Hindawi
Discrete Dynamics in Nature and Society
Volume 2017, Article ID 9610609, 10 pages
https://doi.org/10.1155/2017/9610609

Research Article

Preventing Noise-Induced Extinction in Discrete Population Models

Irina Bashkirtseva

Ural Federal University, Lenina 51, Ekaterinburg 620000, Russia

Correspondence should be addressed to Irina Bashkirtseva; irina.bashkirtseva@urfu.ru

Received 7 May 2017; Revised 10 June 2017; Accepted 18 June 2017; Published 17 July 2017

Academic Editor: Pavel Rehak

A problem of the analysis and prevention of noise-induced extinction in nonlinear population models is considered. For the solution of this problem, we suggest a general approach based on the stochastic sensitivity analysis. To prevent the noise-induced extinction, we construct feedback regulators which provide a low stochastic sensitivity and keep the system close to the safe equilibrium regime. For the demonstration of this approach, we apply our mathematical technique to the conceptual but quite representative Ricker-type models. A variant of the Ricker model with delay is studied along with the classic widely used one-dimensional system.

1. Introduction

Population models, even in the deterministic case, demonstrate a wide variety of dynamic regimes, both equilibrium and oscillatory. These regular and chaotic regimes can be changed in consequence of the different-type bifurcations [1, 2]. Any living system is subject to the inevitable noise which can significantly complicate system's dynamics and cause abrupt ecological shifts and catastrophes [3–5].

In the population systems, an investigation of the underlying reasons of the transformation from the persistence to extinction is a challenging problem. Here, an analysis of the noise-induced extinction attracts the attention of many researchers [6–12]. In the modern population dynamics, along with the problem of the analysis of unwanted shifts caused by noise, control problems for ecosystems are also highly important.

Stable regimes in the population dynamics are provided by intrinsic natural feedback mechanisms of the regulation. If such mechanisms are destroyed under various disturbances, then this desired stabilization can be achieved by the control based on additional artificial feedback [13–17].

Controlling nonlinear systems with regular and chaotic oscillations is an actively developed research domain. Following the pioneering work of Ott et al. [18], many researchers are involved in the investigations concerning the suppression of chaos to equilibria or periodic orbits [19–21].

The presence of stochastic disturbances significantly complicates the solution of control problems [22]. In nonlinear dynamic systems, even with simple regular attractors, a small noise can cause unexpected and undesired noise-induced transitions. The underlying reason of such transitions is the high stochastic sensitivity of the initial deterministic attractors [23, 24]. A control approach based on the idea of reducing the stochastic sensitivity was developed in [25, 26].

The discrete-time population models, even in the one-dimensional case, are quite representative in the simulation of both regular and chaotic behavior. One of them is the well-known Ricker system [27]. A theoretical analysis of the stochastic Ricker system in terms of Markov chains was carried out in [28, 29]. In this paper, we use the Ricker system as a conceptual model to study the probabilistic mechanisms of phenomena of the noise-induced extinction and contraction of the persistence zone.

In Section 2, we consider the one-dimensional stochastic Ricker system as a primary model. The main features of the uncontrolled model such as noise-induced extinction and contraction of the persistence zone are discussed. To prevent the extinction, we use a control approach oriented on the synthesis of the stable equilibrium with the small stochastic sensitivity. The mathematical background of this approach is briefly presented in the Appendix.

In Section 3, we study the Ricker system with delay as a secondary model. Here, we demonstrate how the mathematical theory of the stochastic sensitivity synthesis can be applied to the systems of higher dimensions.

2. Controlling Stochastic Sensitivity in 1D Ricker Model

Consider the deterministic Ricker model [27]

$$x_{t+1} = x_t \exp\left(\mu\left(1 - x_t\right)\right), \tag{1}$$

where x is the size of the population and $\mu > 0$ is a parameter of the intrinsic growth rate. System (1) has two equilibria: $\bar{x}_0 = 0$ and $\bar{x}_1 = 1$. The equilibrium $\bar{x}_0$ is unstable for any $\mu > 0$, and the equilibrium $\bar{x}_1$ is stable for $0 < \mu < 2$. For $\mu > 2$, the system performs Feigenbaum's scenario of period-doubling bifurcations and alternations of regular and chaotic zones. The attractors of the deterministic model (1) are shown in Figure 1. Note that for any μ the population exists in various forms, both regular and chaotic, with the unlimited growth of the amplitude under increasing μ. So, in the framework of this deterministic model, the persistence μ-zone is unbounded: $0 < \mu < \infty$.

It can be underlined that minimum values of the population size in oscillatory modes tend to zero as μ increases. In the presence of random disturbances, such proximity of the population size to zero can cause noise-induced extinction. To study this phenomenon, we consider the following stochastic variant of the Ricker model:

$$x_{t+1} = x_t \left[\exp\left(\mu\left(1 - x_t\right)\right) + \varepsilon\xi_t\right]^{+}, \tag{2}$$

where ξ_t is a standard uncorrelated Gaussian random process with parameters $E(\xi_t) = 0$ and $E(\xi_t^2) = 1$ and ε is the noise intensity. Here, the environmental noise scales proportionally with the population size. This model takes into account the biological fact that the population cannot withstand the negative environmental disturbance that is more than its own size. Mathematically, this obvious feature of the interplay of the population and environment is modeled in (2) by the function

$$[y]^{+} = \begin{cases} y, & y \geq 0, \\ 0, & y < 0. \end{cases} \tag{3}$$

In the framework of this model, the population size x is nonnegative even for large values of random disturbances.

Consider how noise can destroy the persistence regime of the population system. For small noise, random trajectories slightly deviate from the initial deterministic attractor (see blue curves in Figure 2). With increasing noise, these deviations increase too, and the random trajectory can intersect $x = \bar{x}_0$ and further be zero for any t (see red curves in Figure 2). Such annihilation can be interpreted as the noise-induced extinction of the population.

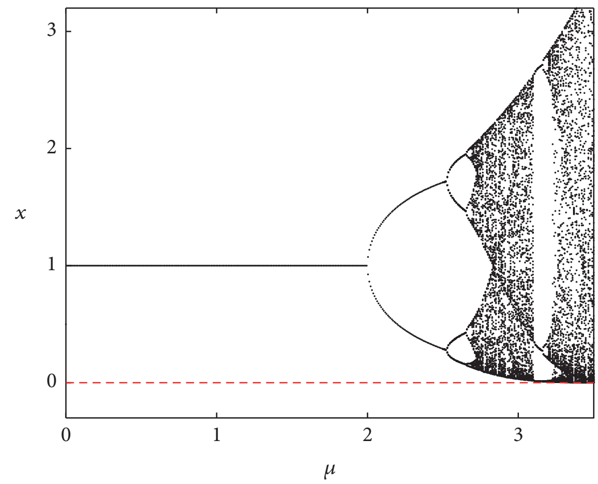

Figure 1: Bifurcation diagram of the deterministic system (1). The unstable equilibrium $\bar{x}_0$ is shown by the red dashed line.

