Research Article

Numerical Approach Based on Two-Dimensional Fractional-Order Legendre Functions for Solving Fractional Differential Equations

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Received 23 August 2016; Revised 2 December 2016; Accepted 18 December 2016; Published 23 January 2017

Academic Editor: Vincenzo Scalzo

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In this paper, a robust, effective, and accurate numerical approach is proposed to obtain the numerical solution of fractional differential equations. The principal characteristic of the approach is the new orthogonal functions based on shifted Legendre polynomials to the fractional calculus. Also the fractional differential operational matrix is driven. Then the matrix with the Tau method is utilized to transform this problem into a system of linear algebraic equations. By solving the linear algebraic equations, the numerical solution is obtained. The approach is tested via some examples. It is shown that the FLF yields better results. Finally, error analysis shows that the algorithm is convergent.

1. Introduction

Fractional-order calculus has a long history. Compared with integer order, fractional differential equations show many advantages over the simulation of physical phenomena [1]. In recent decades, fractional partial differential equations (FPDEs) [2] are widely used to describe engineering processes and dynamical systems; more and more researchers have devoted to study a variety of methods for solving fractional differential equations. The most commonly used methods are generalized differential transform method [3], Adomian decomposition method [4], finite difference method [5], and so on. Recently, the authors have proposed wavelet methods for solving a class of fractional convection-diffusion equations. In [6], different families of wavelets [7–11] have been widely applied for solving problems of partial differential equations. In [12], the authors have obtained the approximate solutions of fractional differential equations using the generalized block pulse operational matrix.

The methods based on the orthogonal functions [13–15] are powerful and wonderful for solving FPDEs and have achieved great success in this field. Now the operational

matrices of fractional derivative and integral [14–16] for Bernstein polynomials and Jacobi polynomials [15] have been derived. However, these polynomials use integer power series to approximate fractional ones; it cannot accurately represent properties of fractional integral and differential. Recently, in [17], Rida and Yousef have proposed a fractional extension of classical Legendre polynomials by replacing the integer order derivative in Rodrigues formula [18] with fractional-order derivatives. These functions are complex, so they are not unsuitable for solving FPDEs. Subsequently, Kazem et al. put forward the orthogonal fractional-order Legendre functions based on shifted Legendre polynomials. In [19], they used these functions to obtain the numerical solutions of fractional partial differential equations. Numerical results show that their methods are effective, accurate, and easy to implement. In this paper, FLF is suitable to characterize the properties of fractional partial differential equations; thus we attempt to use fractional-order Legendre functions to acquire numerical solution of the FPDEs.

The article is organized as follows. In Section 2, we introduce some basic definitions of fractional derivatives and integrals. In Section 3, we construct the FLFs and introduce

some properties. In Section 4, the fractional operational matrices of FLFs are obtained. In Section 5, we use the proposed method to solve FPDEs. In Section 6, the proposed method is tested through several numerical examples. In Section 7, the error analysis is given. Finally Section 8 concludes the paper.

2. Preliminaries and Notations

In this section, we recall the essentials of the fractional calculus theory that will be used in this article.

Definition 1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined as

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad \alpha > 0, \quad (1)$$

$$I^0 u(t) = u(t).$$

Definition 2. The Riemann-Liouville definition of fractional differential operator is given as follows:

$$D_*^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{u(s)}{(t-s)^{\alpha-m+1}} ds, & \alpha > 0, m-1 \leq \alpha < m, \\ \frac{d^m u(t)}{dt^m}, & \alpha = m. \end{cases} \quad (2)$$

$$(i) \quad D^\alpha C = 0,$$

$$(ii) \quad D^\alpha t^\nu = \begin{cases} 0, & \text{for } \nu \in N_0, \nu < [\alpha], \\ \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\alpha)} t^{\nu-\alpha}, & \text{for } \nu \in N_0, \nu \geq [\alpha] \text{ or } \nu \notin N_0, \nu > [\alpha]. \end{cases} \quad (5)$$

$$(iii) \quad D^\alpha \left(\sum_{i=0}^m c_i u_i(t) \right) = \sum_{i=0}^m c_i D^\alpha u_i(t), \quad \text{where } \{c_i\}_{i=0}^m \text{ are constants.}$$

Definition 4 (generalized Taylor's formula). Suppose that $D^{i\alpha} u(t) \in C[0, 1]$ for $i = 0, 1, \dots, m-1$, then one has

$$u(t) = \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} u(0^+) + \frac{t^{m\alpha}}{\Gamma(m\alpha+1)} D^{m\alpha} u(\xi), \quad (6)$$

where $0 < \xi \leq t, \forall t \in [0, 1]$. Also, one has

$$\left| u(t) - \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} u(0^+) \right| \leq M_\alpha \frac{t^{m\alpha}}{\Gamma(m\alpha+1)}, \quad (7)$$

where $M_\alpha \geq |D^{m\alpha} u(\xi)|$.

