Research Article

Existence and Uniqueness of Solutions to the Wage Equation of Dixit-Stiglitz-Krugman Model with No Restriction on Transport Costs

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In spatial economics, the distribution of wages is described by a solution to the *wage equation of Dixit-Stiglitz-Krugman model*. The wage equation is a discrete equation that has a double nonlinear singular structure in the sense that the equation contains a discrete nonlinear operator whose kernel itself is expressed by another discrete nonlinear operator with a singularity. In this article, no restrictions are imposed on the maximum of transport costs of the model and on the number of regions where economic activities are conducted. Applying Brouwer fixed point theorem to this discrete double nonlinear singular operator, we prove sufficient conditions for the wage equation to have a solution and a unique one.

1. Introduction

A large number of discrete dynamic models have been constructed in order to describe various dynamic phenomena observed in nature and society. In particular, the applications of discrete dynamic models have been broadened to various disciplines of economics. The applications help the advance of economics. In turn, it accelerates the progress of theory of discrete dynamics itself to study new discrete dynamic models constructed in economics.

In light of the close and cooperative interaction between discrete dynamics and economics, we should extend applications of discrete dynamics further to new disciplines of economics. In particular, noting that many mathematical sciences (game theory, financial engineering, and so on) have been born from the Nobel Prize research in economic sciences, we should focus on applications of discrete dynamics in *spatial economics*.

Spatial economics is an interdisciplinary area between economics and geography. Its purpose is to study the location, distribution, and self-organization of economic activities.

In about 1990, Krugman began important seminal research and established a useful analytical framework with emphasis laid on the formation of a large variety of economic agglomeration and the clustering of economic activities. A large number of economists have conducted various types of research within his new framework. Their research has since grown into one of the most major branches of spatial economics. Now it is known as *New Economic Geography* (NEG). In 2008, Krugman was awarded the Nobel Memorial Prize in Economic Sciences for his great contribution to spatial economics (see, e.g., [1–4]).

In NEG a large number of discrete dynamic models have been constructed, and many of them are quite new and very impressive. Hence, NEG is one of the most promising fields of applications of discrete dynamics. Among those many discrete models in NEG, *Dixit-Stiglitz-Krugman model* (DSK model) is one of the most fundamental models. Hence, this paper deals with DSK model. In this model, economic activities are conducted in a set of N regions, the economy consists of agriculture and manufacturing, and the population consists of farmers and workers [5, pp. 61–77].

There have been important related studies, and several analytically solvable models have been developed, in order to analyze economic agglomeration and bifurcation (see, e.g., [6–11]).

If we regard the distributions of workers and farmers as known functions, then we define *short-run equilibrium* of DSK model as a solution to the *wage equation*. The wage equation is a discrete nonlinear singular equation whose unknown function denotes the distribution of wages. It should be noted that the wage equation has a double nonlinear singular structure in the sense that the equation contains a discrete nonlinear operator whose kernel itself is expressed by another discrete nonlinear operator with a singularity.

This strong nonlinearity causes great difficulty when attempting to solve the wage equation. In particular, this difficulty increases as the number N of regions increases. In fact, there are ample analytical results when $N = 2$, but there are much fewer ones when $N \geq 3$ (see, e.g., [12–16]). For $N = 2$ or 3, it is customary to deal mainly with a specific case where the competition is between uniform distribution and a complete agglomeration. For these reasons we should seek sufficient conditions for the existence and uniqueness of solutions to the wage equation with no restriction on the number N of regions.

In [17], as a first step, we proved that if the maximum of transport costs of DSK model is sufficiently small, then the wage equation has a unique solution for an arbitrary integer $N \geq 2$. However, this sufficient condition is very restrictive, since it cannot be applied to DSK model whose transport costs are high. Hence, in this article, we prove sufficient conditions for the existence and uniqueness of solutions to the wage equation with no restriction on the maximum of transport costs for an arbitrary integer $N \geq 2$. The main result of this article is Theorem 3. Theorem 3(i) gives a sufficient condition for the wage equation to have a solution, Theorem 3(ii) gives estimates for each solution to the wage equation, and Theorem 3(iii) gives a sufficient condition for the wage equation to have a unique solution.

Indeed the operator contained in the wage equation is very complicated. However, by applying only Brouwer fixed point theorem to this complicated operator, we can prove the main result. Hence, it is easy for readers without a full knowledge of discrete nonlinear equations to understand this article, since Brouwer fixed point theorem is one of the most elementary tools to prove existence of solutions of nonlinear equations (see, e.g., [18]). Moreover, in this paper we do not make use of the method developed in [17]. Hence, readers can understand this article without reading [17] carefully. This paper has four sections in addition to this introduction. In Section 2 we introduce the wage equation of DSK model. In Section 3 we state and discuss Theorem 3. In Section 4, we prove Theorem 3(i)(ii). In Section 5 we prove Theorem 3(iii).

