Research Article

Generalized Rough Fuzzy Ideals in Quantales

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The paper examines the generalized rough fuzzy ideals of quantales. There are some intrinsic relations between fuzzy prime (primary) ideals of quantales and generalized rough fuzzy prime (primary) ideals of quantales. Homomorphic images of “generalized rough ideals, generalized rough prime (primary) ideals, and generalized rough fuzzy prime (primary) ideals” which are incited by quantale homomorphism are examined.

1. Introduction

The idea of “theory of rough sets” proposed by Pawlak [1, 2] to manage uncertainty and granularity in the information system has attracted the concern and attention of scientists and experts in different fields of science and technology. Late years have seen its wide applications in algebraic systems, knowledge discovery, data mining, expert systems, pattern recognition, granular computing, graph theory, machine learning, partially ordered sets, and so forth [3–15]. It is noted that the significant concepts in the classical theory of rough set are the lower and upper approximations obtained from equivalence relation on a universal set. In many cases, as pointed out by numerous researchers, the implementation of theory of rough set becomes restrictive if we use the condition of the equivalence relation in the model of Pawlak rough set. To get control of this issue, several authors generalized the classical rough set theory by using more general binary relations [16–20] or by employing coverings [21, 22]. Besides, theory of rough sets can also be generalized to the fuzzy environment by employing the notion of fuzzy sets of Zadeh [23], and the resulting notions are called fuzzy rough sets [24–27].

Recently, researchers have connected the ideas and techniques for rough set hypothesis to different algebraic structures. Biswas and Nanda [4] took a group as a ground set and presented the notions of rough groups and rough subgroups. Kuroki and Mordeson [28] discussed the structure of rough

sets and rough groups. At that point in [29], Kuroki presented the thought of rough ideals in semigroup. Rough prime ideals and rough fuzzy prime ideals in semigroups were proposed by Xiao and Zhang [17]. Davvaz [30] gave the concepts of rough ideals in rings. He also wrote a short note on algebraic T -rough sets [31]. Kazancı and Davvaz in [32] gave rough prime (primary) ideals and rough fuzzy prime (primary) ideals in commutative rings. To overcome the confinement of equivalence relations in the process of establishing rough sets in a ring, Yamak et al. [19] introduced the concept of set-valued mappings as the basis of the generalized upper and lower approximations of a ring with the help of an ideal. Roughness in modules was researched by Davvaz and Mahdavi pour [33]. Rasouli and Davvaz [14] presented the notion of rough ideals in MV-algebra. In BCK-algebras, rough ideals were defined by Jun [34]. Classical approximation theory has also been applied to some partially ordered structures. For instance, in [10], Estaji et al. investigated the concepts of upper and lower rough ideals in a lattice by introducing the relationships between lattice theory and rough sets. Zhan et al. discussed “a new rough set theory: rough soft hemirings” in [35]. Ma et al. gave the “applications of a kind of novel Z -soft fuzzy rough ideals to hemirings” and investigated “a survey of decision making methods based on certain hybrid soft set models” [36, 37].

The combination of rough set theory to soft set theory is very important. Feng et al. proposed rough soft sets by

combining Pawlak rough sets and soft sets. In particular, Feng et al. put forth a novel concept of soft rough fuzzy sets by combining rough sets, soft sets, and fuzzy sets and we call it Feng-soft rough fuzzy set [38]. And in 2011, Meng et al. further discussed the Feng-soft rough fuzzy sets and put forward another kind of soft rough fuzzy sets, which is called Meng-soft rough fuzzy set [39]. These sets are limited and have a rigorous restrictive condition. Based on the above reason, Zhan and Zhu provided a novel concept of soft rough fuzzy sets, which is called a Z -soft rough fuzzy set [40]. As reported in [41, 42], characterizations of two kinds of hemirings based on probability spaces and reviews on decision making methods based on (fuzzy) soft sets and rough soft sets are discussed, respectively.

The structure of quantale was proposed by Mulvey [43] to study the spectrum of C^* -algebras. The idea of ideals (prime, primary) of quantale was given by Wang and Zhao [44, 45]. Xiao and Li [16] generalized the ideals of quantale by means of set-valued mappings. The start of theory of rough sets for applying in algebraic structures, for example, semigroups, rings, modules, and groups, has been focused on a congruence relation. However, we obtain the restricted applications by using the congruence relation. To take care of this issue, Davvaz [19, 31] introduced the idea of a set-valued homomorphism for rings and groups. In this paper, we intend to generalize the results which have been proved in [46].

The arrangement of the paper is as per the following. In Section 2, we review some principal properties of rough sets, rough fuzzy sets, and ideals of quantale. In Section 3, we have introduced “generalized rough fuzzy ideal” and “generalized rough fuzzy prime (semiprime, primary) ideals” of quantales and give a few properties of such ideals. In Section 4, we will describe the images of generalized rough ideals and discuss how they are related. We will explain the relation between lower (upper) generalized rough and lower (upper) generalized approximations of their homomorphic images by using quantale homomorphism and set-valued homomorphism of quantale. In Section 5, we will discuss generalized rough fuzzy prime (primary) ideals based on quantale homomorphism. At last, the conclusion is given in Section 6.