Note that, for larger μ, the noise with smaller intensity ε causes the noise-induced extinction. It can be clearly seen in Figure 3 where random states of system (2) are shown for three values of ε. For any μ, we started from the deterministic attractor and plotted 100 iterations of the stochastic system (2) after the transient 100 steps. Here, it can be seen that the increasing noise contracts the persistence zone. For instance, in the parametric zone where the deterministic system (1) exhibits persistence with chaotic oscillations, the stochastic system (2) with the noise intensity $\varepsilon = 0.1$ demonstrates total extinction.

Consider now how to avoid this noise-induced extinction using the appropriate control procedures. Here, our aim is to stabilize the equilibrium $\bar{x}_1$ in a wide range of the parameter $\mu > 0$ and provide small-amplitude stochastic oscillations near $\bar{x}_1$.

Consider the stochastic model with control input

$$x_{t+1} = x_t \left[\exp\left(\mu\left(1 - x_t\right)\right) + \varepsilon\xi_t\right]^{+} + u_t, \tag{4}$$

where u_t is formed by the feedback regulator

$$u_t = k\left(x_t - \bar{x}_1\right). \tag{5}$$

To stabilize the stochastic oscillations of system (4) with regulator (5), we will use a theory of the stochastic sensitivity synthesis presented in the Appendix. Here,

$$\begin{gathered} f(x, u, \eta) = x \exp\left(\mu(1 - x)\right) + u + x\eta, \\ \eta = \varepsilon\xi, \\ \bar{x} = \bar{x}_1 = 1, \\ F = \frac{\partial f}{\partial x}\left(\bar{x}_1, 0, 0\right) = 1 - \mu, \\ B = \frac{\partial f}{\partial u}\left(\bar{x}_1, 0, 0\right) = 1, \end{gathered}$$

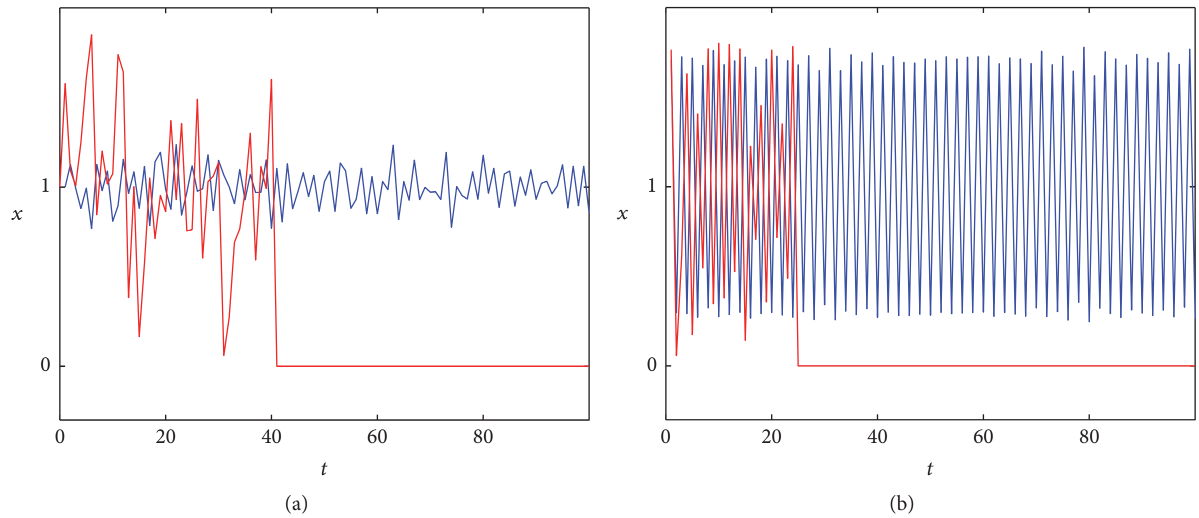

FIGURE 2: Time series of the stochastic system (2) (a) for $\mu = 1.5$ with $\varepsilon = 0.1$ (blue) and $\varepsilon = 0.4$ (red) and (b) for $\mu = 2.5$ with $\varepsilon = 0.01$ (blue) and $\varepsilon = 0.1$ (red).

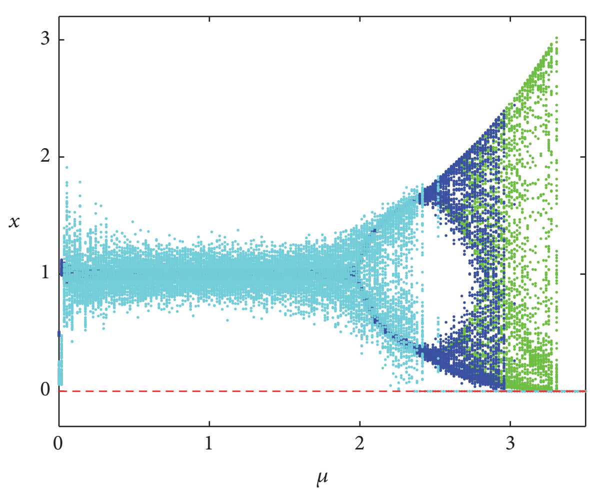

FIGURE 3: Random states of the stochastic system (2) with $\varepsilon = 0.001$ (green), $\varepsilon = 0.01$ (blue), and $\varepsilon = 0.1$ (light blue).

