Research Article

Neimark-Sacker Bifurcation in Demand-Inventory Model with Stock-Level-Dependent Demand

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An analysis of dynamics of demand-inventory model with stock-level-dependent demand formulated as a three-dimensional system of difference equations with four parameters is considered. By reducing the model to the planar system with five parameters, an analysis of one-parameter bifurcation of equilibrium points is presented. By the analytical method, we prove that nondegeneracy conditions for the existence of Neimark-Sacker bifurcation for the planar system are fulfilled. To check the sign of the first Lyapunov coefficient of Neimark-Sacker bifurcation, we use numerical simulations. We give phase portraits of the planar system to confirm the previous analytical results and show new interesting complex dynamical behaviours emerging in it. Finally, the economical interpretation of the system is given.

1. Introduction

Great economic development after the second world war has released a need of development of mathematical methods to support optimization of economic and business processes. Material flow, production, and inventory are aspects of a business, which, to make it profitable, need to be optimized. Therefore, many models of supply chain were created in the mid-20th century. To mention the most noticeable, we ought to list works of Wagner and Whitin [1], Brown [2], and Holt et al. (HMMS model) [3]. They have laid the foundation for supply chain modelling. Those models, despite being relatively simple, have become an inspiration for contemporary researchers: to redefine models in order to fit to the current challenges and to analyse them using available computation power, for example, [4–6].

Prediction of future demand and inventory is an important aspect of running and managing manufacturing or trade company. Methods supporting those tasks have been developed by economists already in the mid-20th century; nonetheless, they are still being improved, as economy still changes and creates new challenges. One of those methods is modelling of economic phenomena using mathematical formulations. The topic of demand and inventory needs a contextual approach, since many factors can influence it and different views may be needed. Therefore, the models are created with the usage of different mathematical tools. We can mention here recent works related to demand and inventory that investigate and describe specific economic cases: Chen and Hu in [7] consider an inventory and pricing model and develop polynomial time algorithms to maximize the total profit; Mondal et al. in [8] consider generic algorithm for the case of inventory of a deteriorating item; Qi et al. in [4] consider supply chain that experiences a disruption in demand during the planning horizon.


The model describes demand and inventory of a product at one echelon of supply chain at the retailer. The considered supply chain consists of three echelons: manufacturer,
retailer, and customers. The following rules are applied to the model: customers buy a good from a retailer; a retailer orders a product in the forecasted amount and forecast is prepared using single exponential smoothing Brown model [12]; a manufacturer produces and delivers exactly the ordered amount and the production capacity is unlimited; customers’ demand depends on a retail price, which can be changed by a discount; price cannot be arbitrarily changed but the retailer can offer a discount depending on stock volume.

The model takes a form of the following system of difference equations:

\[
\begin{align*}
D_{n+1} &= \left[ \frac{AT}{(A + 1)T - S_n} \right]^k D_n, \\
S_{n+1} &= S_n - D_n + \hat{D}_n, \\
\hat{D}_{n+1} &= \alpha D_n + (1 - \alpha) \hat{D}_n,
\end{align*}
\]

where \( n = 0, 1, 2, \ldots \) indicates instance of time; \( S_n \) is a stock volume at \( n \); \( S_n \) is a real number; interpretation of nonnegative values is obvious and the negative value of stock informs price, discount, and demand, price elasticity coefficient that regulates dependence between \( T \) coefficient of Brown model, \( S \) between current stock and forecasted demand.

\[
\begin{align*}
\hat{D}_n &= \frac{\alpha D_n + (1 - \alpha) \hat{D}_n}{(A + 1)T - S_n},
\end{align*}
\]

with \( x^n, z^n \geq 0, y_n < (A + 1)T \) and as \( (x_{n+1}, y_{n+1}, z_{n+1}) = f(x_n, y_n, z_n) \) with a mapping:

\[
f(x, y, z) = \left( \frac{(A T)^k}{((A + 1)T - y)^k} x, y - x + z, \alpha x + (1 - \alpha) z \right).
\]

It is easy to see that \((0, y, 0) \) for \( y < (A + 1)T \) and \((x, T, x) \) for \( x \geq 0 \) are equilibrium points of the system \((x_{n+1}, y_{n+1}, z_{n+1}) = f(x_n, y_n, z_n)\).

Let us shortly elaborate on the meaning of those equilibrium points. The goal of the retailer is to reach target stock \( T \). If additionally actual demand \( x_n \) and forecasted demand \( z_n \) are equal and greater than zero, the retailer's stock level does not change in the next period, which means that system (2) is in the equilibrium point \((x, T, x) \). On the other hand, when \( x_n = z_n = 0 \), the retailer's stock level \( y \) cannot be changed. This means that the retailer remains in equilibrium point \((0, y, 0) \).