2. Equation

By D we denote a finite set consisting of N points. Each point of D represents a region where economic activities are conducted. By L we denote the set of all real-valued functions

$f = f(r)$ of $r \in D$. We can regard L as an N -dimensional Euclidean space and $f = f(r) \in L$ as a point of the N -dimensional Euclidean space. In L we define the following norm instead of the usual Euclidean norm for convenience:

$$\|f\| := \max_{s \in D} |f(s)|. \quad (1)$$

We define the following subsets of L :

$$\begin{aligned} L_{0+} &:= \{f = f(r) \in L; f(r) \geq 0 \text{ for each } r \in D\}, \\ L_+ &:= \{f = f(r) \in L; f(r) > 0 \text{ for each } r \in D\}, \end{aligned} \quad (2)$$

and we denote the complement of L_+ in L_{0+} by L_0 , that is, we define

$$L_0 := L_{0+} \setminus L_+; \quad (3)$$

Moreover we define

$$\begin{aligned} L(\underline{U}, \overline{U}) \\ := \{f = f(r) \in L; \underline{U} \leq f(r) \leq \overline{U} \text{ for each } r \in D\}, \end{aligned} \quad (4)$$

where $\underline{U}$ and $\overline{U}$ are arbitrary constants such that

$$0 < \underline{U} \leq \overline{U}. \quad (5)$$

The wage equation contains the *elasticity of substitution* σ and the *manufacturing expenditure* μ as parameters, and the *transport cost function* $c = c(r, s)$ as known function of $(r, s) \in D \times D$. Moreover, the wage equation contains the *distribution of workers* $\lambda = \lambda(r)$ and the *distribution of farmers* $\phi = \phi(r)$ as known functions of $r \in D$. Throughout the paper we assume the following conditions, which are the most general conditions that are fully accepted in economics (see, e.g., [5, pp. 45–49, 61–65], [15], and [17, (2.3), (2.4), (2.8)–(2.15)]).

Condition 1.

$$N \text{ is an arbitrary integer such that } N \geq 2, \quad (6)$$

$$0 < \mu < 1, \quad (7)$$

$$\sigma > 1, \quad (8)$$

$$\begin{aligned} c(r, r) &= 0, \\ \forall r \in D, \end{aligned} \quad (9)$$

$$\begin{aligned} c(r, s) &> 0, \\ \forall r, s \in D \text{ such that } r \neq s, \end{aligned} \quad (10)$$

$$\lambda = \lambda(r) \in S, \quad (11)$$

$$\phi = \phi(r) \in S, \quad (12)$$

where

$$S := \left\{ f = f(r) \in L_{0+}; \sum_{s \in D} f(s) = 1 \right\}. \quad (13)$$

The wage equation is the following discrete nonlinear equation (see, e.g., [5, (5.3)–(5.5)], [8, pp. 16–19], and [17, (2.5)–(2.7)]):

$$w(r) = \left\{ \sum_{s \in D} e^{-(\sigma-1)c(r,s)} G(w; s)^{\sigma-1} Y(w(s)) \right\}^{1/\sigma}, \quad (14)$$

where $w = w(r)$ denotes an unknown function that describes the distribution of wages, and $Y = Y(w(s))$ and $G = G(w; s)$ are operators that act on $w = w(r)$. These operators are defined as follows:

$$G(w; s)^{\sigma-1} := \frac{1}{\sum_{t \in D} \lambda(t) (1/w(t))^{\sigma-1} e^{-(\sigma-1)c(s,t)}}, \quad (15)$$

$$Y(w(s)) := \mu \lambda(s) w(s) + (1 - \mu) \phi(s). \quad (16)$$

In NEG (15) and (16) are referred to as *price index* and *income*, respectively.

Let us discuss the singularity of the wage equation (14). We see that all $w \in L_+$ can be substituted in (14) and (15) but that (14) has a singularity such that no $w \in L_0$ can be substituted in (15). Noting that $w = w(r)$ describes the distribution of wages, which should be nonnegative-valued in economics, we reasonably seek a solution $w = w(r)$ in L_+ .

Let us discuss the nonlinearity of (14). Regarding $G(w; s)^{\sigma-1}$ as a given function, we can regard the right-hand side of (14) as a nonlinear operator acting on $w = w(s)$. We can regard $e^{-(\sigma-1)c(r,s)} G(w; s)^{\sigma-1}$ as the kernel of this nonlinear operator. However, observing (15), we see that this kernel itself is a discrete nonlinear singular operator acting on $w = w(t) \in L_+$. Therefore, we see that (14) is a discrete double nonlinear singular equation.