In case $\alpha = 1$, the generalized Taylor's formula (6) is the classical Taylors formula.

Definition 3. The Caputo definition of fractional differential operator is defined as

$$D^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{u^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, & m-1 \leq \alpha < m, \\ \frac{d^m u(t)}{dt^m}, & \alpha = m. \end{cases} \quad (3)$$

The relation between the Riemann-Liouville operator and Caputo operator is given by the following expressions:

$$D^\alpha I^\alpha u(t) = u(t),$$

$$I^\alpha D^\alpha u(t) = u(t) - \sum_{k=0}^{m-1} u^{(k)}(0^+) \frac{t^k}{k!}. \quad (4)$$

For $\alpha \geq 0, \nu \geq -1$, and constant C , Caputo fractional derivative has some basic properties which are needed here as follows:

3. Definitions of Fractional-Order Legendre Definition and Function Approximation

3.1. Shifted Legendre Polynomials. The well-known Legendre polynomials are orthogonal with the weight function $\omega(x) = 1$, which is defined on the interval $[-1, 1]$; namely,

$$\int_{-1}^1 P_m(x) P_n(x) \omega(x) dx = \begin{cases} \frac{2}{(2n+1)}, & m = n, \\ 0, & m \neq n. \end{cases} \quad (8)$$

In order to use these polynomials on the interval $[0, 1]$, let $t = 2x - 1$, and then we derived the shifted Legendre polynomials $P_n(2x - 1)$ which are denoted by $L_n(t)$. The

shifted Legendre polynomials are orthogonal with the weight function $\omega_l(t) = 1$ over $[0, 1]$, with the orthogonally property

$$\int_0^1 L_i(t) L_j(t) \omega_l(t) dt = \begin{cases} 0, & i \neq j, \\ \frac{1}{2i+1}, & i = j. \end{cases} \quad (9)$$

Then $L_i(t)$ can be obtained as follows:

$$L_{i+1}(t) = \frac{(2i+1)}{i+1} L_i(t) - \frac{i}{i+1} L_{i-1}(t), \quad i = 1, 2, \dots, \quad (10)$$

$$L_0(t) = 1,$$

$$L_1(t) = 2t - 1.$$

The analytic form of the shifted Legendre polynomial $L_n(t)$ of degree n is given by

$$\begin{aligned} L_n(t) &= \sum_{i=0}^n b_{ni} t^i, \\ L_n(0) &= (-1)^n, \\ L_n(1) &= 1, \end{aligned} \quad (11)$$

where $b_{ni} = (-1)^{n+i}((n+i)!/(n-i)!(i!)^2)$.

3.2. Fractional-Order Legendre Definition. We define the fractional-order Legendre functions (FLFs) by introducing the change of variable $t = x^\alpha$ and $\alpha > 0$ on shifted Legendre polynomials. The fractional-order Legendre function $L_i(x^\alpha)$ is denoted by $Fl_i^\alpha(x)$. The fractional-order Legendre functions are a particular solution of the normalized eigenfunctions of the singular Sturm-Liouville problem.

$$\begin{aligned} \left((x - x^{1+\alpha}) Fl_i^{\prime\alpha}(x) \right)' + \alpha^2 i(i+1) x^{\alpha-1} Fl_i(x) &= 0, \\ x &\in [0, 1]. \end{aligned} \quad (12)$$

Then $Fl_i^\alpha(x)$ can be obtained as follows:

$$\begin{aligned} Fl_{i+1}^\alpha(x) &= \frac{(2i+1)(2x^\alpha - 1)}{i+1} Fl_i^\alpha(x) - \frac{i}{i+1} Fl_{i-1}^\alpha(x), \\ i &= 1, 2, \dots, \end{aligned} \quad (13)$$

$$Fl_0^\alpha(x) = 1,$$

$$Fl_1^\alpha(x) = 2x^\alpha - 1.$$

We can derive the analytic form of $Fl_i^\alpha(x)$ of degree $i\alpha$ as follows:

$$Fl_i^\alpha(x) = \sum_{s=0}^i b_{s,i} x^{s\alpha}, \quad (14)$$

where $b_{s,i} = (-1)^{i+s}(i+s)!/(i-s)!(s!)^2$ and $Fl_i^\alpha(0) = (-1)^i$, $Fl_i^\alpha(1) = 1$.

