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Research Article

Pricing Collar Options with Stochastic Volatility

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This paper studies collar options in a stochastic volatility economy. The underlying asset price is assumed to follow a continuous geometric Brownian motion with stochastic volatility driven by a mean-reverting process. The method of asymptotic analysis is employed to solve the PDE in the stochastic volatility model. An analytical approximation formula for the price of the collar option is derived. A numerical experiment is presented to demonstrate the results.

1. Introduction

The collar option is one kind of exotic option. The payoff of a collar option at expiry time T is $\min(\max(S_T, K_1), K_2)$, where $K_2 > K_1 > 0$ and S_T is the underlying asset price at expiry time. The collar option is useful when institutional investors wish to lock in the profit they already have on the underlying asset. It can be easily seen that the collar option at expiry time T can be expressed as the following:

$$\begin{aligned} \text{Collar}_T &= K_1 - (S_T - K_2)^+ + (S_T - K_1)^+ \\ \text{or Collar}_T &= S_T - (S_T - K_2)^+ + (K_1 - S_T)^+ . \end{aligned} \tag{1}$$

From the first expression it is easily observed that a collar option is equivalent to lending the present value of K_1, shorting a call option with strike price K_2, and purchasing a call option with strike price K_1. From the second expression, we find that a collar option can be seen as a strategy of holding an underlying asset, selling a call option with strike price K_2, and buying a put option with strike price K_1.

The second strategy is very popular among investors or institutions. Collar options can be implemented by investors on the stock they have already owned. Usually investors will obtain the collar when they have enjoyed a decent gain on their investment but they want to hedge against potential downside in their shares. In other words, if they were bullish but cautious on the underlying asset, a collar might be the right strategy. To initiate the trade, they will buy a put option and sell simultaneously a call option. Both options will be out of the money and the call option they sell will be further out of money than the put option they buy. Additionally both contracts will have the same expiration dates. The benefit of opening both options position is that the premium they collect from the selling of the call option can either partially or totally offset the premium pay for the put. In this way they can ensure their investment against major downside move for a little or no cost. There are several outcomes of the collar strategy: if the stock falls below the strike price of the purchase of the put, they have the opportunity to sell those shares above the market value by exercising their option to limit the losses. If the stock price rises above the strike price of their sold call, there is a strong possibility the stock will be called away. Finally it is possible that the stock will remain stuck between the two strike prices at expiration, which means both of their options will expire worthlessly. In summary, a collar strategy is a relatively low cost way to buffer against the downside move of the stock that an investor owns, but there is a tradeoff, namely, that an investor also admits upside potential.

Collar options are also very useful and practical instruments in revenue management and project management. Shan et al. (2010) [1] study the use of collar options to manage revenue risks in real toll public-private partnership transportation projects, in particular how to redistribute the profit, and losses in order to improve the effectiveness of risk management and fulfill the stakeholder's needs. Yim et al. (2011) [2] proposed a material procurement contracts model

to which the zero-cost collar option is applied for hedging the price fluctuation in construction.

The pricing problem of collar option can be solved in the classical Black-Scholes framework in which the volatility of the underlying asset is assumed to be constant. However the constant volatility hypothesis in the classical Black-Scholes framework is hardly satisfied as the volatility smile or smirk is commonly observed in the financial markets. The existence of the volatility smile or smirk which is calculated through the volatility implied by the Black-Scholes formula suggests that a geometric Brownian motion with constant volatility misses some critical features of the data.

Different kinds of methods were proposed to overcome the unrealistic assumption of constant volatility, while two types of models were proved to be successful. One is jump-diffusion model in which the underlying asset returns do not follow a stochastic process with continuous sample path but subject to random jumps. Another is stochastic volatility model in which a random volatility of the underlying asset is assumed to follow a diffusion process. Bakshi et al. (1997) [3] found that, by incorporating stochastic into option pricing, the classical Black-Scholes model can be improved remarkably. After the introduction of the stochastic volatility, introducing random jumps leads to only small reduction in option pricing error. Hence this paper will concentrate on the stochastic volatility model.

Stochastic volatility models of the underlying assets are supported by empirical studies of stock returns. High frequency stock returns data displays excess kurtosis and skewness. Additionally the phenomenon of sustained periods of low-variability alternating with high-variability and the nonconstant implied volatility across time have been observed based on S&P 500 index by Fouque et al. (2000) [4] and Chernov et al. (2003) [5] found strong evidence for stochastic volatility in the returns of stock price in financial markets. Through the study of the stock returns of China's stock market, Wu et al. (2014) [6] argued that stochastic volatility model improved significantly in model fitting. Besides stock returns, other underlying asset prices such as oil spot price display the characteristics of stochastic volatility. Larsson and Nossman (2011) [7] showed that an acceptable representation of oil price dynamics can be fit by combining stochastic volatility and jump processes.