2. Preliminaries

Here, we review a few ideas and results which will be vital in the following.

Definition 1 (see [2]). Let (U, σ) be an approximation space, where U is a nonempty set, and let σ be an equivalence relation on U . For $x \in U$, the equivalence class of x , containing x , is denoted by $[x]_\sigma$. For $A \subseteq U$, the upper and lower approximations of A are, respectively, defined as $\bar{\sigma}(A) = \{x \in U \mid [x]_\sigma \cap A \neq \emptyset\}$, $\underline{\sigma}(A) = \{x \in U \mid [x]_\sigma \subseteq A\}$. It is easy to verify that $\underline{\sigma}(A) \subseteq A \subseteq \bar{\sigma}(A)$ for all $A \subseteq U$.

For more details on rough sets, rough fuzzy sets, and fuzzy rough sets, we refer to [2, 24, 26, 27]. Throughout this paper, we shall use Q_t and Q'_t for quantales, unless stated otherwise.

Definition 2 (see [47]). A complete lattice Q_t having associative binary operation $*$ is called a quantale if it satisfies

$$y * (\bigvee_{i \in I} z_i) = \bigvee_{i \in I} (y * z_i); (\bigvee_{i \in I} y_i) * z = \bigvee_{i \in I} (y_i * z),$$

for all $y, z, y_i, z_i \in Q_t$ ($i \in I$).

We will represent the top element of Q_t by 1 and the bottom element by 0 throughout the paper. Let $A, B \subseteq Q_t$, and we define $A * B$ by the set $\{x * y \mid x \in A, y \in B\}$, $A \vee B$ by $\{x \vee y \mid x \in A, y \in B\}$ and $\bigvee_{i \in I} A_i = \{\bigvee_{i \in I} x_i \mid x_i \in A_i\}$.

Definition 3 (see [44]). Let Q_t be a quantale. A nonempty subset A of Q_t is said to be an ideal of Q_t if the following conditions hold:

- (1) For all $z_1, z_2 \in A$, $z_1 \vee z_2 \in A$ is implied.
- (2) If $z \in Q_t$, $z' \in A$ and $z \leq z'$ imply $z \in A$.
- (3) For all $z \in Q_t$ and $z' \in A$, then $z * z' \in A$ and $z' * z \in A$.

An ideal A is said to be a prime ideal if $z * z' \in A$ implies $z \in A$ or $z' \in A$ for all $z, z' \in Q_t$.

An ideal A is said to be a semiprime ideal if $z * z \in A$ implies $z \in A$ for all $z \in Q_t$.

Primary ideal is an ideal A of Q_t if for all $x, z \in Q_t$, $x * z \in A$ and $x \notin A$ imply $z^n \in A$ for some positive integer n , where $z^n = \underbrace{z * \dots * z}_n$.

As it is well known in the fuzzy theory established by Zadeh [23], a fuzzy subset g of Q_t is defined as a map from Q_t to the unit interval $[0, 1]$. The symbols $\wedge$ and $\vee$ will denote the respective infimum and supremum.

Definition 4 (see [24]). Let (W, σ) be an approximation space. A fuzzy subset g is a mapping from W to $[0, 1]$, then for $x \in W$, one defines

$$\begin{aligned} \underline{\sigma}(g)(z) &= \bigwedge_{p \in [z]_\sigma} g(p); \\ \bar{\sigma}(g)(z) &= \bigvee_{p \in [z]_\sigma} g(p). \end{aligned} \tag{1}$$

They are called the lower and upper approximations of g , respectively. If $\underline{\sigma}(g) \neq \bar{\sigma}(g)$, then $\sigma(g) = (\underline{\sigma}(g), \bar{\sigma}(g))$ is called a rough fuzzy set with respect to σ . For $\alpha \in [0, 1]$, the sets

$$\begin{aligned} g_\alpha &= \{x \in W \mid g(x) \geq \alpha\}; \\ g_{\alpha^+} &= \{x \in W \mid g(x) > \alpha\} \end{aligned} \tag{2}$$

are called α -cut and strong α -cut of the fuzzy set g , respectively.

Definition 5 (see [46]). A nonempty fuzzy subset g of Q_t is called a fuzzy ideal of Q_t , if the following conditions are satisfied:

- (1) If $c \leq d$, then $g(d) \leq g(c)$.
- (2) $g(c) \wedge g(d) \leq g(c \vee d)$.
- (3) $g(c) \vee g(d) \leq g(c * d)$.