Theorem 5. The FLFs are orthogonal with the weight function $\omega_l^\alpha(x) = x^{\alpha-1}$ on the interval $[0, 1]$; then the orthogonally condition is

$$\int_0^1 Fl_n^\alpha(x) Fl_m^\alpha(x) \omega_l^\alpha(x) dx = \frac{1}{(2n+1)\alpha} \delta_{nm}. \quad (15)$$

Proof. With $\int_0^1 L_n(t) L_m(t) \omega_l(t) dt = (1/(2i+1)) \delta_{nm}$, where δ_{nm} is the Kronecker function, let $t = x^\alpha$, and then we have

$$\begin{aligned} \int_0^1 L_n(t) L_m(t) dt &= \int_0^1 L_n(x^\alpha) L_m(x^\alpha) \alpha x^{\alpha-1} dx \\ &= \frac{1}{2n+1} \delta_{nm}, \\ \int_0^1 L_n(x^\alpha) L_m(x^\alpha) \alpha x^{\alpha-1} dx &= \frac{1}{2n+1} \delta_{nm}, \end{aligned} \quad (16)$$

$$\int_0^1 Fl_n^\alpha(x) Fl_m^\alpha(x) \alpha x^{\alpha-1} dx = \frac{1}{2n+1} \delta_{nm},$$

$$\int_0^1 Fl_n^\alpha(x) Fl_m^\alpha(x) x^{\alpha-1} dx = \frac{1}{(2n+1)\alpha} \delta_{nm}.$$

□

3.3. Function Approximation. Suppose $u(x) \in L^2[0, 1]$; it can be expanded in terms of the fractional-order Legendre functions as follows:

$$u(x) = \sum_{i=0}^{\infty} c_i Fl_i^\alpha(x), \quad (17)$$

where the coefficients c_i are obtained by

$$\begin{aligned} c_i &= \alpha(2i+1) \int_0^1 Fl_i^\alpha(x) u(x) \omega_l^\alpha(x) dx, \\ i &= 0, 1, 2, \dots \end{aligned} \quad (18)$$

If we consider truncated series in (17), we obtain

$$u(x) \approx u_m(x) = \sum_{i=0}^{m-1} c_i Fl_i^\alpha(x) = \mathbf{C}^T \mathbf{\Phi}(x), \quad (19)$$

where $\mathbf{C}$ and $\mathbf{\Phi}(x)$ are $m \times 1$ vector given by

$$\begin{aligned} \mathbf{C} &= [c_0, c_1, \dots, c_{m-1}]^T, \\ \mathbf{\Phi}(x) &= [Fl_0^\alpha(x), Fl_1^\alpha(x), \dots, Fl_{m-1}^\alpha(x)]^T. \end{aligned} \quad (20)$$

Theorem 6. Suppose $D^{i\alpha}u(x) \in C[0, 1]$ for $i = 0, 1, 2, \dots, m-1$. $(2m+1)\alpha \geq 1$ and $\mathbf{P}_m^\alpha = \text{span}\{Fl_0^\alpha(x), Fl_1^\alpha(x), \dots, Fl_{m-1}^\alpha(x)\}$. If $u_m(x) = \mathbf{C}^T \mathbf{\Phi}(x)$ is the best approximation to $u(x)$ from $\mathbf{P}_m^\alpha$, then the error bound is presented as follows:

$$\|u(x) - u_m(x)\|_\omega \leq \frac{M_\alpha}{\Gamma(m\alpha+1)} \sqrt{\frac{1}{(2m+1)\alpha}}, \quad (21)$$

where $M_\alpha \geq |D^{m\alpha}u(x)|$, $x \in [0, 1]$.

Proof. Consider the generalized Taylor's formula

$$u(x) = \sum_{i=0}^{m-1} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} u(0^+) + \frac{x^{m\alpha}}{\Gamma(m\alpha+1)} D^{m\alpha} u(\xi), \quad (22)$$

where $0 < \xi < x$, $x \in [0, 1]$.

With Definition 4

$$\left| u(x) - \sum_{i=0}^{m-1} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+) \right| \leq M_\alpha \frac{x^{m\alpha}}{\Gamma(m\alpha+1)}. \quad (23)$$

Since $u_m(x) = \mathbf{C}^T \Phi(x)$ is the best approximation to u from $\mathbf{P}_m^\alpha$ and $\sum_{i=0}^{m-1} (x^{i\alpha}/\Gamma(i\alpha+1)) D^{i\alpha} f(0^+) \in \mathbf{P}_m^\alpha$, hence

$$\begin{aligned} \|u(x) - u_m(x)\|_\omega^2 &\leq \left\| u(x) - \sum_{i=0}^{m-1} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} u(0^+) \right\|_\omega^2 \\ &\leq \frac{M_\alpha^2}{\Gamma(m\alpha+1)^2} \int_0^1 x^{2m\alpha} x^{\alpha-1} dx, \\ \|u(x) - u_m(x)\|_\omega^2 &\leq \frac{M_\alpha^2}{\Gamma(m\alpha+1)^2} \int_0^1 x^{(2m+1)\alpha-1} dx \\ &= \frac{M_\alpha^2}{\Gamma(m\alpha+1)^2 (2m+1)\alpha}. \end{aligned} \quad (24)$$