Various stochastic volatility models are presented by making different assumptions of the process of the underlying asset's volatility. A typical case of these processes is the mean-reverting Ornstein-Uhlenbeck process. Hull and White (1987) [8] and Heston (1993) [9] are among the most widely popular stochastic volatility models. These two models are also proved to be very successful in practical implement, particularly in Heston model the call price is available in closed form. When volatility follows a fast mean-reverting process, the singular perturbation analysis can be employed to solve the partial differential equation of the pricing problem. Fouque et al. (2003) [10] performed singular perturbation analysis on the fast mean-reverting stochastic volatility model and the accuracy of the corresponding approximation in the presence of the nonsmoothness of payoff functions (such as call option) is also studied. In recent years many studies on option pricing under stochastic volatility have been carried out such as Wong and Chan (2007) [11], (2008) [12], and Yang et al. (2014) [13]. To the best of our knowledge, research on the collar options has not been carried out under the stochastic volatility.

Under the assumption that volatility of the underlying asset obeys a fast mean-reverting Ornstein-Uhlenbeck process, we study the pricing problem of the collar option by singular perturbation analysis. The advantages of this method are as follows: firstly the number of parameters in the stochastic volatility model can be effectively reduced and secondly an analytic approximation formula for the collar option can be obtained. The rest of the paper is organized as follows. The model setting of the underlying stochastic volatility model for collar option is presented in Section 2. The partial differential equation for the collar option and an approximation formula are derived in Section 3. A numerical illustration is presented in Section 4, and Section 5 concludes the paper.

2. Stochastic Volatility Model Setting

Denote S_t as the underlying asset price at time t. The underlying asset price is assumed to follow a geometric Brownian motion with a constant μ in drift and its volatility $f(Y_t)$ depends on the mean-reverting process Y_t. The function $f(y)$ is assumed to be sufficiently regular. The mean-reverting process Y_t evolves as an Ornstein-Uhlenbeck process with a positive mean-reverting rate α, an equilibrium level m, and the volatility of the volatility β. Denote W_t and Z_t as two independent standard Brownian motions and ρ is the correlation coefficient between these two Brownian motions. Under the real world probability measure $\mathbb{P}$, the model can be written as

$$\begin{aligned} dS_t &= \mu S_t dt + f(Y_t) S_t dW_t \\ dY_t &= \alpha(m - Y_t) + \beta\left(\rho dW_t + \sqrt{1-\rho^2} dZ_t\right). \end{aligned} \tag{2}$$

Given $Y_0 = y$, Y_t is normally distributed with mean $m + (y - m)e^{-\alpha t}$ and variance $\beta^2(1 - e^{-2\alpha t})/2\alpha$. As $t \to +\infty$, a unique invariant distribution of Y_t is derived. Such invariant distribution is normally distributed with mean m and variance $v^2 = \beta^2/(2\alpha)$. In other words, denoting $\Phi(y) = \exp(-(y - m)^2/(2v^2))/(v\sqrt{2\pi})$, then $\Phi(y)$ is the invariant distribution of Y_t. In order to perform asymptotic analysis, a small parameter ε is introduced and the mean-reverting rate α is defined by $\alpha = 1/\varepsilon$. Using Girsanov's theorem, under the risk-neutral world probability measure $\mathbb{P}^*$, the model can be written as

$$\begin{aligned} dS_t &= rS_t dt + f(Y_t) dW_t^* \\ dY_t &= \left[\frac{1}{\varepsilon}(m - Y_t) + \frac{v\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t)\right] dt \\ &\quad + \frac{v\sqrt{2}}{\sqrt{\varepsilon}}\left(\rho dW_t^* + \sqrt{1-\rho^2} dZ_t^*\right) \\ \Lambda(y) &= \rho\frac{\mu - r}{f(y)} + \gamma(y)\sqrt{1-\rho^2}. \end{aligned} \tag{3}$$

We assume that the market price of volatility risk (or risk premium factor) γ is a bounded function of y only, and r is a constant risk-free interest rate. W_t^* and Z_t^* are two independent standard Brownian motions under the risk-neutral world probability measure $\mathbb{P}^*$.