From (1) and (2) in Definition 5, it is observed that $g(c \vee d) = g(c) \wedge g(d)$ for all $c, d \in Q_t$. Thus, a fuzzy set g is a fuzzy ideal of Q_t if and only if $g(c \vee d) = g(c) \wedge g(d)$ and $g(c * d) \geq g(c) \vee g(d)$ for all $c, d \in Q_t$.

Definition 6 (see [46]). A nonconstant fuzzy ideal g of a quantale Q_t is called a fuzzy prime ideal of Q_t if for all $c, d \in Q_t$,

$$g(c * d) = g(c) \text{ or } g(c * d) = g(d).$$

Note that we require a fuzzy prime ideal of a quantale to be a nonconstant in order to keep consistent with the definition of prime ideals of quantales [45]. Therefore, throughout this paper, a fuzzy ideal of a quantale is always assumed to be nonconstant. For fuzzy semiprime and fuzzy primary ideals, see [46].

Proposition 7 (see [46]). Let g be a fuzzy subset of a quantale Q_t . Then g is a fuzzy (prime, semiprime, primary) ideal of Q_t if and only if for each $\alpha \in [0, 1]$, g_α (resp., g_{α^+}) is either empty or (prime, semiprime, primary) ideal of Q_t .

Throughout this paper, f -ideal, f -prime, f -semiprime, and f -primary ideals will denote fuzzy ideal, fuzzy prime, fuzzy semiprime, and fuzzy primary ideals of quantales, unless stated otherwise. We use $\mathcal{F}(Q_t)$ to denote the set of all fuzzy subsets of Q_t .

The concept of generalized rough sets is a generalization of Pawlak's rough set. In rough set theory, an equivalence relation is the basic requirement for lower and upper approximations. Sometimes it is difficult to find such an equivalence relation among the elements of the set under investigation. In such situations, generalized rough set approach can be useful.

Definition 8 (see [19]). Let U and W be two nonempty universes. Let H be a set-valued mapping given by $H : U \rightarrow P(W)$, where $P(W)$ is the power set of W . Then the triple (U, W, H) is referred to as a generalized approximation space or generalized rough set. Any set-valued function from U to $P(W)$ defines a binary relation from U to W by setting $\sigma_H = \{(x, y) \mid y \in H(x)\}$. Obviously, if σ is an arbitrary relation from U to W , then a set-valued mapping $H_\sigma : U \rightarrow P(W)$ can be defined by $H_\sigma(x) = \{y \in W \mid (x, y) \in \sigma\}$, where $x \in U$. For any set $A \subseteq W$, the lower and upper approximations represented by $\underline{H}(A)$ and $\overline{H}(A)$, respectively, are defined as

$$\begin{aligned} \underline{H}(A) &= \{z \in U \mid H(z) \subseteq A\}, \\ \overline{H}(A) &= \{z \in U \mid H(z) \cap A \neq \emptyset\}. \end{aligned} \quad (3)$$

We call the pair $(\underline{H}(A), \overline{H}(A))$ generalized rough set, and $\underline{H}$, $\overline{H}$ are termed as lower and upper generalized approximation operators, respectively.

If $W = U$ and $\sigma_H = \{(x, y) \mid y \in H(x)\}$ is an equivalence relation on U , then the pair (U, σ_H) is the Pawlak approximation space. Therefore, a generalized rough set is an extended notion of Pawlak's rough set [16].

Definition 9 (see [16]). Let $(Q_t, *_1)$ and $(Q'_t, *_2)$ be two quantales. A set-valued mapping $H : Q_t \rightarrow P^*(Q'_t)$, where $P^*(Q'_t)$ represents the collection of all nonempty subsets of Q'_t , is called a set-valued homomorphism if, for all $a_i, b, a \in Q_t$ ($i \in I$),

- (1) $H(a) *_2 H(b) \subseteq H(a *_1 b)$,
- (2) $\bigvee_{i \in I} H(a_i) \subseteq H(\bigvee_{i \in I} a_i)$.

A set-valued mapping $H : Q_t \rightarrow P^*(Q'_t)$ is called a strong set-valued homomorphism if we replace inclusion by equality in (1) and (2).

From here onwards by SV-Hom, we will mean the set-valued homomorphism. For strong set-valued homomorphism, we will use SSV-Hom. Besides H will mean the map $H : Q_t \rightarrow P^*(Q'_t)$, unless stated otherwise.

3. Generalized Rough Fuzzy Prime (Primary) Ideals in Quantale

In this section, we will introduce the generalized rough fuzzy ideal in quantales and resulting properties of such ideals are presented. Now we use the concept from Definition 4 and generalized it in the following.

Definition 10. Let $(Q_t, *_1)$ and $(Q'_t, *_2)$ be two quantales and let H be a SV-Hom. Let g be any fuzzy subset of Q'_t . Then for every $z \in Q_t$, one defines

$$\begin{aligned} \underline{H}(g)(z) &= \inf_{a \in H(z)} g(a); \\ \overline{H}(g)(z) &= \sup_{a \in H(z)} g(a). \end{aligned} \quad (4)$$

Here $\underline{H}(g)$ is the generalized lower approximation and $\overline{H}(g)$ is the generalized upper approximation of the fuzzy subset g . The pair $(\underline{H}(g), \overline{H}(g))$ is called generalized rough fuzzy set of Q_t if $\underline{H}(g) \neq \overline{H}(g)$.