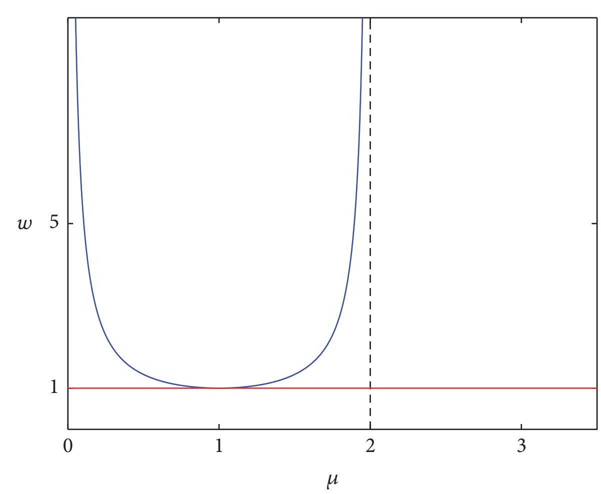

FIGURE 4: Stochastic sensitivity of the equilibrium $\overline{x}_1$.

$$G = \frac{\partial f}{\partial \eta}\left(\overline{x}_1, 0, 0\right) = 1,$$
$$S = 1. \tag{6}$$

The set of the feedback coefficients k of regulator (5) stabilizing $\overline{x}_1 = 1$ of the deterministic system is given by the inequality $\mu - 2 < k < \mu$. Following (A.14), we have

$$w_k = \frac{1}{1 - \left(1 - \mu + k\right)^2}, \tag{7}$$

where w_k is the stochastic sensitivity of the equilibrium $\overline{x}_1 = 1$ in stochastic model (4) with regulator (5). In absence of the control ($k = 0$), the stochastic sensitivity (see the blue curve in Figure 4) of $\overline{x}_1$ in the zone $0 < \mu < 2$ of its stability is given by the formula

$$w_0 = \frac{1}{\mu\left(2 - \mu\right)}. \tag{8}$$

Formally, for $\mu \geq 2$, w_0 is equal to infinity. A problem of the synthesis of the assigned stochastic sensitivity w is solved by regulator (5) with the feedback coefficient (see (A.13))

$$k = \mu - 1 \pm \sqrt{1 - \frac{1}{w}}. \tag{9}$$

Here, the attainability set of the stochastic sensitivity is defined by a simple inequality $w \geq 1$. Results of the modeling of system (2) without control and system (4) and (5) with control which synthesizes different values of the stochastic sensitivity w are shown in Figure 5. In Figure 5(a), random

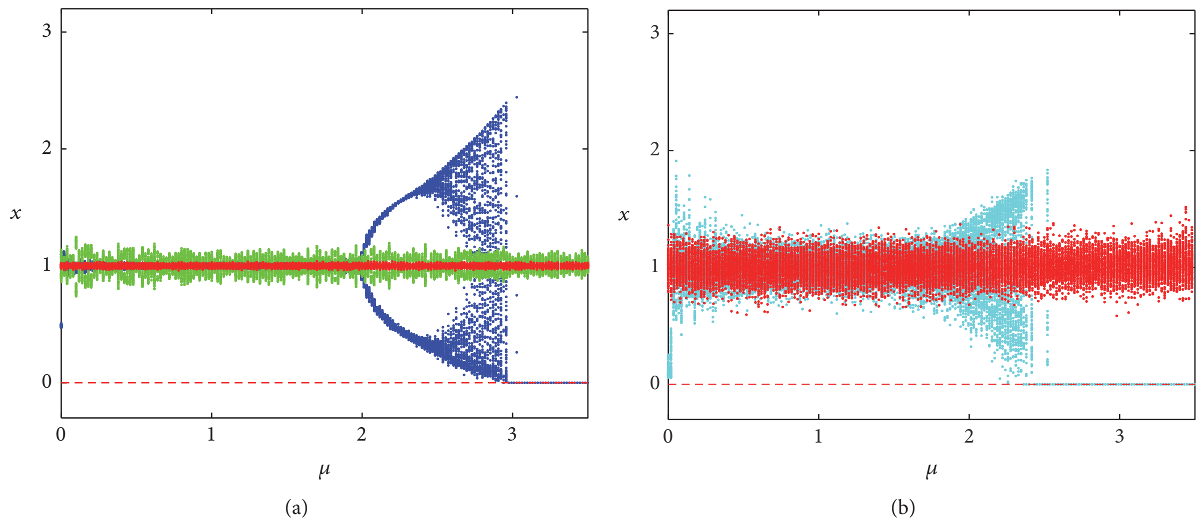

FIGURE 5: Random states of the stochastic system (4) and (5) (a) for $\varepsilon = 0.01$ without control (blue), with control providing $w = 10$ (green) and $w = 1$ (red); (b) for $\varepsilon = 0.1$ without control (light blue) and with control providing $w = 1$ (red).

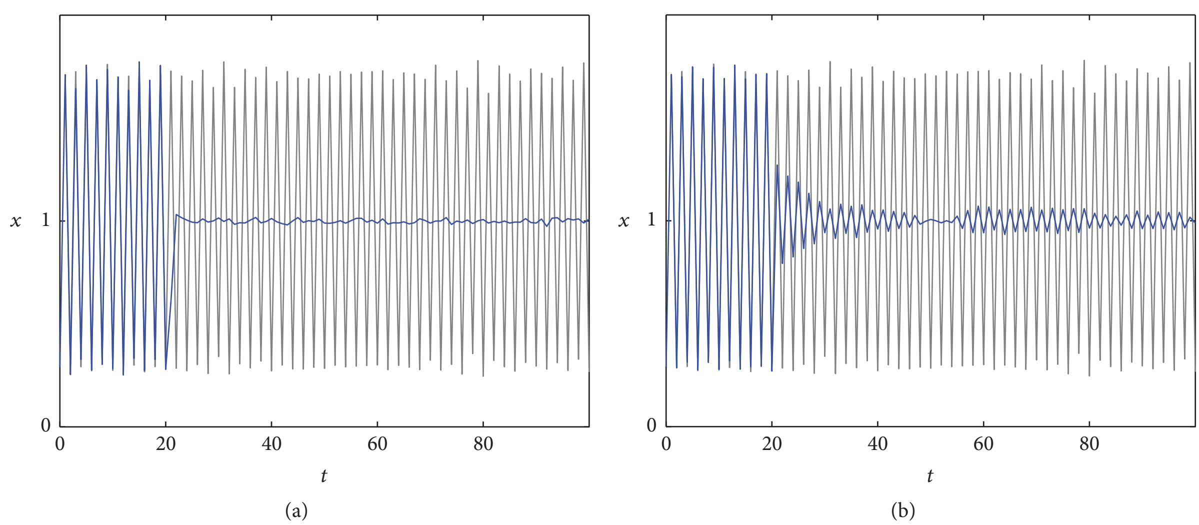

FIGURE 6: Time series of the stochastic system for $\mu = 2.5$ and $\varepsilon = 0.01$ without control (grey) and with control (blue) providing (a) $w = 1$ and (b) $w = 10$. Control is switched on at $t = 20$.

states are plotted for $\varepsilon = 0.01$ with control providing $w = 10$ (green points) and $w = 1$ (red points). As can be seen, the regulator stabilizes $\overline{x}_1 = 1$ and provides uniform dispersion in a wide range of the parameter μ. A value of the dispersion can be controlled by the assigned w. Here, the value $w = 1$ is the minimum attainable. In Figure 5(b), random states are plotted for larger noise intensity $\varepsilon = 0.1$ with control providing this minimum value $w = 1$.