We recall that \( B^{f}_{INV} \) is the positive invariant set of the system \((x_{n+1}, y_{n+1}, z_{n+1}) = f(x_n, y_n, z_n) \) if \( B^{f}_{INV} \subseteq D_f \) and \( f(B^{f}_{INV}) \subseteq B^{f}_{INV} \).

**Proposition 1** (see [11]). Here,

\[
B^{f}_{INV} = \{(x, y, z) \in \mathbb{R}^3 : x, z \geq 0, y + \alpha^{-1} z < (A + 1)T\}
\]

is the positive invariant set of the system \((x_{n+1}, y_{n+1}, z_{n+1}) = f(x_n, y_n, z_n) \), where \( f \) is given by (3).

One can observe that the second and the third equation of system (2) are linear ones. Therefore, we treat them with some linear transformations in order to obtain a more convenient form for analysis: both sides of the second equation.
multiplied by \( \alpha \) and the second and the third equation added. As an output, we get
\[
\alpha (y_{n+1} - y_n) = z_n - z_{n+1},
\]
which enables us to state the following.

**Corollary 2.** System (2) has the invariant plane \( \alpha y + z = \alpha y_1 + z_1 \) for any \( y_1, z_1 \in \mathbb{R} \), \( z_1 \geq 0 \), \( \alpha y_1 + z_1 < \alpha(A+1)T \). Hence, system (2) can be reduced to a family of planar systems:
\[
x_{n+1} = \left[ \frac{AT}{(A+1)T - y_n} \right]^k x_n, \quad y_{n+1} = (1 - \alpha) y_n - x_n + as,
\]
with \( s < (A+1)T \), where \( s := y_1 + z_1/\alpha \), \( y_1 \in \mathbb{R} \), \( z_1 \geq 0 \).

**Proof.** From (5), we get the first-order recursion of the form \( \alpha y_{n+1} + z_{n+1} = \alpha y_1 + z_1 \) which implies that \( \alpha y_n + z_n = \alpha y_1 + z_1 \). Constraint of the expression \( \alpha y_1 + z_1 \) is a direct consequence of the positive invariant set construction.

Let us later write our systems (6) in the form of a family of mappings:
\[
g_0(x, y) = \left( \left[ \frac{AT}{(A+1)T - y} \right]^k x, (1 - \alpha) y - x + as \right),
\]
for \( s < (A+1)T \). Taking into consideration economic interpretation, \( g_0 : D_{g_0} \rightarrow \mathbb{R}^2 \), where \( D_{g_0} := [0, \infty) \times (-\infty, (A+1)T) \).

**Proposition 3.** Let \( A, T, k > 0, \alpha \in (0, 1), \) and \( s < (A+1)T \). The set \( B_{\text{inv}}^{g_0} = D_{g_0} \) is the positive invariant set of the system \((x_{n+1}, y_{n+1}) = g_0(x_n, y_n)\), where \( g_0 \) is given by (7).

**Proof.** Let \( s < (A+1)T \). For any \( x \geq 0, y < (A+1)T \), we have \((g_0)_1(x, y) \geq 0 \) and \((g_0)_2(x, y) = (1 - \alpha) y - x + as < (1 - \alpha)(A+1)T + \alpha(A+1)T = (A+1)T \).

Mapping (7) may have at most two equilibrium points:

(i) \((0, s)\)

(ii) \((\alpha(s - T), T)\) if and only if \( s > T \)

Stability of the equilibrium points of (2) has been examined in [11]. Based on the findings of [11], one can formulate the conclusions about the existence and stability of equilibrium points of planar systems (6), which are presented in Table 1. Now, using these results, we analyse the dynamics of the planar systems using bifurcation theory. We are especially interested in proving the existence of Neimark-Sacker bifurcation.

Graphs illustrating the cases in Table 1 are presented in Figures 1–4.

The planar system is dependent on five parameters \( \alpha \), \( A, T, k, \) and \( s \). The most interesting case in the bifurcation analysis for the planar system (6) is dependence on parameter \( s \) with fixed \( A_0, T_0, k_0 > 0, \alpha_0 \in (0, 1) \), which now we will carry on.

From the conditions in Table 1, in order to examine bifurcations on parameter \( s \), we have to consider two cases dependent on \( k_0 \).