From here onward by GLA, GUA, and GRE, we will mean generalized lower approximation, generalized upper approximation, and generalized rough fuzzy set, respectively.

Lemma 11. Let H be a SV-Hom. Then for every collection $\{g_i\}_{i \in I} \subseteq \mathcal{F}(Q'_t)$,

- (1) $\underline{H}(\inf_{i \in I} g_i) = \inf_{i \in I} \underline{H}(g_i)$;
- (2) $\overline{H}(\sup_{i \in I} g_i) = \sup_{i \in I} \overline{H}(g_i)$.

Proof. (1) For $x \in Q_t$, we have

$$\begin{aligned} \underline{H}\left(\inf_{i \in I} g_i\right)(x) &= \inf_{a \in H(x)} \inf_{i \in I} g_i(a) = \inf_{i \in I} \inf_{a \in H(x)} g_i(a) \\ &= \inf_{i \in I} \underline{H}(g_i)(x). \end{aligned} \quad (5)$$

The other item has the similar proof. $\square$

Proposition 12. Let $(Q_t, *_1)$ and $(Q'_t, *_2)$ be two quantales and let H be a SV-Hom. Let g be a fuzzy subset of Q'_t . Then for each $\alpha \in [0, 1]$, one has the following:

- (1) $(\underline{H}(g))_\alpha = \underline{H}(g_\alpha)$;
- (2) $(\overline{H}(g))_\alpha = \overline{H}(g_\alpha)$;
- (3) $(\underline{H}(g))_{\alpha^+} = \underline{H}(g_{\alpha^+})$;
- (4) $(\overline{H}(g))_{\alpha^+} = \overline{H}(g_{\alpha^+})$.

Proof. (1) Let

$$\begin{aligned}
 z \in (\underline{H}(g))_\alpha &\iff \\
 \underline{H}(g)(z) &\geq \alpha \iff \\
 \inf_{a \in H(z)} g(a) &\geq \alpha \iff \\
 g(a) &\geq \alpha \quad \forall a \in H(z); \iff \\
 H(z) &\subseteq g_\alpha \iff \\
 z &\in \underline{H}(g_\alpha).
 \end{aligned} \tag{6}$$

Axioms (2), (3), and (4) are similar to the proof of (1). $\square$

Definition 13. Let H be a SV-Hom. A fuzzy subset g of the quantale Q'_t is called a lower [an upper] GRF ideal of Q'_t if $\underline{H}(g)$ [$\overline{H}(g)$] is a f -ideal of Q_t . A fuzzy subset g of Q'_t , which is both an upper and a lower GRF ideal of Q'_t , is called GRF ideal of Q'_t .

Now, lower approximations and upper approximations of f -ideals of quantales are being studied in the following.

Theorem 14. Let H be a SSV-Hom and let g be a f -ideal of Q'_t . Then $\underline{H}(g)$ is a f -ideal of Q_t .

Proof. Since g is a f -ideal of Q'_t , by Definition 5, we have $g(avb) = g(a) \wedge g(b)$ and $g(a * b) \geq g(a) \vee g(b) \quad \forall a, b \in Q'_t$. As H is a SSV-Hom, so $H(z_1 \vee z_2) = H(z_1) \vee H(z_2)$, $\forall z_1, z_2 \in Q_t$.

Therefore,

$$\underline{H}(g)(z_1 \vee z_2) = \inf_{e \in H(z_1 \vee z_2)} g(e) = \inf_{e \in H(z_1) \vee H(z_2)} g(e). \tag{7}$$

Since $e \in H(z_1) \vee H(z_2)$, there exist $c_1 \in H(z_1)$ and $c_2 \in H(z_2)$ such that $e = c_1 \vee c_2$.

Hence,

$$\begin{aligned}
 \underline{H}(g)(z_1 \vee z_2) &= \inf_{c_1 \vee c_2 \in H(z_1) \vee H(z_2)} g(c_1 \vee c_2) \\
 &= \inf_{c_1 \in H(z_1), c_2 \in H(z_2)} (g(c_1) \wedge g(c_2)) \\
 &= \inf \left[\inf_{c_1 \in H(z_1)} g(c_1), \inf_{c_2 \in H(z_2)} g(c_2) \right] \\
 &= \underline{H}(g)(z_1) \wedge \underline{H}(g)(z_2).
 \end{aligned} \tag{8}$$

Hence,

$$\underline{H}(g)(z_1 \vee z_2) = \underline{H}(g)(z_1) \wedge \underline{H}(g)(z_2) \quad \forall z_1, z_2 \in Q_t. \tag{9}$$

Again since H is a SSV-Hom, hence $H(z_1 * z_2) = H(z_1) * H(z_2) \quad \forall z_1, z_2 \in Q_t$.