In Figure 6, time series of the stochastic system with $\mu = 2.5$ and $\varepsilon = 0.01$ without control are plotted by a grey color and those with control are plotted by blue for regulators providing $w = 1$ (a) and $w = 10$ (b). Here, the control is switched on at $t = 20$.

Thus, our approach, based on the synthesis of the regulator minimizing the stochastic sensitivity, provides structural stabilization, decreases the amplitude of oscillations, and prevents the noise-induced extinction in the population system. Note that these results can be also interpreted as the suppression of chaos in the corresponding μ-zone.

Further, we will show how this approach can be applied to the population system of higher dimension. In the next section, we demonstrate the effectiveness of our control approach on the example of the Ricker model with delay.

3. Stabilization of the Stochastic Ricker Model with Delay

Consider the following modification of the Ricker model (1) with delay:

$$x_{t+1} = x_t \exp\left(\mu\left(1 - x_{t-1}\right)\right). \tag{10}$$

The Ricker model with delay was studied in [30, 31].

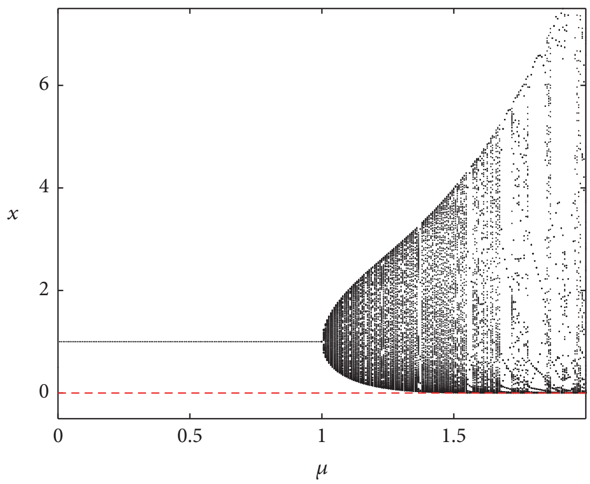

FIGURE 7: Bifurcation diagram of the deterministic system (11). The unstable equilibrium M_0 is shown by the red dashed line.

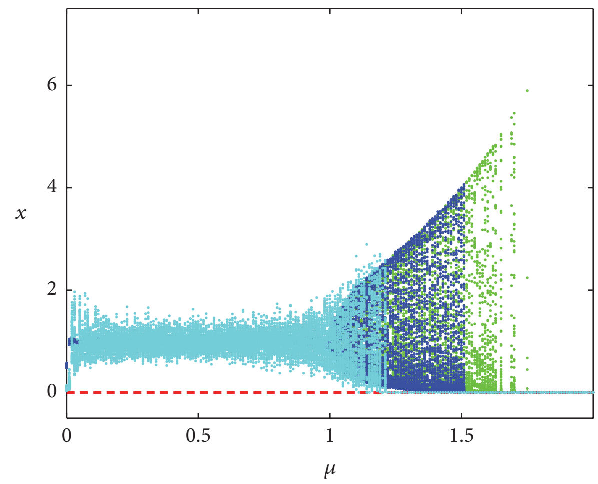

FIGURE 8: Random states of the stochastic system (12) with $\varepsilon = 0.001$ (green), $\varepsilon = 0.01$ (blue), and $\varepsilon = 0.1$ (light blue).

Equation (10) can be rewritten as the following system:

$$\begin{aligned} x_{t+1} &= x_t \exp\left(\mu\left(1 - y_t\right)\right), \\ y_{t+1} &= x_t. \end{aligned} \tag{11}$$

This system has two equilibria: $M_0(0,0)$ and $M_1(1,1)$. For any $\mu > 0$, the equilibrium M_0 is unstable. The equilibrium M_1 is stable for $0 < \mu < 1$. When the parameter μ passes the value $\mu = 1$, the equilibrium M_1 becomes unstable, and system (11) exhibits the Neimark-Sacker bifurcation [32] with the birth of a closed invariant curve. In Figure 7, x-coordinates of attractors of model (11) are plotted. With increasing μ, in system (11), an alternation of regular and chaotic oscillations with increasing amplitude is observed. Here, similar to one-dimensional Ricker model (1), the persistence μ-zone is unbounded: $0 < \mu < \infty$.

Consider now the stochastic variant of the Ricker model with delay:

$$\begin{aligned} x_{t+1} &= x_t \left[\exp\left(\mu\left(1 - y_t\right)\right) + \varepsilon\xi_t\right]^+, \\ y_{t+1} &= x_t. \end{aligned} \tag{12}$$

Here, ξ_t is a standard uncorrelated Gaussian random process with parameters $E(\xi_t) = 0$ and $E(\xi_t^2) = 1$, and ε is the noise intensity.

For this model with delay, the noise contracts the persistence zone as well. In the presence of noise, this persistence μ-zone becomes finite: the larger the noise intensity ε, the smaller the size of the persistence zone. It is clearly seen in Figure 8 where random states of system (12) are shown for three values of ε. Consider now how to stabilize the equilibrium M_1 for the whole interval $\mu > 0$ and provide small-amplitude stochastic oscillations near this equilibrium.

Consider the stochastic model with control input

$$\begin{aligned} x_{t+1} &= x_t \left[\exp\left(\mu\left(1 - y_t\right)\right) + \varepsilon\xi_t\right]^+ + u_t, \\ y_{t+1} &= x_t, \end{aligned} \tag{13}$$

where u_t is formed by the feedback regulator

$$u_t = k_1\left(x_t - \overline{x}_1\right) + k_2\left(y_t - \overline{y}_1\right). \tag{14}$$

Here, $\overline{x}_1 = 1$ and $\overline{y}_1 = 1$ are coordinates of the equilibrium M_1.