Now by taking the square roots, then the theorem is proved. $\square$

For arbitrary function $u(x, t) \in L^2([0, 1] \times [0, 1])$, they can be expanded as the following formula:

$$u(x, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij} Fl_i^\alpha(x) Fl_j^\beta(t), \quad (25)$$

where

$$u_{ij} = (2i+1)(2j+1)\alpha\beta \int_0^1 \int_0^1 u(x, t) Fl_i^\alpha(x) Fl_j^\beta(t) \cdot \omega_l^\alpha(x) \omega_l^\beta(t) dx dt, \quad i, j = 0, 1, \dots \quad (26)$$

Theorem 7. If the series $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij} Fl_i^\alpha(x) Fl_j^\beta(t)$ converges uniformly to $u(x, t)$ on the square $[0, 1] \times [0, 1]$, then one has

$$u_{ij} = (2i+1)(2j+1)\alpha\beta \int_0^1 \int_0^1 u(x, t) Fl_i^\alpha(x) Fl_j^\beta(t) \cdot \omega_l^\alpha(x) \omega_l^\beta(t) dx dt. \quad (27)$$

Proof. By multiplying $\omega_l^\alpha(x) \omega_l^\beta(t) Fl_m^\alpha(x) Fl_n^\beta(t)$ on both sides of (25), where i and j are fixed and integrating term wise with regard to x and t on $[0, 1] \times [0, 1]$, then

$$\begin{aligned} \int_0^1 \int_0^1 u(x, t) Fl_m^\alpha(x) Fl_n^\beta(t) \omega_l^\alpha(x) \omega_l^\beta(t) dx dt &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij} \int_0^1 \int_0^1 Fl_i^\alpha(x) Fl_j^\beta(t) Fl_m^\alpha(x) Fl_n^\beta(t) \omega_l^\alpha(x) \omega_l^\beta(t) dx dt \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij} \int_0^1 \omega_l^\alpha(x) Fl_i^\alpha(x) Fl_m^\alpha(x) dx \int_0^1 \omega_l^\beta(t) Fl_j^\beta(t) Fl_n^\beta(t) dt = u_{ij} \int_0^1 \omega_l^\alpha(x) [Fl_i^\alpha(x)]^2 dx \int_0^1 \omega_l^\beta(t) [Fl_j^\beta(t)]^2 dt \\ &= u_{ij} \frac{1}{(2i+1)\alpha} \frac{1}{(2j+1)\beta}. \end{aligned} \quad (28)$$

$\square$

Theorem 8. If the function $u(x, t)$ is a continuous function on $[0, 1] \times [0, 1]$ and the series $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij} Fl_i^\alpha(x) Fl_j^\beta(t)$ converges uniformly to $u(x, t)$, then $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij} Fl_i^\alpha(x) Fl_j^\beta(t)$ is the FLFs expansion of $u(x, t)$.

Proof. Using contradiction, let

$$\begin{aligned} u(x, t) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij} Fl_i^\alpha(x) Fl_j^\beta(t), \\ u(x, t) &\approx \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} g_{ij} Fl_i^\alpha(x) Fl_j^\beta(t). \end{aligned} \quad (29)$$

Then there is at least one coefficient such that $u_{nm} \neq g_{nm}$; however

$$\begin{aligned} u_{nm} &= (2m+1)(2n+1)\alpha\beta \int_0^1 \int_0^1 u(x, t) Fl_m^\alpha(x) \\ &\cdot Fl_n^\beta(t) \omega_l^\alpha(x) \omega_l^\beta(t) dx dt = g_{nm}. \end{aligned} \quad (30)$$

$\square$

Theorem 9. If two continuous functions defined on $[0, 1] \times [0, 1]$ have the identical FLFs expansions, then these two functions are identical.

Proof. Suppose that $u(x, t)$ and $f(x, t)$ can be expanded by FLFs as follows:

$$\begin{aligned} u(x, t) &\approx \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij} Fl_i^a(x) Fl_j^b(t), \\ f(x, t) &\approx \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij} Fl_i^a(x) Fl_j^b(t). \end{aligned} \quad (31)$$

By subtracting the above two equations with each other, we have

$$\begin{aligned} u(x, t) - f(x, t) &\approx \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (u_{ij} - u_{ij}) Fl_i^a(x) Fl_j^b(t) \\ &= 0 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} 0 Fl_i^a(x) Fl_j^b(t). \end{aligned} \quad (32)$$

□

Theorem 10. If the sum of the absolute values of the FLFs coefficients of a continuous function $u(x, t)$ forms a convergent series, then the FLFs expansion is absolutely uniformly convergent and converges to the function $u(x, t)$.