Case 1 \((k_0 \in (0, 1])\). For \( s < T_0 \), the equilibrium point \((0, s)\) is the only equilibrium point of the system \((x_{n+1}, y_{n+1}) = g_0(x_n, y_n)\) which is stable. For \( s > T_0 \), the considered system possesses two equilibrium points: unstable \((0, s)\) and stable \((\alpha_0(s - T_0), T_0)\). This suggests that \( s = T_0 \) is the bifurcation parameter for the transcritical bifurcation.

Case 2 \((k_0 > 1)\). For \( s = T_0 \), probably the transcritical bifurcation occurs like in the previous case. For \( s < (1 + A_0/k_0)T_0 \), the equilibrium point \((\alpha_0(s - T_0), T_0)\) is stable and loses its stability for \( s > (1 + A_0/k_0)T_0 \). We prove that, for \( s = (1 + A_0/k_0)T_0 \), the Neimark-Sacker bifurcation occurs. We use classical analytical results from [26] and numerical simulations. For the sake of convenience, let us quote from [26] Theorem 4.5 as Lemma 4 and Theorem 4.6 as Lemma 5.

**Lemma 4** (Theorem 4.5, [26]). Suppose that a two-dimensional discrete-time system \( x \rightarrow f(x, \alpha) \), \( x \in \mathbb{R}^2, \alpha \in \mathbb{R}^1 \), with smooth \( f \), has, for all sufficiently small \( |\alpha| \), the fixed point \( x = 0 \) with multipliers \( \mu_1(\alpha) = r(\alpha)e^{i\phi(\alpha)} \), where \( r(0) = 1, \phi(0) = \theta_0 \). Let the following conditions be satisfied:

\[(C.1) \quad r'(0) \neq 0.
\]
\[(C.2) \quad e^{ik\theta_0} \neq 1 \quad \text{for } k = 1, 2, 3, 4.
\]

Then, there are smooth invertible coordinate and parameter changes transforming the system into
\[
\begin{pmatrix}
y_1' \\
y_2'
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 + \beta \\
\beta
\end{pmatrix} \begin{pmatrix}
\cos \theta(\beta) & -\sin \theta(\beta) \\
\sin \theta(\beta) & \cos \theta(\beta)
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
\]
\[
+ \begin{pmatrix}
y_1^2 + y_2^2
\end{pmatrix}
\begin{pmatrix}
\cos \theta(\beta) & -\sin \theta(\beta) \\
\sin \theta(\beta) & \cos \theta(\beta)
\end{pmatrix}
\begin{pmatrix}
\cos \theta(\beta) & -\sin \theta(\beta) \\
\sin \theta(\beta) & \cos \theta(\beta)
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
\]
\[
+ O(\|y\|^4),
\]
with \( \theta(0) = \theta_0 \) and \( a(0) = \text{Re}(e^{-i\theta_0}c_1(0)) \), where \( c_1(0) \) is given by formula (4.21) in [26].
Sample orbits for $s = 800$

Sample orbits for $s = T = 1000$

Sample orbits for $s = T = 1000$ and surrounding of the equilibrium point

Sample orbits for $s = 1100$ and $k < 1$

Sample orbits for $s = 1200$

Figure 1: Examples illustrating conditions (C1) and (C2) from Table 1.

Figure 2: Example illustrating condition (C2) from Table 1.

Figure 3: Examples illustrating conditions (C3) and (C4) from Table 1.
Lemma 5 ((Theorem 4.6, [26]) (generic Neimark-Sacker bifurcation)). For any generic two-dimensional one-parameter system $x \rightarrow f(x, \alpha)$, having at $\alpha = 0$ the fixed point $x_0 = 0$ with complex multipliers $\mu_{1,2}(\alpha) = e^{i\theta_\alpha}$, there is a neighbourhood of $x_0$ in which a unique closed invariant curve bifurcates from $x_0$ as $\alpha$ passes through zero.

Remark 6 (see [26]). The genericity conditions assumed in the theorem are the transversality condition (C.1) and the nondegeneracy condition (C.2) from Lemma 4 and the additional nondegeneracy condition:

$$(C.3) \ a(0) \neq 0.$$ 

We start with translation mapping $g_s$ to the mapping which possesses fixed point in $(0, 0)$ for sufficiently small $s$. We consider the $C^2$ mapping $G_s : \mathbb{R} \times (-\infty, A_0 T_0) \rightarrow \mathbb{R}^2$ with parameter $s \in \mathbb{R}$:

$$G_s(x, y) := \left( \frac{(A_0 T_0)^s x}{(A_0 T_0 - y)} (x + a_0 \left(s + \frac{A_0 T_0}{k_0}\right)) \right) - a_0 \left(s + \frac{A_0 T_0}{k_0}\right), (1 - a_0)y - x \right).$$