We employ the following symbols in this paper.

Definition 2. (i)

$$\bar{\alpha} := \frac{\bar{\phi}}{\bar{\lambda}}, \quad (17)$$

$$\underline{\alpha} := \frac{\underline{\phi}}{\underline{\lambda}},$$

$$\bar{w} := \frac{\bar{\alpha} + 1/\bar{\lambda}}{1 - \mu/((1 - \mu)\bar{\lambda})}, \quad (18)$$

$$\underline{w} := \frac{\underline{\alpha}}{1 + 1/((1 - \mu)\underline{\lambda})},$$

where

$$\bar{\lambda} := \max_{s \in D} \lambda(s), \quad (19)$$

$$\underline{\lambda} := \min_{s \in D} \lambda(s),$$

$$\bar{\phi} := \max_{s \in D} \phi(s), \quad (20)$$

$$\underline{\phi} := \min_{s \in D} \phi(s).$$

(ii)

$$\begin{aligned} \mathbf{k}(\underline{w}, \bar{w}) \\ := \left(\frac{1}{\sigma} \right) \left\{ \left(\frac{\bar{w}}{\underline{w}} \right)^{\sigma-1} \left(\frac{1}{\bar{\lambda}} \right) + \left(\frac{\bar{w}}{\underline{w}} \right)^{2\sigma-1} \left(\frac{1}{\underline{\lambda}} \right)^2 \right\}. \end{aligned} \quad (21)$$

3. Result and Discussion

Let us review previous research on the wage equation (14). Restrictive assumptions are imposed on the previous research in addition to Condition 1. If the equality

$$\mu = 1 \quad (22)$$

holds instead of (7), that is, if DSK model has no agriculture, then the wage equation has a solution [19, Assumption 1.2, (2.10), Theorem 3.1]. This result is too restrictive, since it cannot be applied to DSK model with (7). It is proved in [17, Theorem 3.1] that if $\mathbf{C}$ is so small such that

$$0 < \mathbf{C} < \frac{\log(1/\mu)}{\sigma - 1}, \quad (23)$$

where

$$\mathbf{C} := \max_{(r,s) \in D \times D} c(r,s), \quad (24)$$

then the wage equation has a solution. Moreover, it is proved in [17, Lemma 3.2, Theorem 3.3] that if $\mathbf{C}$ is so small such that

$$0 < \mathbf{C} < \frac{\log V(\mu, \sigma)}{\sigma - 1}, \quad (25)$$

then the wage equation (14) has a unique solution, where $V(\mu, \sigma)$ is a function of $(\mu, \sigma) \in (0, 1) \times (1, +\infty)$ such that

$$\inf_{(\mu, \sigma) \in (0, 1) \times (1, +\infty)} V(\mu, \sigma) \geq 1. \quad (26)$$

It follows from (23) and (25) that the previous research [17] cannot be applied to the case where $\mathbf{C}$ is large. Hence, as mentioned in Section 1, we seek sufficient conditions for the existence and uniqueness of solutions to the wage equation with no restriction on $\mathbf{C}$. Under Condition 1 we prove the following main theorem.

Theorem 3. *If*

$$\underline{\lambda} > 0, \quad (27)$$

$$\underline{\phi} > 0 \quad (28)$$

and μ is so small such that

$$0 < \mu < \frac{\underline{\lambda}}{1 + \underline{\lambda}}, \quad (29)$$

then the following statements (i)–(iii) hold.

(i) The wage equation (14) has a solution $w = w(r)$ such that

$$w = w(r) \in L(\underline{w}, \bar{w}). \quad (30)$$

(ii) If $w = w(r) \in L_+$ is a solution to (14), then the solution satisfies (30).

(iii) If $\mu \in (0, 1)$ and $\sigma - 1 > 0$ are so small that

$$\max\{\mu, \sigma - 1\} < \frac{1}{\mathbf{k}(\underline{w}, \bar{w})}, \quad (31)$$

then (14) has a unique solution in L_+ .

We prove Theorems 3(i)(ii) and 3(iii) in Sections 4 and 5, respectively. Let us discuss this theorem in this section. It follows from (27) and (28) that workers and farmers live in all regions. Theorem 3 does not contain the transport cost function $c = c(r, s)$, in contrast to the fact that (23) and (25) restrict $\mathbf{C}$ strongly in [17]. Hence, Theorem 3 can be applied to the case where $\mathbf{C}$ is large. Moreover, Theorem 3(i)(ii) does not contain the elasticity of substitution σ , either. Hence, we can apply Theorem 3(i)(ii) also to the case where σ is large.

