} \|\tilde{g}_k\| = 0. \quad (88)$$

Proof. Suppose that (88) does not hold. Then there exists $\tau > 0$, such that

$$\|\tilde{g}_k\| > \tau \quad \text{holds, } \forall k. \quad (89)$$

By (18), we have that

$$\text{Pred}_k \geq \frac{\tau^2}{4} \min \left\{ \frac{1}{\|\tilde{B}_k\|}, \frac{1}{\mu_k} \right\} \quad \text{holds } \forall k. \quad (90)$$

As $\{\tilde{B}_k\}$ is bounded above, similar to the second part in the proof of Lemma 10, we can conclude that

$$\frac{1}{\mu_k} \rightarrow 0, \quad \text{as } k \rightarrow \infty, \quad (91)$$

and thus

$$\text{Pred}_k \geq \frac{\tau^2}{4} \frac{1}{\mu_k} \quad \text{holds for all sufficiently large } k. \quad (92)$$

By (75) and (92), we have that $\sum_{k \in S} (1/\mu_k)$ is convergent and thus $\sum_{k \in S} (\|\tilde{g}_k\|/\mu_k)$ is also convergent as $\{\tilde{g}_k\}$ is bounded. However, Lemma 9, (91), (92), and the boundedness of $\{\tilde{g}_k\}$ deduce the divergence of $\sum_{k \in S} (\|\tilde{g}_k\|/\mu_k)$. This contradiction completes the proof. $\square$

With the help of Lemmas 10 and 11, we can easily obtain the following result.

Theorem 12. Suppose that (A1)-(A2) hold and $\sigma_k = \sigma_0$ for all $k \geq 0$; there exists an accumulation point of $\{x_k\}$ at which the KKT condition holds.

Note that, in Theorem 12, we do not suppose that RCPLD holds.

TABLE 1: Results of Algorithm 2 and ALGENCAN.

Name	n	m	Algorithm 2		ALGENCAN	
			n_f	n_g	n_f	n_g
AIRCRAFT	8	5	5	5	20	21
ARGTRIG	200	200	69	69	15	16
BOOTH	2	2	6	6	5	6
BROWNALE	200	200	8	8	21	22
BROYDN3D	5000	5000	23	23	22	23
BYRDSRPH	3	2	42	21	80	78
BT1	2	1	17	17	67	45
BT2	3	1	40	37	74	71
BT4	3	2	17	17	34	34
BT5	3	1	12	11	28	28
BT7	5	3	5	5	97	98
BT8	5	2	19	15	76	51
BT9	4	2	18	15	106	100
BT10	2	2	15	15	95	95
BT11	5	3	15	15	75	75
BT12	5	3	10	10	20	21
CHNRSNBE	50	98	100	37	19	20
CLUSTER	2	2	11	11	26	27
CUBENE	2	2	8	6	17	18
DECONVNE	63	40	6	6	3982	1195
EIGENB	7	7	2550	2550	2451	608
GOTTFR	2	2	6	6	24	24
HATFLDF	3	3	5	5	55	45
HEART6	6	6	122	61	387	216
HEART8	8	8	29	10	87	43
HS39	4	2	65	61	106	100
HS48	5	2	46	19	6	7
HYDCAR6	29	29	267	85	212	154
INTEGREQ	502	500	6	6	14	15
MARATOS	2	1	13	11	42	43
MWRIGHT	5	3	14	14	29	29
ORTHREGB	27	6	6	6	23	24
RECIPE	3	3	6	6	39	40
RSNBRNE	2	2	6	6	16	17
SINVALNE	2	2	6	6	8	8
TRIGGER	7	6	481	290	757	270
YFITNE	3	17	11	11	805	520
ZANGWIL3	3	3	6	6	5	6

4. Numerical Experiment

In this Section, we investigate the performance of Algorithm 2. We compare Algorithm 2 with the famous Fortran package ALGENCAN. In our computer program, the parameters in Algorithm 2 are chosen as follows:

$$\begin{aligned}
 \theta_1 &= 0.1, \\
 \theta_2 &= 1, \\
 \theta_3 &= 4, \\
 \gamma_1 &= 0.1, \\
 \gamma_2 &= 0.9, \\
 \sigma_0 &= 1,
 \end{aligned}$$

$$\delta_0 = 1,$$

$$\bar{\lambda} = 10^{20},$$

$$\underline{\lambda} = 10^{-20}.$$

(93)

We set B_k to be the exact Hessian of the Lagrangian $f(x) - \lambda^T c(x)$ at the point x_k . The Matlab subroutine *minres* is used to solve (15). All algorithms are terminated when one of the following conditions holds: (1) $\|g_k + A_k^T \lambda_k\| \leq 10^{-8}$ and $\|c_k\| \leq 10^{-8}$; (2) $\|g_k + A_k^T \lambda_k\| \leq 10^{-8}$ and $\|A_k^T c_k\| \leq 10^{-8}$; (3) $\|s_k\| \leq 10^{-8}$. All test problems are chosen from CUTEst collection [22].

The numerical results are listed in Table 1 where the name of problem is denoted by *Name*, the number of its variables

is denoted by n , the number of constraints is denoted by m , the number of function evaluations is denoted by n_f , and the number of gradient evaluations is denoted by n_g . In Table 1, we list the results of 38 test problems. Considering the numbers of function evaluations (n_f), Algorithm 2 is better than *ALGENCAN* for 30 cases (78.9%). Considering the numbers of gradient evaluations (n_g), Algorithm 2 is better than *ALGENCAN* for 31 cases (81.6%).

5. Conclusions

In this paper, we present a new algorithm for equality constrained optimization. We add an adaptive quadratic term to the quadratic model of the augmented Lagrangian function. In each iteration, we solve a simple unconstrained subproblem to obtain the trial step. The global convergence is established under reasonable assumptions.

From the numerical results and the theoretical analysis, we believe that the new algorithm can efficiently solve equality constrained optimization problems.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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