Proof. Consider

$$\begin{aligned} \left| \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij} Fl_i^a(x) Fl_j^b(t) \right| &\leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |u_{ij}| |Fl_i^a(x)| |Fl_j^b(t)| \\ &\leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |u_{ij}|. \end{aligned} \quad (33)$$

If we consider truncated series in (25), we obtain

$$u(x, t) \approx \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} u_{ij} Fl_i^a(x) Fl_j^b(t) = \Phi^T(x) \mathbf{U} \Phi(t), \quad (34)$$

where

$$\begin{aligned} \Phi(x) &= [Fl_0^a(x), Fl_1^a(x), \dots, Fl_{m-1}^a(x)]^T; \\ \Phi(t) &= [Fl_0^b(t), Fl_1^b(t), \dots, Fl_{n-1}^b(t)]^T, \\ \mathbf{U} &= \{u_{ij}\}_{i,j=0}^{m-1, n-1}. \end{aligned} \quad (35)$$

In this paper, we use the Tau method [20] to compute the coefficients u_{ij} . □

4. Fractional Differential Operational Matrix of FLFs

The derivative of the $\Phi(x)$ can be approximated

$$D^\gamma \Phi(x) \approx \mathbf{D}^\gamma \Phi(x), \quad (36)$$

where $\mathbf{D}^\gamma$ is called the FLFs operational matrix of derivative.

Theorem 11. Suppose $\mathbf{D}^\gamma$ is the $m \times m$ operational matrix of Caputo fractional derivatives of order $\gamma > 0$, $\alpha > \gamma/2$, when $\alpha \notin \mathbb{N}$; then the elements of $\mathbf{D}^\gamma$ are obtained as

$$\begin{aligned} \{d_{ij}\}_{i,j}^{m-1, m-1} &= (2j+1) \\ &\cdot \alpha \sum_{s=0}^i \sum_{r=0}^j b_{r,j} b'_{s,i} \frac{\Gamma(s\alpha+1)}{\Gamma(s\alpha-\gamma+1)} \frac{1}{(s+r+1)\alpha-\gamma}, \end{aligned} \quad (37)$$

where

$$\begin{aligned} b'_{s,i} &= \begin{cases} 0, & s\alpha \in N_0, \quad s\alpha < \gamma, \\ b'_{s,i} = b_{s,i} & s\alpha \notin N_0, \quad s\alpha \geq [\gamma] \text{ or } s\alpha \in N_0, \quad s\alpha \geq \gamma. \end{cases} \end{aligned} \quad (38)$$

Proof. With the properties of the derivative (ii) and the orthogonally of FLFs, we have

$$D^\gamma Fl_i^a(x) = \sum_{s=0}^i b'_{s,i} \frac{\Gamma(s\alpha+1)}{\Gamma(s\alpha-\gamma+1)} x^{s\alpha-\gamma}. \quad (39)$$

Let

$$x^{s\alpha-\gamma} = \sum_{j=0}^{m-1} d_j Fl_j^a(x). \quad (40)$$

Multiplying $\omega_l^a(x) Fl_j^a(x)$ on both sides of (40), we have

$$d_j = (2j+1) \alpha \sum_{r=0}^j b_{r,j} \frac{1}{(s+r+1)\alpha-\gamma}. \quad (41)$$

Substituting (40) and (41) into (39), we have

$$\begin{aligned} D^\gamma Fl_i^a(x) &= (2j+1) \\ &\cdot \alpha \sum_{j=0}^{m-1} \sum_{s=0}^i \sum_{r=0}^j b_{r,j} b'_{s,i} \frac{\Gamma(s\alpha+1)}{\Gamma(s\alpha-\gamma+1)} \frac{1}{(s+r+1)\alpha-\gamma} \\ &\cdot Fl_j^a(x). \end{aligned} \quad (42)$$

Hence,

$$\begin{aligned} d_{ij} &= (2j+1) \\ &\cdot \alpha \sum_{s=0}^i \sum_{r=0}^j b_{r,j} b'_{s,i} \frac{\Gamma(s\alpha+1)}{\Gamma(s\alpha-\gamma+1)} \frac{1}{(s+r+1)\alpha-\gamma}, \end{aligned} \quad (43)$$

$i, j = 0, 1, \dots, m-1.$

□

5. Description of the Proposed Method

We consider a class of fractional partial differential equations as follows:

$$\frac{\partial^\gamma u(x, t)}{\partial x^\gamma} + \frac{\partial^\gamma u(x, t)}{\partial t^\gamma} = g(x, t), \quad (44)$$

subject to the initial conditions

$$u(0, t) = u(x, 0) = 0, \quad (45)$$

where $0 \leq x, t \leq 1$, $n-1 < \nu, \gamma \leq n$, $\partial^\nu u(x, t)/\partial x^\nu$ and $\partial^\gamma u(x, t)/\partial t^\gamma$ are fractional derivative in Caputo sense, $g(x, t)$ is the known continuous function, and $u(x, t)$ is the unknown function.