Thus we have

$$\begin{aligned}
 \underline{H}(g)(z_1 * z_2) &= \inf_{e \in H(z_1) * H(z_2)} g(e) \\
 &= \inf_{e \in H(z_1) * H(z_2)} g(e).
 \end{aligned} \tag{10}$$

Now since $e \in H(z_1) * H(z_2)$, there exist $c_1 \in H(z_1)$, $c_2 \in H(z_2)$ such that $e = c_1 * c_2$.

Thus,

$$\begin{aligned}
 \underline{H}(g)(z_1 * z_2) &= \inf_{c_1 * c_2 \in H(z_1) * H(z_2)} g(c_1 * c_2) \\
 &\geq \inf_{c_1 * c_2 \in H(z_1) * H(z_2)} [g(c_1) \vee g(c_2)] \\
 &= \inf_{c_1 \in H(z_1), c_2 \in H(z_2)} [g(c_1) \vee g(c_2)] \\
 &= \sup \left[\inf_{c_1 \in H(z_1)} g(c_1), \inf_{c_2 \in H(z_2)} g(c_2) \right] \\
 &= \underline{H}(g)(z_1) \vee \underline{H}(g)(z_2).
 \end{aligned} \tag{11}$$

Hence,

$$\underline{H}(g)(z_1 * z_2) \geq \underline{H}(g)(z_1) \vee \underline{H}(g)(z_2) \quad \forall z_1, z_2 \in Q_t. \tag{12}$$

Thus, by (9) and (12), $\underline{H}(g)$ is a f -ideal of Q_t . $\square$

Theorem 15. Let H be a SSV-Hom and let g be a f -ideal of Q'_t . Then $\overline{H}(g)$ is a f -ideal of Q_t .

Proof. Since H is a SSV-Hom, therefore $H(z_1 \vee z_2) = H(z_1) \vee H(z_2) \quad \forall z_1, z_2 \in Q_t$. Also g is f -ideal of Q'_t ; hence $g(a \vee b) = g(a) \wedge g(b) \quad \forall a, b \in Q'_t$.

Consider

$$\overline{H}(g)(z_1 \vee z_2) = \sup_{c \in H(z_1 \vee z_2)} g(c) = \sup_{c \in H(z_1) \vee H(z_2)} g(c). \tag{13}$$

For $c \in H(z_1) \vee H(z_2)$, we have $a \in H(z_1)$ and $b \in H(z_2)$ such that $c = a \vee b$.

Hence,

$$\begin{aligned}
 \overline{H}(g)(z_1 \vee z_2) &= \sup_{a \vee b \in H(z_1) \vee H(z_2)} g(a \vee b) \\
 &= \sup_{a \in H(z_1), b \in H(z_2)} [g(a) \wedge g(b)] \\
 &= \inf \left[\sup_{a \in H(z_1)} g(a), \sup_{b \in H(z_2)} g(b) \right] \\
 &= \overline{H}(g)(z_1) \wedge \overline{H}(g)(z_2).
 \end{aligned} \tag{14}$$

Thus,

$$\overline{H}(g)(z_1 \vee z_2) = \overline{H}(g)(z_1) \wedge \overline{H}(g)(z_2) \quad \forall z_1, z_2 \in Q_t. \tag{15}$$

Now,

$$\begin{aligned}
 \overline{H}(g)(z_1 * z_2) &= \sup_{c \in H(z_1) * H(z_2)} g(c) \\
 &= \sup_{c \in H(z_1) * H(z_2)} g(c).
 \end{aligned} \tag{16}$$

For $c \in H(z_1) *_2 H(z_2)$, there exist $a \in H(z_1)$ and $b \in H(z_2)$ such that $c = a *_2 b$.

Hence,

$$\begin{aligned} \overline{H}(g)(z_1 *_1 z_2) &= \sup_{a *_2 b \in H(z_1) *_2 H(z_2)} g(a *_2 b) \\ &\geq \sup_{a \in H(z_1), b \in H(z_2)} [g(a) \vee g(b)] \\ &= \sup \left(\sup_{a \in H(z_1)} g(a), \sup_{b \in H(z_2)} g(b) \right) \\ &= \overline{H}(g)(z_1) \vee \overline{H}(g)(z_2). \end{aligned} \quad (17)$$

Thus,

$$\overline{H}(g)(z_1 *_1 z_2) \geq \overline{H}(g)(z_1) \vee \overline{H}(g)(z_2) \quad \forall z_1, z_2 \in Q_t. \quad (18)$$

Hence by (15) and (18), we have $\overline{H}(g)$ is a f -ideal of Q_t . $\square$

By the above two theorems, we have immediately the following corollary.

Corollary 16. Let H be a SSV-Hom and let g be a f -ideal of Q'_t . Then g is a GRF ideal of Q'_t .