3. Collar Option Pricing with Stochastic Volatility

Under the risk-neutral world probability measure $\mathbb{P}^*$, the process (S_t, Y_t) is a Markov process. The price of the collar option at time t is a function of the present value of the underlying asset $S_t = s$ and the present value of the process driving the volatility $Y_t = y$. Denote the price of the collar option at time $t < T$ (T is the contract's maturity time) as $V(t, s, y)$; by the risk-neutral pricing method, the price of the collar option with stochastic volatility is given as

$$V(t,s,y) = \mathbb{E}^* \left\{ e^{-r(T-t)} \min\left(\max\left(S_T, K_1\right), K_2\right) \mid S_t = s,\ Y_t = y \right\}. \tag{4}$$

By the Feynman-Kac formula, the pricing function given can be obtained as the solution of the partial differential equation below:

$$\begin{aligned} &\frac{\partial V}{\partial t} + r\left(s\frac{\partial V}{\partial s} - V\right) + \frac{1}{2}f^2(y)s^2\frac{\partial^2 V}{\partial s^2} \\ &\quad + \frac{1}{\varepsilon}\left[(m-y)\frac{\partial V}{\partial y} + v^2\frac{\partial^2 V}{\partial y^2}\right] \\ &\quad + \frac{1}{\sqrt{\varepsilon}}\left[v\sqrt{2}\rho f(y)s\frac{\partial^2 V}{\partial s\partial y} - v\sqrt{2}\Lambda(y)\frac{\partial V}{\partial y}\right] = 0 \end{aligned} \tag{5}$$

with the terminal condition

$$V(T,s,y) = \min\left(\max\left(S_T, K_1\right), K_2\right). \tag{6}$$

By introducing the following operators

$$\begin{aligned} \mathscr{L}_0 &= (m-y)\frac{\partial}{\partial y} + v^2\frac{\partial^2}{\partial y^2}, \\ \mathscr{L}_1 &= v\sqrt{2}\left[\rho f(y)s\frac{\partial^2}{\partial s\partial y} - \Lambda(y)\frac{\partial}{\partial y}\right] \\ \mathscr{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2}f^2(y)s^2\frac{\partial^2}{\partial s^2} + rs\frac{\partial}{\partial s} - r. \end{aligned} \tag{7}$$

the partial differential equation (5) becomes

$$\left(\frac{1}{\varepsilon}\mathscr{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathscr{L}_1 + \mathscr{L}_2\right)V = 0 \tag{8}$$

with the terminal condition (6).

Problem (8) is called a singular perturbation problem. In the following, we are going to solve this problem in asymptotic expansions under the assumption that $\alpha > 0$ is a fast mean-reverting rate, in particular $0 < \varepsilon \ll 1$. The price of the collar option V can be expanded in the powers of $\sqrt{\varepsilon}$ as below

$$V = V_0 + \sqrt{\varepsilon}V_1 + \varepsilon V_2 + \varepsilon\sqrt{\varepsilon}V_3 + \cdots, \tag{9}$$

where $V_0, V_1, \ldots$ are functions of (t, s, y) that will be solved one by one, and we are primarily interested in the first two terms. Denote V_0 and $V_1^\varepsilon = \sqrt{\varepsilon}V_1$ as the zero-order approximation and first-order approximation for the collar option. We will use $\widetilde{V} = V_0 + \sqrt{\varepsilon}V_1$ to approximate the price of the collar option in a fast mean-reverting economy. The terminal conditions for the first two terms in (9) are $V_0(T, s, y) = \min(\max(S_T, K_1), K_2)$ and $V_1(T, s, y) = 0$.

Substituting (9) into (8) leads to

$$\begin{aligned} &\frac{1}{\varepsilon}\mathscr{L}_0V_0 + \frac{1}{\sqrt{\varepsilon}}\left(\mathscr{L}_0V_1 + \mathscr{L}_1V_0\right) \\ &\quad + \left(\mathscr{L}_0V_2 + \mathscr{L}_1V_1 + \mathscr{L}_2V_0\right) \\ &\quad + \sqrt{\varepsilon}\left(\mathscr{L}_0V_3 + \mathscr{L}_1V_2 + \mathscr{L}_2V_1\right) + \cdots = 0. \end{aligned} \tag{10}$$