It follows from (6), (11)–(13), (19), and (20) that

$$\begin{aligned} 0 &\leq \underline{\lambda} \leq \frac{1}{N} \leq \bar{\lambda} \leq 1, \\ 0 &\leq \underline{\phi} \leq \frac{1}{N} \leq \bar{\phi} \leq 1. \end{aligned} \quad (32)$$

Applying this inequality (7) and (27)–(29) to (18), we see easily that

$$0 < \underline{w} < 1 < \bar{w} < +\infty. \quad (33)$$

Making use of this inequality (8) and (27), we can define (21).

From (18) we see that

$$\lim_{\mu \rightarrow 0+0} (\bar{w}, \underline{w}) = \left(\bar{\alpha} + \frac{1}{\underline{\lambda}}, \frac{\alpha}{1 + 1/\underline{\lambda}} \right). \quad (34)$$

Applying this result to (21), we see easily that

$$\begin{aligned} &\lim_{(\mu, \sigma) \rightarrow (0+0, 1+0)} \mathbf{k}(\underline{w}, \bar{w}) \\ &= \frac{1}{\underline{\lambda}} + \left(\frac{(\bar{\alpha} + 1/\underline{\lambda})(1 + 1/\underline{\lambda})}{\alpha} \right) \left(\frac{1}{\underline{\lambda}} \right)^2. \end{aligned} \quad (35)$$

It follows from (27) and (28) that the right-hand side of this equality is positive and finite. Applying this result to (31), we see that if $\mu \in (0, 1)$ and $\sigma - 1$ are sufficiently small, then (31) holds.

The following condition is referred to as the *assumption of no-black-hole*, which is famous in NEG:

$$\frac{\sigma - 1}{\sigma} > \mu. \quad (36)$$

When this condition is violated, there is no agglomeration (core-periphery pattern). In order to avoid treating economies in which increasing returns are extremely strong, this inequality is assumed in [5, p. 59]. We see easily that this condition is consistent with (29) and (31). Hence, we can apply Theorem 3 to DSK model that satisfies the assumption of no-black-hole.

4. Existence

The purpose of this section is to prove Theorem 3(i)(ii). Defining the operator,

$$\begin{aligned} f(u) &= f(u)(r) \\ &:= \sum_{s \in D} e^{-(\sigma-1)c(r,s)} G(u; s)^{\sigma-1} Y(u(s)), \end{aligned} \quad (37)$$

we can rewrite (14) equivalently as follows:

$$w(r) = f(w)(r)^{1/\sigma}. \quad (38)$$

Hence, in order to prove Theorem 3(i), we have to only seek a fixed point of the operator $f(u)(r)^{1/\sigma}$. For this purpose we obtain estimates for (15).

Lemma 4. *If (27) holds and $u = u(r)$ satisfies that*

$$\underline{U} \leq u(r), \quad \forall r \in D, \quad (39)$$

where $\underline{U}$ is a positive constant, then

$$\begin{aligned} \frac{1}{\lambda(s)/u(s)^{\sigma-1} + 1/\underline{U}^{\sigma-1}} &\leq G(u; s)^{\sigma-1} \leq \frac{u(s)^{\sigma-1}}{\lambda(s)}, \\ &\forall s \in D. \end{aligned} \quad (40)$$

Proof. Note that if (27) does not hold, then we cannot define the right-hand side of (40). Applying (9) to (15) with $w(t) = u(t)$, we can divide $1/G(u; s)^{\sigma-1}$ as follows:

$$\begin{aligned} \frac{1}{G(u; s)^{\sigma-1}} &= \frac{\lambda(s)}{u(s)^{\sigma-1}} \\ &+ \sum_{t \in D, t \neq s} \lambda(t) \left(\frac{1}{u(t)} \right)^{\sigma-1} e^{-(\sigma-1)c(s,t)}. \end{aligned} \quad (41)$$

Dropping the second term from the right-hand side of this equality, we obtain the right-hand inequality of (40).

It follows immediately from (8)–(10) that

$$e^{-(\sigma-1)c(s,t)} \leq 1, \quad \forall s, t \in D. \quad (42)$$

Making use of this inequality and (39), we replace $(1/u(t))^{\sigma-1} e^{-(\sigma-1)c(s,t)}$ with $1/\underline{U}^{\sigma-1}$ in the second term of the right-hand side of (41). Applying (11) and (13) to the inequality thereby obtained, we obtain the left-hand inequality of (40). $\square$

Making use of Lemma 4, we obtain estimates for (37).