Proposition 17. Let H be a SSV-Hom. Let $\{g_i\}_{i \in I}$ be a family of f -ideals of Q'_t . Then $\underline{H}(\inf_{i \in I} g_i)$ is a f -ideal of Q_t .

Proof. Since every g_i is a f -ideal for $i \in I$, therefore $\forall x, y \in Q_t$,

$$\begin{aligned} \underline{H}\left(\inf_{i \in I} g_i\right)(x \vee y) &= \left(\inf_{i \in I} \underline{H}(g_i)\right)(x \vee y) \\ &= \inf_{i \in I} \underline{H}(g_i)(x \vee y) \\ &= \inf_{i \in I} [\inf (H(g_i)(x), H(g_i)(y))] \\ &= \inf \left[\inf_{i \in I} \underline{H}(g_i)(x), \inf_{i \in I} \underline{H}(g_i)(y) \right] \\ &= \inf \left[\underline{H}\left(\inf_{i \in I} g_i\right)(x), \underline{H}\left(\inf_{i \in I} g_i\right)(y) \right] \\ &= \underline{H}\left(\inf_{i \in I} g_i\right)(x) \wedge \underline{H}\left(\inf_{i \in I} g_i\right)(y). \end{aligned} \quad (19)$$

Hence,

$$\begin{aligned} \underline{H}\left(\inf_{i \in I} g_i\right)(x \vee y) &= \underline{H}\left(\inf_{i \in I} g_i\right)(x) \wedge \underline{H}\left(\inf_{i \in I} g_i\right)(y) \quad \forall x, y \in Q_t, \\ \underline{H}\left(\inf_{i \in I} g_i\right)(x *_1 y) &= \left(\inf_{i \in I} \underline{H}(g_i)\right)(x *_1 y) \\ &= \inf_{i \in I} \underline{H}(g_i)(x *_1 y) \\ &\geq \inf_{i \in I} [\sup (H(g_i)(x), H(g_i)(y))] \end{aligned}$$

$$\begin{aligned} &= \sup \left[\left(\inf_{i \in I} \underline{H}(g_i)(x) \right), \left(\inf_{i \in I} \underline{H}(g_i)(y) \right) \right] \\ &= \sup \left[\underline{H}\left(\inf_{i \in I} g_i\right)(x), \underline{H}\left(\inf_{i \in I} g_i\right)(y) \right] \\ &= \underline{H}\left(\inf_{i \in I} g_i\right)(x) \vee \underline{H}\left(\inf_{i \in I} g_i\right)(y). \end{aligned} \quad (20)$$

Hence,

$$\begin{aligned} \underline{H}\left(\inf_{i \in I} g_i\right)(x *_1 y) &\geq \underline{H}\left(\inf_{i \in I} g_i\right)(x) \\ &\vee \underline{H}\left(\inf_{i \in I} g_i\right)(y) \quad \forall x, y \in Q_t. \end{aligned} \quad (21)$$

Therefore, $\underline{H}(\inf_{i \in I} g_i)$ is a f -ideal of Q_t . $\square$

Theorem 18. Let H be a SSV-Hom and let g be a f -ideal of Q'_t . Then $\underline{H}(g)$ (respectively, $\overline{H}(g)$) is a f -ideal of Q_t if and only if for each $\alpha \in [0, 1]$, $\underline{H}(g_\alpha)$ (respectively, $\overline{H}(g_\alpha)$), where $g_\alpha \neq \emptyset$, is an ideal of Q_t .

Proof. Suppose $\underline{H}(g)$ is a f -ideal of Q_t . We need to show that $\underline{H}(g_\alpha)$ is an ideal of Q_t . Let $z_1, z_2 \in \underline{H}(g_\alpha)$. Then $\underline{H}(g)(z_1) \geq \alpha$, $\underline{H}(g)(z_2) \geq \alpha$. But since $\underline{H}(g)$ is a f -ideal, so $\underline{H}(g)(z_1 \vee z_2) = \underline{H}(g)(z_1) \wedge \underline{H}(g)(z_2) \geq \alpha$. Hence $z_1 \vee z_2 \in \underline{H}(g_\alpha)$. Let $y \in \underline{H}(g_\alpha)$, $z \in Q_t$, and $z \leq y$. Then $\underline{H}(g)(z) \geq \underline{H}(g)(y) \geq \alpha$. Thus $z \in \underline{H}(g_\alpha)$. Suppose $y \in \underline{H}(g_\alpha)$ and $\forall z \in Q_t$, then $\underline{H}(g)(z *_1 y) \geq \underline{H}(g)(z) \vee \underline{H}(g)(y) = \underline{H}(g)(y) \geq \alpha$, and we get $z *_1 y \in \underline{H}(g_\alpha)$. Similarly, $y *_1 z \in \underline{H}(g_\alpha)$. Hence, $\underline{H}(g_\alpha)$ is an ideal of Q_t .