In notations of the Appendix, for system (13) and the equilibrium M_1, we have the following:

$$\begin{aligned} f(x, y, u, \eta) &= \begin{bmatrix} x\exp\left(\mu\left(1 - y\right)\right) + u + x\eta \\ y \end{bmatrix}, \\ F &= \begin{bmatrix} 1 & -\mu \\ 1 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ G &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ S &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ K &= \begin{bmatrix} k_1 & k_2 \end{bmatrix}, \\ F + BK &= \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}, \quad a = 1 + k_1, \; b = -\mu + k_2. \end{aligned} \tag{15}$$

The stability condition $\rho(F + BK) < 1$ is equivalent to the system of inequalities: $|b| < 1$, $|a| < 1 - b$. So, parameters k_1 and k_2 of the stabilizing regulator (14) are

$$\begin{aligned} \left|k_2 - \mu\right| &< 1, \\ \left|1 + k_1\right| &< 1 + \mu - k_2. \end{aligned} \tag{16}$$

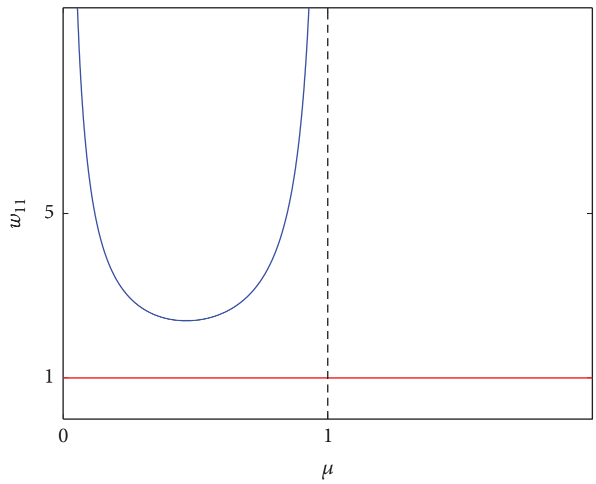

FIGURE 9: Plot of the element $w_{11}(\mu)$ of the stochastic sensitivity matrix W of the equilibrium M_1 in system (12).

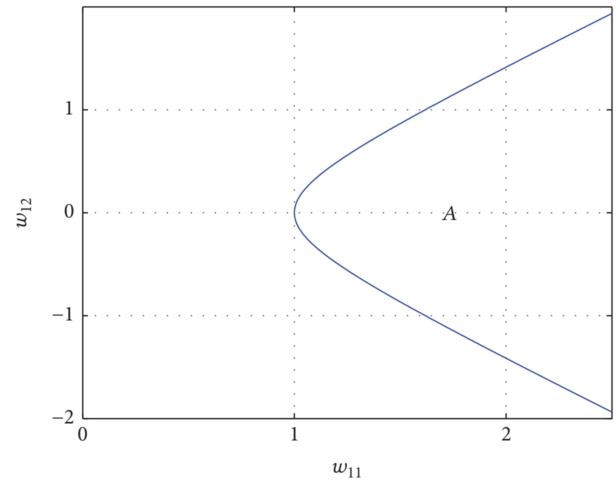

FIGURE 10: Attainability domain.

System (A.8) for the elements of the stochastic sensitivity matrix $W = \left[\begin{smallmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{smallmatrix}\right]$ can be written as

$$\begin{aligned} w_{11} &= a^2 w_{11} + 2ab w_{12} + b^2 w_{22} + 1, \\ w_{12} &= a w_{11} + b w_{12}, \\ w_{22} &= w_{11}. \end{aligned} \tag{17}$$

Here, due to the symmetry of the matrix W, we have $w_{21} = w_{12}$.

System (17) has the solution

$$\begin{aligned} w_{11} = w_{22} &= \frac{1-b}{(1+b)\left((1-b)^2 - a^2\right)}, \\ w_{12} &= \frac{a}{(1+b)\left((1-b)^2 - a^2\right)}. \end{aligned} \tag{18}$$

In the absence of the control ($k_1 = k_2 = 0$), we have

$$\begin{aligned} w_{11} = w_{22} &= \frac{1+\mu}{\mu(1-\mu)(2+\mu)}, \\ w_{12} &= \frac{1}{\mu(1-\mu)(2+\mu)}. \end{aligned} \tag{19}$$

The function $w_{11}(\mu)$ is plotted in the stability zone $0 < \mu < 1$ of the equilibrium M_1 in Figure 9. It can be noted that the introduced delay increases the stochastic sensitivity of the stable equilibrium (compare Figures 4 and 9).

The problem of the synthesis of the assigned stochastic sensitivity matrix W is reduced to finding a and b from system (17). The corresponding solution is given by the formulas

$$\begin{aligned} b &= \pm\sqrt{1 - \frac{w_{11}}{w_{11}^2 - w_{12}^2}}, \\ a &= \frac{w_{12}}{w_{11}}\left(1 \mp \sqrt{1 - \frac{w_{11}}{w_{11}^2 - w_{12}^2}}\right). \end{aligned} \tag{20}$$

Note that this solution cannot be found for any matrix W. Elements of the attainable matrices satisfy the following conditions:

$$\begin{aligned} w_{11} &= w_{22} > 0, \\ w_{11} &\le w_{11}^2 - w_{12}^2. \end{aligned} \tag{21}$$

The attainability domain A is plotted in Figure 10. Any attainable matrix can be provided by regulator (14) with parameters

$$\begin{aligned} k_1 &= \frac{w_{12}}{w_{11}}\left(1 \mp \sqrt{1 - \frac{w_{11}}{w_{11}^2 - w_{12}^2}}\right) - 1, \\ k_2 &= \mu \pm \sqrt{1 - \frac{w_{11}}{w_{11}^2 - w_{12}^2}}. \end{aligned} \tag{22}$$

As one can see, in the attainability domain, the minimum value $w_{11} = 1$ corresponds to $w_{12} = 0$. So, we get the stochastic matrix $W = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $k_1 = -1$, $k_2 = \mu$.

Results of the modeling of system (12) without control and system (13) with regulator (14) which synthesizes minimum values of the attainable stochastic sensitivity $w_{11} = 1$ are shown in Figures 11 and 12. In Figure 11(a), for $\varepsilon = 0.01$, random states of the uncontrolled system are plotted by blue color, and for the system with control they are shown by red. In Figure 11(b), for larger noise intensity $\varepsilon = 0.1$, random states of the uncontrolled system are plotted by light blue, and for the system with control they are shown by red. As can be seen, the regulator stabilizes M_1 and provides uniform dispersion in a wide range of the parameter μ.