By collecting the terms of order $1/\varepsilon$, we end up with equation $\mathscr{L}_0V_0 = 0$. The fact that the operator $\mathscr{L}_0$ acts only on the variable y confirms that V_0 must be a function independent of y. Similarly, collecting the terms of order $1/\sqrt{\varepsilon}$ yields $\mathscr{L}_0V_1 + \mathscr{L}_1V_0 = 0 \Rightarrow \mathscr{L}_0V_1 = 0$, since $\mathscr{L}_1$ involves y differentials. It is clear that V_1 is also independent of y. This implies that the first and the second term (V_0 and V_1) in (9) will not depend on the present volatility y. In the same vein, we can continue this approach to obtain the order-1 terms: $\mathscr{L}_0V_2 + \mathscr{L}_1V_1 + \mathscr{L}_2V_0 = 0$. Because V_1 is independent of y it is obvious that $\mathscr{L}_1V_1 = 0$, so we have the following equation:

$$\mathscr{L}_0V_2 + \mathscr{L}_2V_0 = 0. \tag{11}$$

Equation (11) is of the form of a Poisson equation for V_2 with respect to the operator $\mathscr{L}_0$ in the variable y. The unique solution of this equation exists if

$$\left\langle \mathscr{L}_2V_0 \right\rangle = \int \mathscr{L}_2V_0\Phi(y)\,dy = 0; \tag{12}$$

the operator $\langle\cdot\rangle$ denotes the expectation with respect to the invariant distribution of Y. This fact is known as the Fredholm solvability for Poisson equations.

Theorem 1. *The zero-order approximation term for the price of the collar option in a fast mean-reverting stochastic volatility economy is the solution of the Black-Scholes equation with constant volatility.*

Proof. Since V_0 is independent of y, by the direct calculation we have $\langle\mathscr{L}_2V_0\rangle = \langle\mathscr{L}_2\rangle V_0 = 0$. From the definition of $\mathscr{L}_2$, we can deduce that $\langle\mathscr{L}_2\rangle = \partial/\partial t + (1/2)\langle f^2(y)\rangle s^2(\partial^2/\partial s^2) + r(s(\partial/\partial s) - \cdot)$; by defining the effective volatility $\overline{\sigma}^2$ as $\overline{\sigma}^2 = \langle f^2(y)\rangle = \int f^2(y)\Phi(y)dy$, the operator $\langle\mathscr{L}_2\rangle$ can be written as

$$\left\langle \mathscr{L}_2 \right\rangle = \frac{\partial}{\partial t} + \frac{1}{2}\overline{\sigma}^2 s^2\frac{\partial^2}{\partial s^2} + r\left(s\frac{\partial}{\partial s} - \cdot\right); \tag{13}$$

hence the zero-order approximation term $V_0(t,s)$ is the solution of the Black-Scholes partial differential equation with constant volatility $\overline{\sigma}^2$,

$$\frac{\partial V_0}{\partial t}+\frac{1}{2}\overline{\sigma}^2 s^2\frac{\partial^2 V_0}{\partial s^2}+r\left(s\frac{\partial V_0}{\partial s}-V_0\right)=0 \tag{14}$$

and final condition $V_0(T,s,y)=\min(\max(S_T,K_1),K_2)$. □

Considering the next order term in $\sqrt{\varepsilon}$, we have $\mathscr{L}_0V_3+\mathscr{L}_1V_2+\mathscr{L}_2V_1=0$. This is a Poisson equation for V_3 with respect to $\mathscr{L}_0$. After applying the Fredholm solvability condition and the fact that V_1 is independent of y, the last equation reduces to $\langle\mathscr{L}_1V_2\rangle+\langle\mathscr{L}_2\rangle V_1=0$. From the equation $\langle\mathscr{L}_2\rangle V_0=0$ we have $\mathscr{L}_2V_0=\mathscr{L}_2V_0-\langle\mathscr{L}_2V_0\rangle=(1/2)(f^2(y)-\overline{\sigma}^2)s^2(\partial^2V_0/\partial s^2)$; hence V_2 the solution of the equation $\mathscr{L}_0V_2-\mathscr{L}_2V_0=0$ is given by

$$\begin{aligned}V_2(t,s,y)&=-\frac{1}{2}\mathscr{L}_0^{-1}\left(f^2(y)-\overline{\sigma}^2\right)s^2\frac{\partial^2V_0}{\partial s^2}\\&=-\frac{1}{2}\left(\phi(y)+c(t,x)\right)s^2\frac{\partial^2V_0}{\partial s^2}.\end{aligned} \tag{15}$$

where $\phi(y)$ is defined as the solution of the Poisson equation $\mathscr{L}_0\phi=f^2(y)-\overline{\sigma}^2$.