Conversely, assume $\underline{H}(g_\alpha)$ is an ideal of Q_t . We will show that $\underline{H}(g)$ is a f -ideal of Q_t . For any $z_1, z_2 \in Q_t$, let $\alpha = \underline{H}(g)(z_1) \wedge \underline{H}(g)(z_2) \in \text{rang}(\underline{H}(g))$. Then $\underline{H}(g)(z_1) \geq \alpha$ and $\underline{H}(g)(z_2) \geq \alpha$; that is, $z_1 \in \underline{H}(g_\alpha)$ and $z_2 \in \underline{H}(g_\alpha)$. Hence, $z_1 \vee z_2 \in \underline{H}(g_\alpha)$.

Consider

$$\underline{H}(g)(z_1 \vee z_2) = \inf_{c \in H(z_1 \vee z_2)} g(c) = \inf_{c \in H(z_1) \vee H(z_2)} g(c). \quad (22)$$

Since H is a SSV-Hom, for $c \in H(z_1) \vee H(z_2)$, there exist $a_1 \in H(z_1)$ and $a_2 \in H(z_2)$ such that $c = a_1 \vee a_2$.

Hence we obtain

$$\begin{aligned} \underline{H}(g)(z_1 \vee z_2) &= \inf_{a_1 \vee a_2 \in H(z_1) \vee H(z_2)} g(a_1 \vee a_2) \\ &= \inf_{a_1 \vee a_2 \in H(z_1) \vee H(z_2)} [g(a_1) \wedge g(a_2)] \\ &= \inf_{a_1 \in H(z_1), a_2 \in H(z_2)} [g(a_1) \wedge g(a_2)] \\ &= \inf \left(\inf_{a_1 \in H(z_1)} g(a_1), \inf_{a_2 \in H(z_2)} g(a_2) \right) \\ &= \underline{H}(g)(z_1) \wedge \underline{H}(g)(z_2). \end{aligned} \quad (23)$$

So $\underline{H}(g)(z_1 \vee z_2) = \underline{H}(g)(z_1) \wedge \underline{H}(g)(z_2) \quad \forall z_1, z_2 \in \underline{H}(g_\alpha)$.

Now for $z \in \underline{H}(g_\alpha)$ and $\forall y \in Q_t$, we obtain $z *_1 y \in \underline{H}(g_\alpha)$ and $y *_1 z \in \underline{H}(g_\alpha)$. Hence, $\underline{H}(g)(z *_1 y) \geq \alpha$ and

FIGURE 1: Illustration of Q_t . FIGURE 2: Illustration of Q'_t .

$\underline{H}(g)(z) \geq \alpha$. If either $\underline{H}(g)(y) \geq \alpha$ or $\underline{H}(g)(y) < \alpha$, in both the cases, $(\underline{H}(g)(x) \vee \underline{H}(g)(y)) \geq \alpha$. We suppose $(\underline{H}(g)(x) \vee \underline{H}(g)(y)) = \alpha$. So $\underline{H}(g)(x *_1 y) \geq (\underline{H}(g)(x) \vee \underline{H}(g)(y))$. Hence, $\underline{H}(g)$ is a f -ideal of Q_t . $\square$

Example 19. Let $(Q_t, *_1)$ and $(Q'_t, *_2)$ be two quantales, where Q_t and Q'_t are depicted in Figures 1 and 2 and the binary operations $*_1$ and $*_2$ on both the quantales are the same as the meet operation in the lattices Q_t and Q'_t as shown in Tables 1 and 2.

Let H be a SSV-Hom as defined by $H(0) = \{0'\}$, $H(a) = \{i, j\}$, $H(1) = \{1'\}$. Let g be a f -ideal of Q'_t defined by $g = 0.9/0' + 0.6/i + 0.7/j + 0.6/1'$. Then GLA and GUA of the f -ideal g of Q'_t are as follows: $\underline{H}(g) = 0.9/0 + 0.6/a + 0.6/1$ and $\overline{H}(g) = 0.9/0 + 0.7/a + 0.6/1$. It is easily verified that $\underline{H}(g)$ and $\overline{H}(g)$ are f -ideals of Q_t .

Consider $H : Q'_t \rightarrow P^*(Q'_t)$ defined by $H(0') = H(i) = H(j) = \{0'\}$ and $H(1') = Q'_t$. Then H is a SV-Hom.

Let μ be a fuzzy subset of Q'_t defined by $\mu(x) = \{1, \text{ if } x = 0'; 0.7, \text{ if } x \neq 0'\} \forall x \in Q'_t$. Then μ is a f -ideal of Q'_t . Hence GLA and GUA of f -ideal μ of Q'_t are $\underline{H}(\mu) = 1/0' + 1/i + 1/j + 0.7/1'$ and $\overline{H}(\mu) = 1/0' + 1/i + 1/j + 1/1'$. It is observed that $\underline{H}(\mu)$ is not a f -ideal of Q'_t and $\overline{H}(\mu)$ is a constant f -ideal. Hence it is important to take SSV-Hom.