In Figure 12, phase trajectories and time series of the x-coordinate of the stochastic system with $\mu = 1.2$ without control are plotted by blue color, and with control they are plotted by red color for the optimal regulator providing $w_{11} = 1$. Here, we consider $\varepsilon = 0.01$ (a) and $\varepsilon = 0.1$ (b). For small noise ($\varepsilon = 0.01$), our regulator sharply decreases the amplitude of stochastic oscillations. For larger noise ($\varepsilon = 0.1$), in the system without control, the population becomes extinct. The

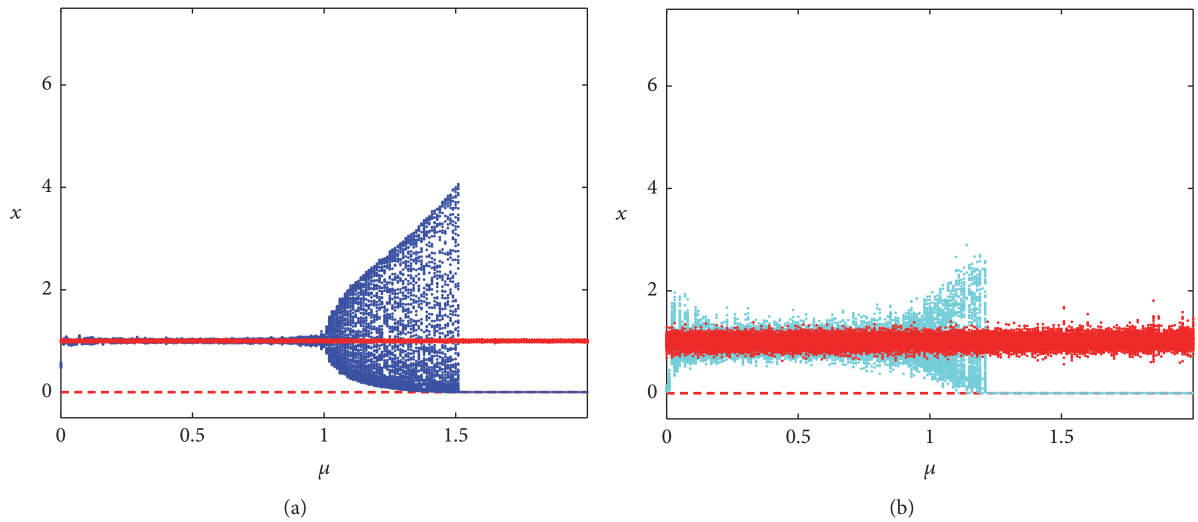

Figure 11: Random states of the controlled stochastic system (12) without control (blue) and with control providing $w = 1$ (red) for (a) $\varepsilon = 0.01$ and (b) $\varepsilon = 0.1$.

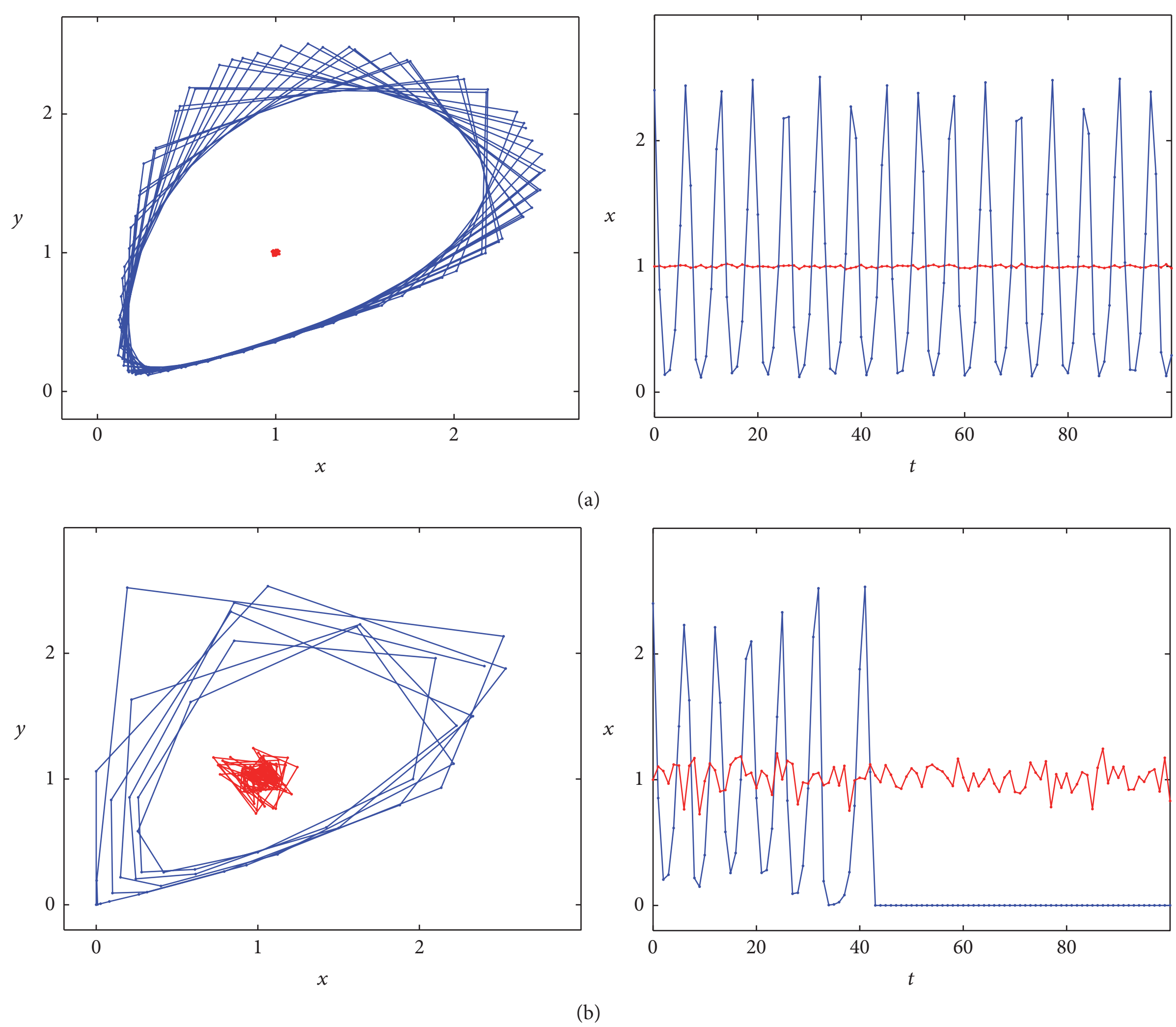

Figure 12: Random trajectories and time series of the stochastic system for $\mu = 1.2$ without control (blue) and with control providing $w = 1$ (red) for (a) $\varepsilon = 0.01$ and (b) $\varepsilon = 0.1$.

constructed regulator prevents this unwanted ecological shift, stabilizes M_1, and provides admissible stochastic oscillations near M_1.