Theorem 2. *The first-order approximation term V_1^ε can be expressed by the zero-order approximation term V_0 as the following: $V_1^\varepsilon=-(T-t)(C_1s^2(\partial^2V_0/\partial s^2)+C_2s^3(\partial^3V_0/\partial s^3))$, where $C_1=(2\rho\langle f\phi'\rangle-\langle\Lambda\phi'\rangle)\nu\sqrt{\varepsilon}/\sqrt{2}$ and $C_1=\langle f\phi'\rangle\rho\nu\sqrt{\varepsilon}/\sqrt{2}$; $\phi'=d\phi/dy$.*

Proof. Because $\langle\mathscr{L}_1V_2\rangle+\langle\mathscr{L}_2\rangle V_1=0$, the fact that V_1 is independent of y and the result of (15), we can obtain $\langle\mathscr{L}_2\rangle V_1=(1/2)\langle\mathscr{L}_1\phi(y)\rangle s^2(\partial^2V_0/\partial s^2)$. By the definition of $\mathscr{L}_1$, we can derive the operator

$$\langle\mathscr{L}_1\phi(y)\cdot\rangle=\sqrt{2}\rho\nu\left\langle f\phi'(y)\right\rangle s\frac{\partial^2}{\partial s}-\sqrt{2}\nu\left\langle\Lambda\phi'(y)\right\rangle\cdot \tag{16}$$

By the operator (16), we obtain the equation for V_1 as the following:

$$\begin{aligned}\langle\mathscr{L}_2\rangle V_1&=\frac{\sqrt{2}}{2}\rho\nu\left\langle f\phi'\right\rangle s^3\frac{\partial^3V_0}{\partial s^3}\\&\quad+\left(\sqrt{2}\rho\nu\left\langle f\phi'\right\rangle-\frac{\sqrt{2}}{2}\nu\left\langle\Lambda\phi'\right\rangle\right)s^2\frac{\partial^2V_0}{\partial s^2}\end{aligned} \tag{17}$$

with terminal condition $V_1(T,s)=0$. Because we denote $V_1^\varepsilon=\sqrt{\varepsilon}V_1$, (17) could be written as

$$\begin{aligned}\langle\mathscr{L}_2\rangle V_1^\varepsilon&=\frac{\nu\sqrt{\varepsilon}}{\sqrt{2}}\left(2\rho\left\langle f\phi'\right\rangle-\left\langle\Lambda\phi'\right\rangle\right)s^2\frac{\partial^2V_0}{\partial s^2}\\&\quad+\frac{\rho\nu\sqrt{\varepsilon}}{\sqrt{2}}\left\langle f\phi'\right\rangle s^3\frac{\partial^3V_0}{\partial s^3}\\&=C_1s^2\frac{\partial^2V_0}{\partial s^2}+C_2s^3\frac{\partial^3V_0}{\partial s^3}.\end{aligned} \tag{18}$$

Using the fact that $\langle\mathscr{L}_2\rangle s^n(\partial^nV_0/\partial s^n)=s^n(\partial^n/\partial s^n)\langle\mathscr{L}_2\rangle V_0=0$, the following is easily obtained:

$$\begin{aligned}&\langle\mathscr{L}_2\rangle\left(-(T-t)\left(C_1s^2\frac{\partial^2V_0}{\partial s^2}+C_2s^3\frac{\partial^3V_0}{\partial s^3}\right)\right)\\&\quad=\left(C_1s^2\frac{\partial^2V_0}{\partial s^2}+C_2s^3\frac{\partial^3V_0}{\partial s^3}\right)\\&\qquad-(T-t)\langle\mathscr{L}_2\rangle\left(C_1s^2\frac{\partial^2V_0}{\partial s^2}+C_2s^3\frac{\partial^3V_0}{\partial s^3}\right)\\&\quad=C_1s^2\frac{\partial^2V_0}{\partial s^2}+C_2s^3\frac{\partial^3V_0}{\partial s^3}.\end{aligned} \tag{19}$$

Hence $V_1^\varepsilon=-(T-t)(C_1s^2(\partial^2V_0/\partial s^2)+C_2s^3(\partial^3V_0/\partial s^3))$. □

According to the result of Theorem 1, the zero-order approximation term for the price of the collar option V_0 is the solution of the Black-Scholes equation with constant volatility $\overline{\sigma}$, hence following the derivation by Zhang (1997) [14]; the zero-order approximate term V_0 is given by the following formula:

$$V_0(t,s)=K_1e^{-r(T-t)}+C(t,s;K_1)-C(t,s;K_2), \tag{20}$$

where $C(t,s;K_i)=sN(d_1(t,s;K_i))-K_ie^{-r(T-t)}N(d_2(t,s;K_i))$ and $N(\cdot)$ denotes the standard normal cumulative distribution function. $d_1(t,s;K_i)=(\ln(s/K_i)+(r+\overline{\sigma}^2/2)(T-t))/\overline{\sigma}\sqrt{T-t}$ and $d_2(t,s;K_i)=d_2(t,s;K_i)-\overline{\sigma}\sqrt{T-t}(i=1,2)$. And it is easy to check that (20) satisfies the partial differential equation of (14).