The Jacobian of $G_s$ at $(0, 0)$ has the form

$$J G_s(0, 0) = \begin{bmatrix} 1 & a_0 k_0 A_0 T_0 \alpha_0 \left(s + \alpha_0 \right) \\ 1 - a_0 & 1 \\ \end{bmatrix},$$

with the complex eigenvalues for sufficiently small $s$ of the form

$$\lambda_{1,2}(s) = 1 - \frac{1}{2}a_0 + \frac{i}{2} \sqrt{4\alpha_0 - \alpha_0^2} \left(\frac{k_0}{A_0 T_0} s + 1\right) - \alpha_0^2,$$

and modulus

$$r(s) = \sqrt{1 + \frac{\alpha_0 k_0}{A_0 T_0} s},$$

Notice that $r(0) = 1$, $r'(0) = \alpha_0 k_0 / 2A_0 T_0 \neq 0$, which means that condition (C.1) of Lemma 4 is satisfied. Now, we check that condition (C.2) (called no strong resonance) is satisfied. We have to prove that $e^{i\theta_0} \neq 1$, where $I = 1, 2, 3, 4$ and $\theta_0$ is such that $\lambda_{1,2}(0) = r(0)e^{i\theta_0}$. We get for $\alpha_0 \in (0, 1)$

$$1 - \frac{1}{2}a_0 + \frac{i}{2} \sqrt{4\alpha_0 - \alpha_0^2} = e^{i\theta_0} \neq 1,$$

$$+ i \left(1 - \frac{1}{2}a_0\right) \sqrt{4\alpha_0 - \alpha_0^2} = e^{2i\theta_0}$$

$$1 - \frac{1}{2}a_0 (\alpha_0 - 3)^2 + i \left(\alpha_0^2 - 4\alpha_0 + 3\right) \sqrt{4\alpha_0 - \alpha_0^2}$$

$$= e^{3i\theta_0} \neq 1,$$

$$1 + \frac{1}{2}a_0 (\alpha_0^3 - 8\alpha_0^2 + 20\alpha_0 - 16)$$

$$+ i \left(\alpha_0^3 - 3\alpha_0^2 - 5\alpha_0 + 2\right) \sqrt{4\alpha_0 - \alpha_0^2} = e^{4i\theta_0}$$

$$\neq 1,$$

because $2, 2 \pm \sqrt{2}$ are solutions to equation $-(1/2)\alpha_0^3 + 3\alpha_0^2 - 5\alpha_0 + 2 = 0$. Fulfilling conditions (C.1) and (C.2) from Lemma 4 means that nondegeneracy conditions are satisfied. To get the existence of Neimark-Sacker bifurcation, we have to check the last condition (C.3) of the form $a(0) = \text{Re}[\alpha_0 \alpha_0^2 c(0)] \neq 0$, named the first Lyapunov coefficient, given in Lemma 5. If $a(0) < 0$, we get the supercritical Neimark-Sacker bifurcation, and if $a(0) > 0$ we get the subcritical Neimark-Sacker bifurcation. Due to complicated conditions for any parameters $A_0, T_0 > 0$, $\alpha_0 \in (0, 1)$, and $k_0 > 1$, we check the sign of the first Lyapunov coefficient only for specific $A_0, T_0, \alpha_0$, and $k_0$ in Section 3.

Remark 7. In the analogous way, we can prove the existence of the Neimark-Sacker one-parameter bifurcation of the planar system for any other parameters $A, T, k$, and $\alpha$ with the
bifurcation parameter which fulfills the equality \( s = (1 + A/k)T \).

### 3. Numerical Simulations

In this section, by using numerical simulation, we give the bifurcation diagrams and phase portraits of system (6) to confirm the previous analytical results and show new interesting complex dynamical behaviours emerging for system (6).

Now, let us give illustration to the cases provided in Table 1. Graphs in Figures 1–4 depict behaviour of (6) depending on the parameter \( s \). The other parameters remain unchanged at \( A = 1, T = 1000, \alpha = 0.5 \), and, for C1,
Figure 7: Diagram of bifurcation of the equilibrium points with respect to $y$, for the parameter $s$ varying from 800 to 2000, $x_1 = 0.1$, $y_1 = s$ for $A = 1, T = 1000, \alpha = 0.5,$ and $k = 4$.