Lemma 5. *If (27) holds and $u = u(r)$ satisfies*

$$\underline{U} \leq u(r) \leq \bar{U}, \quad \forall r \in D, \quad (43)$$

where $\underline{U}$ and $\bar{U}$ are constants that satisfy (5), then

$$\underline{f}(\underline{U}) \leq f(u)(r) \leq \bar{f}(\bar{U}), \quad \forall r \in D, \quad (44)$$

where

$$\underline{f}(\underline{U}) := \underline{U}^{\sigma-1} \left(\frac{\underline{U}}{1 + 1/\underline{\lambda}} \right), \quad (45)$$

$$\begin{aligned} \bar{f}(\bar{U}) &:= \bar{U}^{\sigma-1} \left(\bar{U} + \frac{\mu \bar{U} + (1-\mu)}{\underline{\lambda}} \right), \\ \underline{U} &:= \mu \underline{U} + (1-\mu) \underline{\alpha}, \\ \bar{U} &:= \mu \bar{U} + (1-\mu) \bar{\alpha}. \end{aligned} \quad (46)$$

Proof. Note that if (27) does not hold, then we cannot define (45). Applying (7), (11), and (12) to (16), we see that

$$\text{each term of } Y(u(s)) \text{ is nonnegative for all } u = u(s) \in I_{0+}. \quad (47)$$

Applying (7), (11)–(13), and (43) to

$$\mathbf{Y}(u) := \sum_{s \in D} Y(u(s)), \quad (48)$$

we obtain

$$\mu \underline{U} + (1-\mu) \leq \mathbf{Y}(u) \leq \mu \bar{U} + (1-\mu). \quad (49)$$

Making use of (9), we divide (37) as follows in the same way as (41):

$$\begin{aligned} f(u)(r) &= G(u; r)^{\sigma-1} Y(u(r)) \\ &+ \sum_{s \in D, s \neq r} e^{-(\sigma-1)c(r,s)} G(u; s)^{\sigma-1} Y(u(s)). \end{aligned} \quad (50)$$

Applying (47) and the right-hand inequality of (40) to the right-hand side of this equality, we replace $G(u; r)^{\sigma-1}$ and $G(u; s)^{\sigma-1}$ with $u(r)^{\sigma-1}/\lambda(r)$ and $u(s)^{\sigma-1}/\lambda(s)$, respectively. Moreover, applying (8), (19), (42), and (43) to the second term of the right-hand side of the inequality thereby obtained, we replace $e^{-(\sigma-1)c(r,s)} u(s)^{\sigma-1}/\lambda(s)$ with $\bar{U}^{\sigma-1}/\underline{\lambda}$. Hence, we obtain

$$f(u)(r) \leq u(r)^{\sigma-1} \left(\frac{Y(u(r))}{\lambda(r)} \right) + \left(\frac{\bar{U}^{\sigma-1}}{\underline{\lambda}} \right) \mathbf{Y}(u). \quad (51)$$

Applying (49) to the second term of the right-hand side of this inequality, we deduce that

$$\begin{aligned} f(u)(r) &\leq u(r)^{\sigma-1} \left(\frac{Y(u(r))}{\lambda(r)} \right) \\ &+ \left(\frac{\bar{U}^{\sigma-1}}{\underline{\lambda}} \right) (\mu \bar{U} + (1-\mu)). \end{aligned} \quad (52)$$

Applying (47) and the left-hand inequality of (40) to the first term of the right-hand side of (50), and dropping the second term from the right-hand side, we obtain

$$f(u)(r) \geq \frac{Y(u(r))}{\lambda(r)/u(r)^{\sigma-1} + 1/\underline{U}^{\sigma-1}}. \quad (53)$$

Applying (7), (16), (17), and (43) to $Y(u(s))/\lambda(s)$, we see easily that

$$\underline{U} \leq \frac{Y(u(s))}{\lambda(s)} \leq \bar{U}. \quad (54)$$

Applying this inequality to (52) and (53), we obtain

$$\begin{aligned} \frac{\underline{U}}{1/u(r)^{\sigma-1} + 1/(\underline{U}^{\sigma-1}\lambda(r))} &\leq f(u)(r) \\ &\leq u(r)^{\sigma-1} \bar{U} + \left(\frac{\bar{U}^{\sigma-1}}{\underline{\lambda}} \right) (\mu \bar{U} + (1-\mu)). \end{aligned} \quad (55)$$

Replace $\lambda(r)$ and $u(r)$ with $\underline{\lambda}$ and $\underline{U}$ in the left-hand side of (55). Replace $u(r)$ with $\bar{U}$ in the right-hand side of (55). Then we obtain (44). $\square$