Theorem 3. *The approximate formula for a collar option at time $t<T$ in a fast mean-reverting stochastic volatility economy can be expressed as the following:*

$$\begin{aligned}\widetilde{V}&=V_0+V_1^\varepsilon=K_1e^{-r(T-t)}+C(t,s;K_1)-C(t,s;K_2)\\&\quad+\frac{sn(d_1(t,s;K_1))}{\overline{\sigma}}\left(C_2\frac{d_1(t,s;K_1)}{\overline{\sigma}}\right.\\&\qquad\left.+(C_2-C_1)\sqrt{T-t}\right)\\&\quad+\frac{sn(d_1(t,s;K_2))}{\overline{\sigma}}\left(C_2\frac{d_1(t,s;K_2)}{\overline{\sigma}}\right.\\&\qquad\left.+(C_2-C_1)\sqrt{T-t}\right),\end{aligned} \tag{21}$$

where $n(\cdot)$ denotes the standard normal probability density function.

Proof. According to the result of Lemma A.2 in the appendix and simple calculation, we can obtain the following results:

$$\begin{aligned}
\frac{\partial V_0}{\partial s} &= \frac{\partial C(t,s;K_1)}{\partial s} - \frac{\partial C(t,s;K_2)}{\partial s} \\
&= N(d_1(t,s;K_1)) - N(d_1(t,s;K_2)); \\
\frac{\partial^2 V_0}{\partial s^2} &= \frac{\partial^2 C(t,s;K_1)}{\partial s^2} - \frac{\partial^2 C(t,s;K_2)}{\partial s^2} \\
&= \frac{1}{s\overline{\sigma}\sqrt{T-t}}\left(n(d_1(t,s;K_1)) - n(d_1(t,s;K_2))\right); \\
\frac{\partial^3 V_0}{\partial s^3} &= \frac{\partial^3 C(t,s;K_1)}{\partial s^3} - \frac{\partial^3 C(t,s;K_2)}{\partial s^3} \\
&= \frac{-n(d_1(t,s;K_1))}{s^2\overline{\sigma}\sqrt{T-t}}\left(\frac{d_1(t,s;K_1)}{\overline{\sigma}\sqrt{T-t}} + 1\right) \\
&\quad - \frac{-n(d_1(t,s;K_2))}{s^2\overline{\sigma}\sqrt{T-t}}\left(\frac{d_1(t,s;K_2)}{\overline{\sigma}\sqrt{T-t}} + 1\right).
\end{aligned} \tag{22}$$

By further computation we have

$$\begin{aligned}
&s^2\frac{\partial^2 V_0}{\partial s^2} \\
&\quad = \frac{s}{\overline{\sigma}\sqrt{T-t}}\left(n(d_1(t,s;K_1)) - n(d_1(t,s;K_2))\right); \\
&s^3\frac{\partial^3 V_0}{\partial s^3} \\
&\quad = \frac{-sn(d_1(t,s;K_1))}{\overline{\sigma}\sqrt{T-t}}\left(\frac{d_1(t,s;K_1)}{\overline{\sigma}\sqrt{T-t}} + 1\right) \\
&\qquad - \frac{-sn(d_1(t,s;K_2))}{\overline{\sigma}\sqrt{T-t}}\left(\frac{d_1(t,s;K_2)}{\overline{\sigma}\sqrt{T-t}} + 1\right);
\end{aligned} \tag{23}$$

by the result of Theorem 2, V_1^ε can be expressed as the following formula:

$$\begin{aligned}
V_1^\varepsilon &= -(T-t)\left(C_1 s^2\frac{\partial^2 V_0}{\partial s^2} + C_2 s^3\frac{\partial^3 V_0}{\partial s^3}\right) \\
&= \frac{sn(d_1(t,s;K_1))}{\overline{\sigma}}\left(C_2\frac{d_1(t,s;K_1)}{\overline{\sigma}}\right. \\
&\qquad \left. + (C_2 - C_1)\sqrt{T-t}\right) \\
&\quad + \frac{sn(d_1(t,s;K_2))}{\overline{\sigma}}\left(C_2\frac{d_1(t,s;K_2)}{\overline{\sigma}}\right. \\
&\qquad \left. + (C_2 - C_1)\sqrt{T-t}\right).
\end{aligned} \tag{24}$$

Combining the result of Lemma A.2, the approximate formula for a collar option at time t can be derived. □

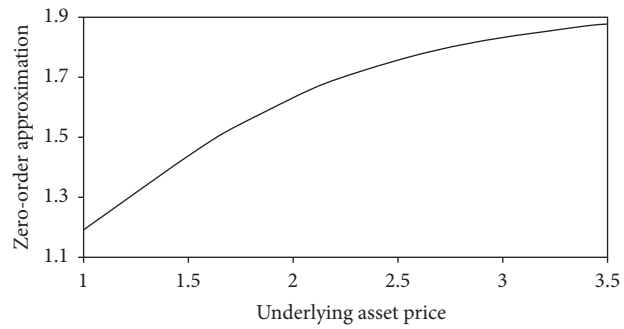

Figure 1: Zero-order approximation of the collar option.