If we approximate function $u(x, t)$ by a terms of the FLFs, it can be written as (34). Then we have

$$\begin{aligned} \frac{\partial^\nu u(x, t)}{\partial x^\nu} &\approx \frac{\partial^\nu (\Phi^T(x) \mathbf{U} \Phi(t))}{\partial x^\nu} \\ &= \frac{\partial^\nu (\Phi^T(x))}{\partial x^\nu} \mathbf{U} \Phi(t) \\ &= \left(\frac{\partial^\nu \Phi(x)}{\partial x^\nu} \right)^T \mathbf{U} \Phi(t) \\ &\approx \Phi^T(x) (\mathbf{D}^\nu)^T \mathbf{U} \Phi(t), \end{aligned} \quad (46)$$

$$\begin{aligned} \frac{\partial^\gamma u(x, t)}{\partial t^\gamma} &\approx \frac{\partial^\gamma (\Phi^T(x) \mathbf{U} \Phi(t))}{\partial t^\gamma} = \Phi^T(x) \mathbf{U} \frac{\partial^\gamma \Phi(t)}{\partial t^\gamma} \\ &\approx \Phi^T(x) \mathbf{U} \mathbf{D}^\gamma \Phi(t), \\ g(x, t) &\approx \Phi^T(x) \mathbf{G} \Phi(t), \end{aligned}$$

where

$$\begin{aligned} \mathbf{G} &= \{g_{ij}\}_{i,j=0}^{m-1;n-1}, \\ g_{ij} &= (2i+1)(2j+1)\alpha\beta \int_0^1 \int_0^1 g(x, t) Fl_i^\alpha(x) Fl_j^\beta(t) \\ &\quad \cdot \omega_i^\alpha(x) \omega_j^\beta(t) dx dt. \end{aligned} \quad (47)$$

Substituting (46) into (44), we have

$$\begin{aligned} \Phi^T(x) (\mathbf{D}^\nu)^T \mathbf{U} \Phi(t) + \Phi^T(x) \mathbf{U} \mathbf{D}^\gamma \Phi(t) \\ \approx \Phi^T(x) \mathbf{G} \Phi(t). \end{aligned} \quad (48)$$

Similar to the typical Tau method [20], then we generate $mn - n - m + 1$ algebraic equations by multiplying $\omega_i^\alpha(x) \omega_j^\beta(t) Fl_i^\alpha(x) Fl_j^\beta(t)$ (for $i = 0, 1, 2, \dots, m-2$; $j = 0, 1, 2, \dots, n-2$), integrating from 0 to 1 and using the orthogonal property, we have

$$\mathbf{H} = (\mathbf{D}^\nu)^T \mathbf{U} + \mathbf{U} \mathbf{D}^\gamma - \mathbf{G} = 0. \quad (49)$$

For condition. (45), we have

$$\begin{aligned} u(0, t) &\approx \Phi^T(0) \mathbf{U} \Phi(t) \approx \mathbf{C}^T \Phi(t), \\ u(x, 0) &\approx \Phi^T(x) \mathbf{U} \Phi(0) \approx \Phi^T(x) \mathbf{F}, \end{aligned} \quad (50)$$

with

$$\begin{aligned} \mathbf{C}^T &= \{c_i\}_{i=0}^{n-1}, \\ c_i &= (2i+1)\beta \int_0^1 u(0, t) Fl_i^\beta(t) \omega_i^\beta(t) dt, \\ \mathbf{F} &= \{f_j\}_{j=0}^{m-1}, \\ f_j &= (2j+1)\alpha \int_0^1 u(x, 0) Fl_j^\alpha(x) \omega_j^\alpha(x) dx. \end{aligned} \quad (51)$$

So we obtain

$$\begin{aligned} \Phi^T(0) \mathbf{U} &\approx \mathbf{C}^T; \\ \mathbf{U} \Phi(0) &\approx \mathbf{F}. \end{aligned} \quad (52)$$

Equations (49) and (52) generate $mn + 1$ linear algebraic equations. By solving the linear algebraic equations, the unknown coefficients u_{ij} can be obtained.

6. Error Analysis

We consider $e(x, t) = u(x, t) - u_{mn}(x, t)$ as the error function of the approximate solution $u_{mn}(x, t)$ for $u(x, t)$, where $u(x, t)$ is the exact solution of (44) [21].