Proof of Theorem 3(i). We prove Theorem 3(i) by making use of Lemma 5. Let

$$u = u(r) \in L(\underline{U}, \bar{U}), \quad (56)$$

where $\underline{U}$ and $\bar{U}$ are constants that satisfy (5). Making use of (33), we can substitute $(\underline{U}, \bar{U}) = (\underline{w}, \bar{w})$ in (45). Substituting (18) in (45) with $(\underline{U}, \bar{U}) = (\underline{w}, \bar{w})$ and making use of (29), we see easily that

$$\left(\underline{f}(\underline{w})^{1/\sigma}, \bar{f}(\bar{w})^{1/\sigma} \right) = (\underline{w}, \bar{w}). \quad (57)$$

Substituting (57) in (44) with $(\underline{U}, \bar{U}) = (\underline{w}, \bar{w})$, we see that $f(u)^{1/\sigma}$ is an operator from $L(\underline{w}, \bar{w})$ to itself. Observing (37), we see easily that $f(u)^{1/\sigma}$ is continuous with respect to $u \in L(\underline{w}, \bar{w})$. Moreover, we see easily that $L(\underline{w}, \bar{w})$ is a convex closed subset of L . Hence, applying Brouwer fixed point theorem to the operator $f(u)^{1/\sigma}$ (see, e.g., [18]), we see that the operator has a fixed point in $L(\underline{w}, \bar{w})$. Hence, we obtain Theorem 3(i). $\square$

Proof of Theorem 3(ii). We make use of Lemma 5. Let $w = w(r) \in L_+$ be a solution. Substituting (38) in (44), when

$$\begin{aligned} \underline{U} &:= \min_{s \in D} w(s), \\ \bar{U} &:= \max_{s \in D} w(s), \end{aligned} \quad (58)$$

$$u(r) := w(r),$$

we obtain the following inequality:

$$\underline{f}(\underline{U}) \leq w(r)^\sigma \leq \bar{f}(\bar{U}), \quad \forall r \in D. \quad (59)$$

Noting that $\underline{f}(\underline{U})$ and $\bar{f}(\bar{U})$ are independent of $r \in D$, we replace $w(r)^\sigma$ with $\underline{U}^\sigma$ and $\bar{U}^\sigma$ in this inequality. Hence we obtain the following inequalities:

$$\begin{aligned} \underline{f}(\underline{U}) &\leq \underline{U}^\sigma, \\ \bar{U}^\sigma &\leq \bar{f}(\bar{U}). \end{aligned} \quad (60)$$

Substitute (45) and (46) in these inequalities. Solving the inequalities thereby obtained with respect to $\underline{U}$ and $\bar{U}$, making use of (29), and recalling (18), we obtain (30). $\square$

5. Uniqueness

The purpose of this section is to prove Theorem 3(iii). For this purpose we make use of the following lemma.

Lemma 6. *If (27) holds and $u = u(r)$ and $v = v(r)$ satisfy*

$$\begin{aligned} \underline{U} &\leq u(r) \leq \bar{U}, \\ \underline{U} &\leq v(r) \leq \bar{U}, \\ \forall r &\in D, \end{aligned} \quad (61)$$

where $\underline{U}$ and $\bar{U}$ are constants such that

$$0 < \underline{U} \leq 1 \leq \bar{U}, \quad (62)$$

then

$$\|f(u) - f(v)\| \leq h(\underline{U}, \bar{U}) \|u - v\|, \quad (63)$$

where

$$h(\underline{U}, \bar{U}) := \mu \bar{U}^{\sigma-1} \left(\frac{1}{\underline{\lambda}} \right) + (\sigma - 1) \left(\frac{\bar{U}^{2\sigma-1}}{\underline{U}^\sigma} \right) \left(\frac{1}{\underline{\lambda}} \right)^2. \quad (64)$$

Proof. Let $a, b \in \mathbb{R} \setminus \{0\}$. Applying the mean value theorem to the function $X^{a/b}$ of $X > 0$, we see easily that if $0 < X_1 \leq X_2$, then

$$\left| X_1^{a/b} - X_2^{a/b} \right| \leq \left| \frac{a}{b} \right| \left(\sup_{X_1 \leq \xi \leq X_2} \xi^{a/b-1} \right) |X_1 - X_2|. \quad (65)$$