Figure 8: Diagrams of bifurcation of the equilibrium points with respect to $x$ for the parameter $k$ varying from 4.95 to 6.15, $x_1 = 100.1$, $y_1 = 1000$ for $A = 1, T = 1000, \alpha = 0.5,$ and $s = 1200$. 
Figure 9: Diagrams of bifurcation of the equilibrium points with respect to \( y \) for the parameter \( k \) varying from 4.95 to 5.85, \( x_1 = 100.1 \), \( y_1 = 1000 \), for \( A = 1 \), \( T = 1000 \), \( \alpha = 0.5 \), and \( s = 1200 \).

Figure 10: Diagrams of bifurcation of the equilibrium points with respect to \( x \) (a) and \( y \) (b), for the parameter \( A \) varying from 0 to 1.2, \( x_1 = 0.1 \), \( y_1 = s \) for \( s = 1250 \), \( T = 1000 \), \( \alpha = 0.5 \), and \( k = 4 \).

C2, C4, and C5, \( k = 4 \); for (C3), \( k = 0.5 \). The graphs clearly confirm the analytical results.

Interesting behaviour can be observed at border point \( s = T \) depicted in Figures 1-2 in Graphs (C2a-b). At the first glance, in Graph (C2a), one can assume stability of the equilibrium point \((0, s)\), which would be a contradiction of the analytical result. However, looking from a closer perspective in Graph (C2b) one can see that the orbits leave the surroundings of the equilibrium point, herewith confirming the instability proved with analytical methods.

Let us have again a look at the border case of \( s = (1 + A/k)T \) depicted in Figure 5. In each of the graphs, the same parameters are applied in order to observe the behaviour of the system for varying initial conditions of \( x \) and \( y \) (\( A = 1 \), \( T = 1000 \), \( \alpha = 0.5 \), \( s = 1250 \), and \( k = 4 \)). Each time, we set the initial conditions closer and closer to the equilibrium
Figure 11: Diagram of bifurcation of the equilibrium points with respect to \( x \) (a) and \( y \) (b), for the parameter \( \alpha \) varying from 0 to 0.3, \( x_1 = 100.1, y_1 = s = 1230 \) for \( A = 1, T = 1000 \), and \( k = 4 \).

Point \((x^{**}, y^{**})\). Appearance of this kind of circle suggests supercritical bifurcation of Neimark-Sacker type. Indeed, negative sign of the first Lyapunov coefficient \( a(0) \) obtained numerically confirms this suggestion.

4. Economical Interpretation

System (6) depends on five parameters: \( A, \alpha, T, k, \) and \( s \). Each of the parameters carries the economical meaning described in Section 1. In a business practice, the parameters \( T \) and \( s \) are set in advance. Nonetheless, as it is shown in Table 1, changes of parameter \( s \) cause the change of behaviour of the equilibrium points. Therefore, we present bifurcation diagrams of \( x \) and \( y \) as \( s \) changes in Figures 6 and 7.

Parameter \( k \), which represents price elasticity, is defined depending on the types of a product and a market and often is determined experimentally and must be adjusted frequently. Therefore, it is interesting to analyse behaviour of system (6) at varying values of this parameter. For that purpose, let us illustrate changes of the dynamics of the equilibrium points in the bifurcation diagrams (see Figures 8 and 9). Model (6) includes steering mechanism, which allows the user to influence the behaviour of the system. With the parameter \( \alpha \), one can decide on the share of influence between the forecast and actuals of sales. The less \( \alpha \) is, the greater the impact of actuals is, and vice versa. Figure 11 presents bifurcation diagram for changing parameter \( \alpha \). Similarly, in Figure 10, we present bifurcation for varying \( A \). A is a parameter to control the influence of the discrepancy between the actual stock and the target one \( T \).

Business Example. Let us consider a product \( H_R \) that follows demand(D)-price(p) dependence \( D_2/D_1 = (p_2/p_1)^k \) with price elasticity \( k = 4 \). The target \( T = 1000 \) is set on stock volume and assumed demand at least \( z_1 = 50 \). Actual demand is insufficient \( x_1 = 1 \), and hence the actual stock exceeds the target \( y_1 = 1100 > T \). Smoothing coefficient \( \alpha = 0.5 \), which means calculation of the amount to be delivered takes into account actuals and forecast with equal wages. Parameter \( A = 1 \). One can easily calculate \( s = y_1/z_1/\alpha = 1200 \). From Table 1, it is observed that the product satisfies condition (C4), and hence the system has two equilibrium points \((0, 1200)\) and \((100, 1000)\), unstable and stable, respectively.

Additional Points

Numerical analysis and graph plotting have been performed using Matcontm (according to [27]) and Matlab R2016a. The bifurcation diagrams depict 200 iterations starting from iteration \( n = 801 \).

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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