Therefore, $u_{mn}(x, t)$ satisfies the following problem:

$$\frac{\partial^\nu u_{mn}(x, t)}{\partial x^\nu} + \frac{\partial^\gamma u_{mn}(x, t)}{\partial t^\gamma} + R_{mn}(x, t) = g(x, t), \quad (53)$$

where $R_{mn}(x, t)$ is the residual function,

$$R_{mn}(x, t) = \frac{\partial^\nu u_{mn}(x, t)}{\partial x^\nu} + \frac{\partial^\gamma u_{mn}(x, t)}{\partial t^\gamma} - g(x, t). \quad (54)$$

We find an approximation $e'_{mn}(x, t)$ to the error function $e_{mn}(x, t)$ in the same way as we did before for the solution of the problem, the error function satisfies the problem

$$\frac{\partial^\nu e'_{mn}(x, t)}{\partial x^\nu} + \frac{\partial^\gamma e'_{mn}(x, t)}{\partial t^\gamma} = R_{mn}(x, t). \quad (55)$$

We should note that, in order to construct the approximate $e'_{mn}(x, t)$ to $e_{mn}(x, t)$, only (55) needs to be recalculated in the same way as we did before for the solution of (44).

7. Numerical Examples

Example 1. We consider the following FPDE:

$$\begin{aligned} \frac{\partial^{1/4} u(x, t)}{\partial x^{1/4}} + \frac{\partial^{1/2} u(x, t)}{\partial t^{1/2}} &= g(x, t), \\ u(0, t) &= 0, \\ u(x, 0) &= 0, \end{aligned} \quad (56)$$

FIGURE 1: Numerical solution with $m = 6$ for Example 1.

where $g(x, t) = 8x^2t^{3/2}/3\sqrt{\pi} + 32t^2x^{7/4}/21\Gamma(3/4)$; using the method presented in Section 5, the equation is transformed as follows:

$$\begin{aligned} (\mathbf{D}^{1/4})^T \mathbf{U} + \mathbf{U} \mathbf{D}^{1/2} &\approx \mathbf{G}, \\ \Phi^T(0) \mathbf{U} &\approx \mathbf{C}^T; \\ \mathbf{U} \Phi(0) &\approx \mathbf{F}. \end{aligned} \quad (57)$$

By solving (57), we can obtain the numerical solution of the problem.

When $n = 6$, $m = 6$, the matrix $\mathbf{U}$ is get as follows:

$$\mathbf{U} = \begin{bmatrix} 0.0198 & 0.0396 & 0.0282 & 0.0098 & 0.0014 & 0 \\ 0.0486 & 0.0973 & 0.0694 & 0.0242 & 0.0034 & 0 \\ 0.054 & 0.108 & 0.077 & 0.027 & 0.0038 & 0 \\ 0.0412 & 0.0824 & 0.0588 & 0.0206 & 0.0029 & 0 \\ 0.0221 & 0.0443 & 0.0316 & 0.0110 & 0.0015 & 0 \\ 0.0062 & 0.0123 & 0.0087 & 0.0030 & 0 & 0 \end{bmatrix}. \quad (58)$$

When $m = 8$, $n = 6$, the matrix $\mathbf{U}$ is get as follows:

$$\mathbf{U} = \begin{bmatrix} 0.0198 & 0.0396 & 0.0282 & 0.0099 & 0.0013 & 0 \\ 0.0486 & 0.0971 & 0.0693 & 0.0241 & 0.0034 & 0 \\ 0.0540 & 0.1079 & 0.0770 & 0.0269 & 0.0038 & 0 \\ 0.0407 & 0.0814 & 0.0580 & 0.0203 & 0.0029 & 0 \\ 0.0225 & 0.0449 & 0.0320 & 0.0112 & 0.0016 & 0 \\ 0.0092 & 0.0183 & 0.0131 & 0.0046 & 0 & 0 \\ 0.0027 & 0.0054 & 0.0038 & 0.0013 & 0 & 0 \\ 0.0005 & 0.0009 & 0 & 0.0002 & 0 & 0 \end{bmatrix}. \quad (59)$$

When $n = 6$, Figures 1, 2, and 3 show the numerical solution of (56) for $\alpha = 1/4$, $\beta = 1/2$ with $m = 6$, $m =$

FIGURE 2: Numerical solution with $m = 8$ for Example 1. FIGURE 3: Numerical solution with $m = 10$ for Example 1.

8, $m = 10$. The exact solution is x^2t^2 which is shown in Figure 4. We can see clearly that the numerical solutions are very good agreement with the exact solution. Particularly, when $m = 10$ the numerical solution matches the exact solution very well.

Example 2. We consider the following equation:

$$\begin{aligned} \frac{\partial^y u(x, t)}{\partial x^y} + \frac{\partial^y u(x, t)}{\partial t^y} &= g(x, t), \\ u(0, t) &= 0. \end{aligned} \quad (60)$$

FIGURE 4: Exact solution for Example 1.

FIGURE 5: The comparison between FLM and ADM with $m = 3$, $n = 3$ for Example 2.