Applying (1) and (61) to this inequality with $\{X_1, X_2\} = \{u(r)^b, v(r)^b\}$, we obtain

$$\|u^a - v^a\| \leq \left| \frac{a}{b} \right| A(\underline{U}, \bar{U}) \|u^b - v^b\|, \quad (66)$$

where

$$\begin{aligned} A(\underline{U}, \bar{U}) &:= \bar{U}^{a-b}, \quad \text{if } a - b \geq 0, \\ A(\underline{U}, \bar{U}) &:= \underline{U}^{a-b}, \quad \text{if } a - b < 0. \end{aligned} \quad (67)$$

Subtracting (15) with $w = v$ from (15) with $w = u$, we see that

$$G(u; s)^{\sigma-1} - G(v; s)^{\sigma-1} = I_1 I_2, \quad (68)$$

where

$$\begin{aligned} I_1 &:= G(v; s)^{\sigma-1} G(u; s)^{\sigma-1}, \\ I_2 &:= \sum_{t \in D} \lambda(t) \left(\frac{1}{v(t)^{\sigma-1}} - \frac{1}{u(t)^{\sigma-1}} \right) e^{-(\sigma-1)c(s,t)}. \end{aligned} \quad (69)$$

It follows from (40) that

$$\frac{\bar{U}^{\sigma-1}}{\underline{\lambda} + 1} \leq G(u; r)^{\sigma-1} \leq \left(\frac{\bar{U}^{\sigma-1}}{\underline{\lambda}} \right). \quad (70)$$

Applying this inequality to I_1 , we obtain

$$\|I_1\| \leq \left(\frac{\bar{U}^{\sigma-1}}{\underline{\lambda}} \right)^2. \quad (71)$$

Applying (8), (11), (13), (42), and (66) with $(a, b) = (-(\sigma - 1), 1)$ to I_2 , we obtain

$$\|I_2\| \leq \left(\frac{\sigma - 1}{\underline{U}^\sigma} \right) \|u - v\|. \quad (72)$$

Applying this inequality and (71) to (68), we see that

$$\begin{aligned} &\|G(u; \cdot)^{\sigma-1} - G(v; \cdot)^{\sigma-1}\| \\ &\leq \left(\frac{\bar{U}^{\sigma-1}}{\underline{\lambda}} \right)^2 \left(\frac{\sigma - 1}{\underline{U}^\sigma} \right) \|u - v\|. \end{aligned} \quad (73)$$

Substituting (37) in $f(u) - f(v)$, we see that

$$f(u)(r) - f(v)(r) = I_3 + I_4, \quad (74)$$

where

$$\begin{aligned} I_3 &:= \sum_{s \in D} \mu \lambda(s) \{u(s) - v(s)\} G(u; s)^{\sigma-1} e^{-(\sigma-1)c(r,s)}, \\ I_4 &:= \sum_{s \in D} Y(v(s)) \{G(u; s)^{\sigma-1} - G(v; s)^{\sigma-1}\} e^{-(\sigma-1)c(r,s)}. \end{aligned} \quad (75)$$

Applying (11), (13), (42), and (70) to I_3 , we obtain

$$\|I_3\| \leq \mu \left(\frac{\bar{U}^{\sigma-1}}{\underline{\lambda}} \right) \|u - v\|. \quad (76)$$

Applying (42), (47), (48), and (73) to I_4 , we see that

$$\|I_4\| \leq Y(u) \left(\frac{\bar{U}^{\sigma-1}}{\underline{\lambda}} \right)^2 \left(\frac{\sigma - 1}{\underline{U}^\sigma} \right) \|u - v\|. \quad (77)$$

Applying (7) and (62) to the right-hand side of (49), we see that

$$Y(u) \leq \bar{U}. \quad (78)$$

Applying this inequality to (77), we obtain

$$\|I_4\| \leq (\sigma - 1) \left(\frac{\bar{U}^{2\sigma-1}}{\underline{U}^\sigma} \right) \left(\frac{1}{\underline{\lambda}} \right)^2 \|u - v\|. \quad (79)$$

Add this inequality to (76). Applying the inequality thereby obtained to (74), we obtain (63). $\square$

Proof of Theorem 3(iii). Assume that (14) has two solutions $u, v \in L_+(\lambda)$. It follows from Theorem 3(ii) that

$$u, v \in L(\underline{w}, \bar{w}). \quad (80)$$

Subtracting (38) with $w = v$ from (38) with $w = u$, we deduce that

$$u^\sigma - v^\sigma = f(u) - f(v). \quad (81)$$

Applying (8), (80), and (66) with $(a, b) = (1, \sigma)$ to the left-hand side of this equality, we see that

$$\|u - v\| \leq \left(\frac{1}{\sigma \underline{w}^{\sigma-1}} \right) \|u^\sigma - v^\sigma\|. \quad (82)$$