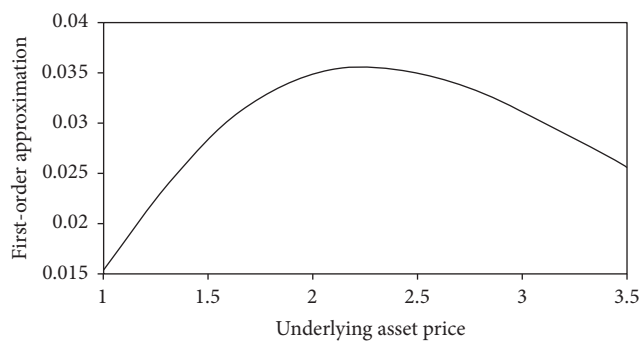

Figure 2: First-order approximation of the collar option.

Under the assumption that $\forall y \in \mathbb{R}$, volatility function $f(y)$ is positive and bounded, and $\forall y \in \mathbb{R}$, the market price of the volatility risk $\gamma(y)$ is also bounded, and the accuracy of this approximation is given by Fouque et al. (2003) [10].

4. Numerical Computation

From the result of Theorem 3, an approximation for the price of a collar option under fast mean-reverting stochastic volatility has been formally obtained, and the parameters in the stochastic volatility model are reduced to C_1 and C_2. As suggested by Fouque et al. (2003) [10] these two parameters can be calibrated from implied volatility surface. In order to illustrate with a numerical example, we consider a collar option with the strike prices of $K_1 = 1$ and $K_2 = 2$. The following parameters are chosen throughout the numerical computation: the (constant) effective volatility $\overline{\sigma} = 0.5$, the time to maturity $T - t = 1$, the risk-free interest rate $r = 6\%$, and the underlying asset price at time 0 is chosen as $s \in [1, 3.5]$. The numerical result of the zero-order approximation of the collar option price at time 0 is shown in Figure 1. The numerical result of the first-order approximation of the collar option price at time 0 is shown in Figure 2 with parameters $C_1 = -0.01$ and $C_2 = 0.004$. The comparison result of the zero-order approximation V_0 and the approximate option price $\widetilde{V}$ is shown in Figure 3.

5. Conclusion

In this paper, by means of singular perturbation analysis, the corresponding partial differential equation for the collar

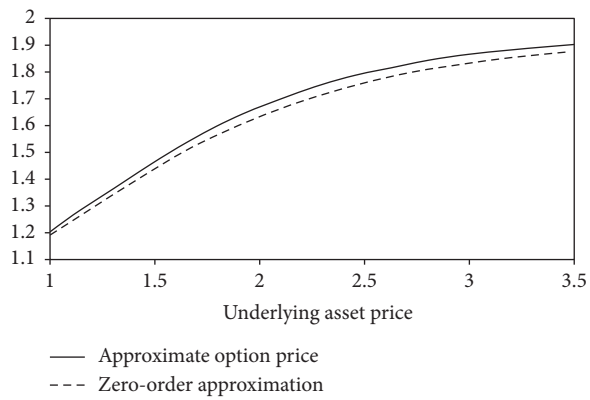

Figure 3: Comparison of V_0 and $\widetilde{V}$.

option of the stochastic volatility model is derived, and an analytical approximation formula for the price of a collar option under a fast mean-reverting stochastic volatility scenario is obtained. From Figure 2 of the numerical experiment one can observe that the first-order approximation V_1^ε has a hump shape around the value of K_2 and it is positive, so the approximate price of the collar option under fast mean-reverting economy is overpriced compared to the zero-order approximation V_0 which is shown in Figure 3. In other words, pricing collar option in traditional Black-Scholes framework may underprice the option value.

Appendix

Lemma A.1. *Denote $N(\cdot)$ as the standard normal cumulative distribution function and $n(\cdot)$ as the standard normal probability density function. Let d_1 and d_2 be as given by*

$$d_1 = \frac{\ln(s/K) + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = \sigma\sqrt{T-t}, \tag{A.1}$$

and then

$$sn(d_1) = Ke^{-r(T-t)}n(d_2), \tag{A.2}$$

where $s, K, r, T, t \in \mathbb{R}^+$ and $T > t$.