4. Conclusions

We studied the noise-induced extinction in nonlinear population models. For the mathematical analysis of this phenomenon, we used a general theoretical approach based on the stochastic sensitivity functions technique. Constructive abilities of this approach were demonstrated on the conceptual 1D and 2D stochastic Ricker models. To prevent the noise-induced extinction, we reduced a level of the stochastic sensitivity of the nontrivial equilibrium by the appropriate control. A mathematical background of the theory of the stochastic sensitivity synthesis by feedback regulators was briefly presented. For Ricker models, it was shown how these regulators provide a low stochastic sensitivity and keep the system close to the required safe equilibrium regime.

Appendix

Controlling Stochastic Sensitivity

Consider a general discrete-time stochastic system with control:

$$x_{t+1} = f(x_t, u_t, \eta_t). \tag{A.1}$$

Here, x is an n-dimensional vector of system state, u is an m-dimensional vector of control parameters, η is an l-dimensional vector of random disturbances, and $f(x)$ is a smooth vector function. It is supposed that $\eta_t = \varepsilon\xi_t$, where ξ_t is an l-dimensional discrete-time uncorrelated random process with parameters

$$\begin{aligned} E\xi_t &= 0, \\ E\xi_t\xi_t^\top &= V, \\ E\xi_t\xi_k^\top &= 0, \\ &(t \neq k), \end{aligned} \tag{A.2}$$

where V is an $l \times l$ matrix and ε is a scalar parameter of the noise intensity.

Let $\bar{x}$ be an equilibrium of the uncontrolled deterministic system (A.1) with $u = 0$ and $\eta = 0$. Stability of $\bar{x}$ is not supposed. The control input will be formed by the appropriate feedback function $u(x)$.

Consider a solution x_t^ε of system (A.1) with the initial condition $x_0^\varepsilon = \bar{x} + \varepsilon z_0$, where z_0 is an n-vector of the initial disturbance. The sensitivity of the equilibrium $\bar{x}$ both to initial data and to current random disturbances is defined by the derivative

$$z_t = \left.\frac{\partial x_t^\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0}. \tag{A.3}$$

The variable z_t is governed by the linear stochastic equation

$$z_{t+1} = (F + BK) z_t + G\xi_t, \tag{A.4}$$

where

$$\begin{aligned} F &= \frac{\partial f}{\partial x}(\bar{x}, 0, 0), \\ B &= \frac{\partial f}{\partial u}(\bar{x}, 0, 0), \\ K &= \frac{\partial u}{\partial x}(\bar{x}), \\ G &= \frac{\partial f}{\partial \eta}(\bar{x}, 0, 0). \end{aligned} \tag{A.5}$$

The dynamics of the moments $m_t = Ez_t$, $M_t = Ez_t z_t^\top$ for the solution z_t of system (A.4) satisfy the equations

$$m_{t+1} = (F + BK) m_t, \tag{A.6}$$

$$M_{t+1} = (F + BK) M_t (F + BK)^\top + S, \quad S = GVG^\top. \tag{A.7}$$

Consider a set of matrices $\mathbb{K} = \{K \in \mathbb{R}^{m\times n},\ \rho(F + BK) < 1\}$, where $\rho(A)$ is a spectral radius of the matrix A. Suppose that the pair (F, B) is stabilized [33], so the set $\mathbb{K}$ is not empty.

For any $K \in \mathbb{K}$, (A.7) has a unique stationary stable solution $W = \lim_{t\to\infty} M_t$. The matrix W satisfies the following algebraic equation:

$$W = (F + BK) W (F + BK)^\top + S. \tag{A.8}$$

The matrix W characterizing the response of system (A.1) to the small random disturbances is called the stochastic sensitivity matrix [34] of the equilibrium $\bar{x}$.

Consider how this stochastic sensitivity matrix depends on the control function $u(x)$. As can be seen, the matrix W is defined only by the local parameters $K = (\partial u/\partial x)(\bar{x})$. So, here we can restrict ourselves by the simple linear regulators in the following form:

$$u(x) = K(x - \bar{x}). \tag{A.9}$$

Consider now a problem of the synthesis of the stochastic sensitivity matrix W by the appropriate regulator (A.9). Let

$$\mathbb{M} = \{M \in \mathbb{R}^{n\times n} \mid M \succ 0\} \tag{A.10}$$

be a set of admissible stochastic sensitivity matrices ($M \succ 0$ means that the matrix M is symmetric and positive definite). Denote by W_K a solution of (A.8) for the fixed matrix $K \in \mathbb{K}$. For any $K \in \mathbb{K}$, the matrix W_K belongs to $\mathbb{M}$ if the pair $(F + BK, S)$ is controllable [33].

The aim of the proposed control design is the synthesis of the desired stochastic sensitivity matrix.

Let $\overline{W}$ be a desired stochastic sensitivity matrix of system (A.1) with feedback (A.9). To find a matrix $K \in \mathbb{K}$ guaranteeing the equality $W_K = \overline{W}$, we have to solve the following quadratic matrix equation:

$$\begin{aligned} \overline{W} = {} & F\overline{W}F^\top + F\overline{W}K^\top B^\top + BK\overline{W}F^\top + BK\overline{W}K^\top B^\top \\ & + S. \end{aligned} \tag{A.11}$$

In some cases, this equation is unsolvable. Here, it is important to analyze the attainability of W (see details in [25]).

For one-dimensional case ($n = m = l = 1$), (A.11) is the following:

$$B^2WK^2 + 2BFWK + F^2W + S - W = 0. \tag{A.12}$$

A discriminant of this quadratic equation is $D = 4B^2W(W - S)$, so the inequality $W \geq S$ is an attainability condition here. For $B \neq 0$, we get an explicit formula for the feedback coefficient:

$$K = -\frac{1}{B}\left(-F \pm \sqrt{1 - \frac{S}{W}}\right). \tag{A.13}$$

The function W_K has the following representation:

$$W_K = \frac{S}{1 - (F + BK)^2}. \tag{A.14}$$

The value of W_K is minimal ($W_K = S$) when $K = -F/B$.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the Russian Science Foundation (no. 16-11-10098).

References

[1] A. D. Bazykin, *Nonlinear Dynamics of Interacting Populations*, vol. 11, World Scientific, River Edge, NJ, USA, 1998.