Making use of (33) and (80), we can substitute $(\underline{U}, \bar{U}) = (\underline{w}, \bar{w})$ in (62) and (63). Applying the inequality thereby obtained and (82) to (81), we see that

$$\|u - v\| \leq k(\underline{w}, \bar{w}) \|u - v\|, \quad (83)$$

where

$$k(\underline{w}, \bar{w}) := \frac{h(\underline{w}, \bar{w})}{\sigma \underline{w}^{\sigma-1}}. \quad (84)$$

Applying (21) and (64) with $(\underline{U}, \bar{U}) = (\underline{w}, \bar{w})$ to $k(\underline{w}, \bar{w})$, we see easily that

$$0 < k(\underline{w}, \bar{w}) \leq \max\{\mu, \sigma - 1\} k(\underline{w}, \bar{w}). \quad (85)$$

Applying (31) to this inequality, we obtain Theorem 3(iii). $\square$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

- [1] M. Fujita and J.-F. Thisse, *Economics of Agglomeration: Cities, Industrial Location, and Globalization*, Cambridge University Press, 2013.
- [2] M. Fujita, "The evolution of spatial economics: from Thöen to the new economic geography," *Japanese Economic Review*, vol. 61, no. 1, pp. 1–32, 2010.
- [3] P. Krugman, "The official homepage of the Nobel Prize in Economic Sciences," 2008, http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/.
- [4] G. I. P. Ottaviano and D. Puga, "Agglomeration in the global economy: a survey of the 'new economic geography,'" *World Economy*, vol. 21, no. 6, pp. 707–731, 1998.
- [5] M. Fujita, P. Krugman, and A. Venables, *The Spatial Economy: Cities, Regions, and International Trade*, MIT Press, 2001.
- [6] T. Akamatsu, Y. Takayama, and K. Ikeda, "Spatial discounting, Fourier, and racetrack economy: a recipe for the analysis of spatial agglomeration models," *Journal of Economic Dynamics & Control*, vol. 36, no. 11, pp. 1729–1759, 2012.
- [7] R. Forslid and G. I. P. Ottaviano, "An analytically solvable core-periphery model," *Journal of Economic Geography*, vol. 3, no. 3, pp. 229–240, 2003.
- [8] K. Ikeda and K. Murota, *Bifurcation Theory for Hexagonal Agglomeration in Economic Geography*, Springer, Tokyo, 2014.
- [9] K. Ikeda, K. Murota, T. Akamatsu, and Y. Takayama, "Agglomeration patterns in a long narrow economy of a new economic geography model: analogy to a racetrack economy," *International Journal of Economic Theory*, vol. 13, no. 1, pp. 113–145, 2017.
- [10] P. Krugman, *The Self-Organizing Economy*, Blackwell Publishers, Cambridge, Mass, USA, 1996.
- [11] M. Pflüger, "A simple, analytically solvable, Chamberlinian agglomeration model," *Regional Science and Urban Economics*, vol. 34, no. 5, pp. 565–573, 2004.
- [12] T. Ago, I. Isono, and T. Tabuchi, "Locational disadvantage of the hub," *Annals of Regional Science*, vol. 40, no. 4, pp. 819–848, 2006.
- [13] P. Krugman and R. L. Elizondo, "Trade policy and the third world metropolis," *Journal of Development Economics*, vol. 49, no. 1, pp. 137–150, 1996.
- [14] T. Mori and K. Nishikimi, "Economies of transport density and industrial agglomeration," *Regional Science and Urban Economics*, vol. 32, no. 2, pp. 167–200, 2002.
- [15] P. Mossay, "The core-periphery model: a note on the existence and uniqueness of short-run equilibrium," *Journal of Urban Economics*, vol. 59, no. 3, pp. 389–393, 2006.
- [16] N. G. Pavlidis, M. N. Vrahatis, and P. Mossay, "Existence and computation of short-run equilibria in economic geography," *Applied Mathematics and Computation*, vol. 184, no. 1, pp. 93–103, 2007.
- [17] M. Tabata, N. Eshima, Y. Kiyonari, and I. Takagi, "The existence and uniqueness of short-run equilibrium of the Dixit-Stiglitz-Krugman model in an urban-rural setting," *IMA Journal of Applied Mathematics*, vol. 80, no. 2, pp. 474–493, 2015.
- [18] I. Farmakis and M. Moskowit, *Fixed Point Theorems and Their Applications*, World Scientific, 2013.
- [19] M. Tabata, N. Eshima, and Y. Sakai, "Existence, uniqueness, and computation of short-run and long-run equilibria of the Dixit-Stiglitz-Krugman model in an urban setting," *Applied Mathematics and Computation*, vol. 234, pp. 339–355, 2014.