The exact solution of this problem is $u(x, t) = x^{1/2}t$, where $\nu = \gamma = 1/2$, $g(x, t) = \sqrt{\pi}t/2 + 2\sqrt{tx}/\sqrt{\pi}$. The equation is transformed as follows:

$$\begin{aligned} (\mathbf{D}^{1/2})^T \mathbf{U} + \mathbf{U} \mathbf{D}^{1/2} &\approx \mathbf{G}; \\ \Phi^T(0) \mathbf{U} &\approx \mathbf{C}^T, \end{aligned} \quad (61)$$

when $m = 3$, $n = 3$, and $m = 5$, $n = 5$. The results obtained using our proposed method and ADM at $t = 0.3$ are shown in Figures 5 and 6. Comparing the two figures, it can be concluded that the FLM has a better advantage than ADM for solving fractional partial differential equations and the numerical results have achieved a good convergence precision when small m and n are selected.

FIGURE 6: The comparison between FLM and ADM with $m = 5$, $n = 5$ for Example 2.

Example 3. Consider the following fractional Poisson equation:

$$\begin{aligned} \frac{\partial^\nu u(x, t)}{\partial x^\nu} + \frac{\partial^\gamma u(x, t)}{\partial t^\gamma} &= g(x, t), \\ u(0, t) &= 0, \\ u(x, 0) &= 0. \end{aligned} \quad (62)$$

(1) When $\nu = \gamma = 5/4$,

$$\begin{aligned} g(x, t) &= \frac{5\Gamma(1/4)}{16} x^{5/2} + \frac{5\Gamma(1/4)}{16} t^{15/4} \\ &+ \frac{30\sqrt{\pi} x^{5/4} t^{5/4}}{16\Gamma(9/4)} + \frac{2t^{5/2} x^{5/4} \Gamma(15/4)}{\sqrt{\pi}}. \end{aligned} \quad (63)$$

The exact solution is $x^{5/2} t^{5/4} + x^{5/4} t^{15/4}$. When $n = 4$, the absolute errors for $m = 5$, $m = 6$, $m = 7$ are shown in Figures 7, 8, and 9.

(2) When $\nu = \gamma = 1/2$, $g(x, t) = \sin(x + t)$. The exact solution is unknown.

Therefore, in order to show the efficiency of the present method, we define the error function as follows:

$$\begin{aligned} e(x, t) &= \Phi^T(x) (\mathbf{D}^\nu)^T \mathbf{U} \Phi(t) + \Phi^T(x) \mathbf{U} \mathbf{D}^\gamma \Phi(t) \\ &- \Phi^T(x) \mathbf{G} \Phi(t). \end{aligned} \quad (64)$$

Let $\alpha = \beta = 1/2$ with $m = n = 8$, $m = n = 9$, and $m = n = 10$; the absolute error $|e(x, t)|$ is shown in Figures 10, 11, and 12.

When $m = n = 7$, we can have the coefficients matrix $\mathbf{U}$.

$$\mathbf{U} = \begin{bmatrix} 0.1570 & 0.1708 & 0.0198 & 0 & 0.0060 & 0 & 0 \\ 0.1708 & 0.1482 & -0.0152 & 0 & -0.0051 & -0.0001 & 0.0002 \\ 0.0198 & -0.0153 & -0.0381 & -0.0007 & 0.0019 & -0.0004 & 0.0001 \\ -0.0003 & 0.0025 & -0.0007 & -0.0031 & 0.0006 & 0.0001 & -0.0001 \\ -0.0061 & -0.0051 & 0.0018 & 0.0006 & -0.0003 & 0.0001 & 0.0001 \\ 0 & -0.0002 & -0.0003 & 0.0001 & 0.0001 & -0.0001 & -3.029 \\ -0.0002 & 0.0002 & 0.0001 & 0 & 0.0001 & -3.029 & 0 \end{bmatrix}. \quad (65)$$

Then we have the numerical solution $u(x, t) = \Phi^T(x) \mathbf{U} \Phi(t)$.

FIGURE 7: The absolute error for Example 3 with $m = 5$. FIGURE 9: The absolute error for Example 3 with $m = 7$. FIGURE 8: The absolute error for Example 3 with $m = 6$. FIGURE 10: The absolute error for Example 3 with $m = n = 8$.

8. Conclusions

In this paper, we define a basis of FLFs and have derived its fractional differential operational matrix. The fractional derivatives are described in the Caputo sense. The matrix together with Tau method is used to effectively approximate the exact solution of fractional partial differential equations. Compared with other numerical approach, the FLFs-Tau method can accurately represent properties of fractional calculus. Moreover, only a small number of FLFs are needed to obtain a satisfactory result. Numerical results which are given in the previous section demonstrate good accuracy of the algorithm. Finally, error analysis shows that the algorithm is convergent.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by Dr. Startup funds of Taiyuan University of Science and Technology (20122054).

FIGURE 11: The absolute error for Example 3 with $m = n = 9$. FIGURE 12: The absolute error for Example 3 with $m = n = 10$.

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