Proof. It is easily seen that $d_1 = (\ln(s/K) + (r+\sigma^2/2)(T-t))/\sigma\sqrt{T-t} = (\ln(s/K) + r(T-t))/\sigma\sqrt{T-t} + \sigma\sqrt{T-t}/2$ and therefore $\ln(s/K) + r(T-t) = d_1\sigma\sqrt{T-t} - \sigma^2(T-t)/2$. It can be observed that

$$\begin{aligned} s &= Ke^{-r(T-t)}\left(\frac{s}{K}\right)e^{r(T-t)} \\ &= Ke^{-r(T-t)}\exp\left(\ln\left(\frac{s}{K}\right) + r(T-t)\right) \\ &= Ke^{-r(T-t)}\exp\left(d_1\sigma\sqrt{T-t} - \frac{\sigma^2(T-t)}{2}\right); \end{aligned} \tag{A.3}$$

hence we have the following:

$$\begin{aligned} &se^{-d_1^2/2} \\ &\quad = Ke^{-r(T-t)}\exp\left(-\frac{d_1^2}{2} + d_1^2\sigma\sqrt{T-t} - \frac{\sigma^2(T-t)}{2}\right) \\ &\quad = Ke^{-r(T-t)}\exp\left(-\frac{1}{2}\left(d_1 - \sigma\sqrt{T-t}\right)^2\right) \\ &\quad = Ke^{-r(T-t)}e^{-d_2^2/2}. \end{aligned} \tag{A.4}$$

So we have $s(1/\sqrt{2\pi})e^{-d_1^2/2} = Ke^{-r(T-t)}(1/\sqrt{2\pi})e^{-d_2^2/2} \Rightarrow sn(d_1) = Ke^{-r(T-t)}n(d_2)$. □

Lemma A.2. *Let $C(s,t) = sN(d_1) - Ke^{-r(T-t)}N(d_2)$ be the plain-vanilla European call option at time t, where d_1 and d_2 and $N(\cdot)$ and $n(\cdot)$ are given in Lemma A.1; then*

$$\begin{aligned} \frac{\partial C}{\partial s} &= N(d_1); \\ \frac{\partial^2 C}{\partial s^2} &= \frac{n(d_1)}{s\sigma\sqrt{T-t}}; \\ \frac{\partial^3 C}{\partial s^3} &= \frac{-n(d_1)}{s^2\sigma\sqrt{T-t}}\left(\frac{d_1}{\sigma\sqrt{T-t}} + 1\right). \end{aligned} \tag{A.5}$$

Proof. Because $N'(x) = n(x)$, we can obtain that

$$\begin{aligned} \frac{\partial C}{\partial s} &= N(d_1) + sN'(d_1)\frac{\partial d_1}{\partial s} \\ &\quad - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial s} \\ &= N(d_1) + sn(d_1)\frac{\partial d_1}{\partial s} - Ke^{-r(T-t)}n(d_2)\frac{\partial d_2}{\partial s}. \end{aligned} \tag{A.6}$$

By the result of Lemma A.1, the equation above could be rewritten as

$$\begin{aligned} \frac{\partial C}{\partial s} &= N(d_1) + sn(d_1)\frac{\partial d_1}{\partial s} - sn(d_1)\frac{\partial d_2}{\partial s} \\ &= N(d_1) + sn(d_1)\left(\frac{\partial d_1}{\partial s} - \frac{\partial d_2}{\partial s}\right). \end{aligned} \tag{A.7}$$

Using the fact that

$$\frac{\partial d_1}{\partial s} = \frac{\partial d_2}{\partial s} = \frac{1}{s\sigma\sqrt{T-t}}, \tag{A.8}$$

we have

$$\frac{\partial C}{\partial s} = N(d_1). \tag{A.9}$$

So

$$\frac{\partial^2 C}{\partial s^2} = N'(d_1)\frac{\partial d_1}{\partial s} = \frac{n(d_1)}{s\sigma\sqrt{T-t}}. \tag{A.10}$$

Because $n'(x) = -xn(x)$ we can obtain the following by direct computation:

$$\begin{aligned}\frac{\partial^3 C}{\partial s^3} &= n'(d_1)\frac{\partial d_1}{\partial s}\cdot\frac{1}{s\sigma\sqrt{T-t}} \\ &\quad + n(d_2)\frac{1}{\sigma\sqrt{T-t}}\left(-\frac{1}{s^2}\right) \\ &= \frac{-n(d_1)}{s^2\sigma\sqrt{T-t}}\left(\frac{d_1}{\sigma\sqrt{T-t}}+1\right).\end{aligned} \tag{A.11}$$

□

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

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