[2] P. Turchin, *Complex Population Dynamics: A Theoretical/Empirical Synthesis*, vol. 35, Princeton University Press, Princeton, NJ, USA, 2003.

[3] R. Lande, S. Engen, and B. E. Saether, *Stochastic Population Dynamics in Ecology and Conservation*, Oxford University Press, Oxford, UK, 2003.

[4] L. Ridolfi, P. D'Odorico, and F. Laio, *Noise-Induced Phenomena in The Environmental Sciences*, Cambridge University Press, Cambridge, UK, 2011.

[5] M. Scheffer, S. Carpenter, J. A. Foley, C. Folke, and B. Walker, "Catastrophic shifts in ecosystems," *Nature*, vol. 413, no. 6856, pp. 591–596, 2001.

[6] R. M. May, "Stability in randomly fluctuating versus deterministic environments," *The American Naturalist*, vol. 107, no. 957, pp. 621–650, 1973.

[7] P. Chesson, "Stochastic population models," in *Ecological Heterogeneity, Ecological Studies: Analysis and Synthesis*, J. Kolasa and S. T. A. Pickett, Eds., vol. 86, pp. 123–143, Springer-Verlag, New York, NY, USA, 1991.

[8] R. Lande, "Risks of population extinction from demographic and environmental stochasticity and random catastrophes," *American Naturalist*, vol. 142, no. 6, pp. 911–927, 1993.

[9] L. J. Allen and P. van den Driessche, "Relations between deterministic and stochastic thresholds for disease extinction in continuous- and discrete-time infectious disease models," *Mathematical Biosciences*, vol. 243, no. 1, pp. 99–108, 2013.

[10] Z. Jin, G. Sun, and H. Zhu, "Epidemic models for complex networks with demographics," *Mathematical Biosciences and Engineering. MBE*, vol. 11, no. 6, pp. 1295–1317, 2014.

[11] X. Luo, L. Chang, and Z. Jin, "Demographics induce extinction of disease in an SIS model based on conditional Markov chain," *Journal of Biological Systems*, vol. 25, no. 1, pp. 145–171, 2017.

[12] I. Bashkirtseva and L. Ryashko, "How environmental noise can contract and destroy a persistence zone in population models with Allee effect," *Theoretical Population Biology*, vol. 115, pp. 61–68, 2017.

[13] A. A. Berryman, "Stabilization or regulation: what it all means!," *Oecologia*, vol. 86, no. 1, pp. 140–143, 1991.

[14] A. A. Berryman, "Limiting factors and population regulation," *Oikos*, vol. 105, no. 3, pp. 667–670, 2004.

[15] C. Folke, T. Hahn, P. Olsson, and J. Norberg, "Adaptive governance of social-ecological systems," *Annual Review of Environment and Resources*, vol. 30, pp. 441–473, 2005.

[16] F. M. Hilker and F. H. Westerhoff, "Preventing extinction and outbreaks in chaotic populations," *American Naturalist*, vol. 170, no. 2, pp. 232–241, 2007.

[17] L. Ryashko and I. Bashkirtseva, "Stochastic sensitivity analysis and control for ecological model with the Allee effect," *Mathematical Modelling of Natural Phenomena*, vol. 10, no. 2, pp. 130–141, 2015.

[18] E. Ott, C. Grebogi, and J. A. Yorke, "Controlling chaos," *Physical Review Letters*, vol. 64, no. 11, pp. 1196–1199, 1990.

[19] A. L. Fradkov and A. Y. Pogromsky, *Introduction to Control of Oscillations and Chaos*, vol. 35, World Scientific Series of Nonlinear Science, River Edge, NJ, USA, 1998.

[20] S. Boccaletti, C. Grebogi, Y.-C. Lai, H. Mancini, and D. Maza, "The control of chaos: theory and applications," *Physics Reports. A Review Section of Physics Letters*, vol. 329, no. 3, pp. 103–197, 2000.

[21] G. Chen and X. Yu, Eds., *Chaos Control: Theory and Applications*, Springer, New York, NY, USA, 2003.

[22] J.-Q. Sun, *Stochastic Dynamics and Control*, vol. 4, Elsevier, Oxford, UK, 2006.

[23] I. Bashkirtseva, L. Ryashko, and P. Stikhin, "Noise-induced backward bifurcations of stochastic 3D-cycles," *Fluctuation and Noise Letters*, vol. 9, no. 1, pp. 89–106, 2010.

[24] I. Bashkirtseva, "Stochastic phenomena in one-dimensional Rulkov model of neuronal dynamics," *Discrete Dynamics in Nature and Society*, vol. 2015, Article ID 495417, 7 pages, 2015.

[25] I. A. Bashkirtseva and L. B. Ryashko, "On stochastic sensitivity control in discrete systems," *Automation and Remote Control*, vol. 71, no. 9, pp. 1833–1848, 2010.

[26] I. Bashkirtseva and L. Ryashko, "Control of equilibria for nonlinear stochastic discrete-time systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 9, pp. 2162–2166, 2011.

[27] E. W. Ricker, "Stock and recruitment," *Journal of the Fisheries Research Board of Canada*, vol. 11, no. 5, pp. 559–623, 1954.

[28] G. Högnäs, "On the quasi-stationary distribution of a stochastic Ricker model," *Stochastic Processes and their Applications*, vol. 70, no. 2, pp. 243–263, 1997.

[29] T. Gadrich and G. Katriel, "A mechanistic stochastic Ricker model: analytical and numerical investigations," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 26, no. 4, Article ID 1650067, 21 pages, 2016.

[30] S. A. Levin and R. M. May, "A note on difference-delay equations," *Theoretical Population Biology. An International Journal*, vol. 9, no. 2, pp. 178–187, 1976.

[31] F. A. Bartha, A. Garab, and T. Krisztin, "Local stability implies global stability for the 2-dimensional Ricker map," *Journal of Difference Equations and Applications*, vol. 19, no. 12, pp. 2043–2078, 2013.

[32] R. J. Sacker, *On Invariant Surfaces and Bifurcation of Periodic Solutions of Ordinary Differential Equations*, vol. 333 of *Report IMM-NYU*, New York University, New York, NY, USA, 1964.

[33] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, vol. 10 of *Applications of Mathematics*, Springer-Verlag, New York, NY, USA, 2nd edition, 1979.

[34] I. Bashkirtseva, L. Ryashko, and I. Tsvetkov, "Sensitivity analysis of stochastic equi-libria and cycles for the discrete dynamic systems," *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, vol. 17, no. 4, pp. 501–515, 2010.