Definition 20. Let H be a SV-Hom and let g be a fuzzy subset of a quantale Q'_t . Then g is called an upper [a lower] GRF prime ideal of Q'_t if $\overline{H}(g)$ [$\underline{H}(g)$] is a f -prime ideal of Q_t . A fuzzy subset g of Q'_t , which is both an upper and a lower GRF prime ideal, is called GRF prime ideal of Q'_t .

TABLE 1: Binary operation $*_1$ subject to Q_t .

$*_1$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

TABLE 2: Binary operation $*_2$ subject to Q'_t .

$*_2$	0'	i	j	1'
0'	0'	0'	0'	0'
i	0'	i	0'	i
j	0'	0'	j	j
1'	0'	i	j	1'

Similarly, we can define upper [lower] GRF semiprime (primary) ideals of quantale. Thus the concept of generalized rough fuzzy ideals of quantales extends the notion of rough fuzzy ideals.

Proposition 21. Let H be a SSV-Hom. If g is a f -prime ideal of Q'_t , then $\underline{H}(g)$ is a f -prime ideal of Q_t .

Proof. As g is a f -prime ideal of Q'_t , therefore $g(c *_2 b) = g(c)$ or $g(c *_2 b) = g(b) \forall c, b \in Q'_t$ and hence, g is a f -ideal of Q'_t , so by Theorem 14, $\underline{H}(g)$ is a f -ideal of Q_t .

Consider

$$\begin{aligned} \underline{H}(g)(x_1 *_1 y_1) &= \inf_{e \in H(x_1 *_1 y_1)} g(e) \\ &= \inf_{e \in H(x_1) *_2 H(y_1)} g(e). \end{aligned} \quad (24)$$

Since H is a SSV-Hom, therefore for $e \in H(x_1) *_2 H(y_1)$ there exist $c \in H(x_1)$ and $b \in H(y_1)$ such that $e = c *_2 b$.

Hence,

$$\begin{aligned} \underline{H}(g)(x_1 *_1 y_1) &= \inf_{c *_2 b \in H(x_1) *_2 H(y_1)} g(c *_2 b) \\ &= \inf_{c \in H(x_1), b \in H(y_1)} g(c *_2 b) \\ &= \inf_{c \in H(x_1), b \in H(y_1)} [g(c) \text{ or } g(b)] \quad (25) \\ &= \inf_{c \in H(x_1)} g(c) \text{ or } \inf_{b \in H(y_1)} g(b) \\ &= \underline{H}(g)(x_1) \text{ or } \underline{H}(g)(y_1). \end{aligned}$$

Thus, $\underline{H}(g)(x_1 *_1 y_1) = \underline{H}(g)(x_1)$ or $\underline{H}(g)(x_1 *_1 y_1) = \underline{H}(g)(y_1) \forall x_1, y_1 \in Q'_t$. Hence $\underline{H}(g)$ is a f -prime ideal of Q_t . $\square$

Proposition 22. Let H be a SSV-Hom. If g is a f -prime ideal of Q'_t , then $\overline{H}(g)$ is a f -prime ideal of Q_t .

Proof. The proof is similar as reported in Proposition 21. $\square$

By the above two theorems, we have immediately the following corollary.

Corollary 23. Let H be a SSV-Hom and let g be a f -prime ideal of Q_t . Then $\underline{H}(g)$ is a GRF prime ideal of Q_t .

Theorem 24. Let H be a SSV-Hom and let $\underline{H}(g)$ be a f -ideal of Q_t . Then $\underline{H}(g)$ is a f -prime ideal of Q_t if and only if $\underline{H}(g)(x *_1 y) = \underline{H}(g)(x) \vee \underline{H}(g)(y) \quad \forall x, y \in Q_t$.

Proof. Let $\underline{H}(g)$ be a f -prime ideal of Q_t . Then $\underline{H}(g)(x *_1 y) = \underline{H}(g)(x)$ or $\underline{H}(g)(x *_1 y) = \underline{H}(g)(y)$.

This implies that

$$\underline{H}(g)(x *_1 y) \leq \underline{H}(g)(x) \vee \underline{H}(g)(y). \quad (26)$$

As $\underline{H}(g)$ is a f -ideal of Q_t , hence by definition of f -ideal, we have

$$\underline{H}(g)(x *_1 y) \geq \underline{H}(g)(x) \vee \underline{H}(g)(y). \quad (27)$$

By (26) and (27), we obtain $\underline{H}(g)(x *_1 y) = \underline{H}(g)(x) \vee \underline{H}(g)(y)$. Conversely, suppose that $\underline{H}(g)(x *_1 y) = \underline{H}(g)(x) \vee \underline{H}(g)(y) \quad \forall x, y \in Q_t$. We have to show that $\underline{H}(g)$ is a f -prime ideal. As $[0, 1]$ is totally ordered, so $\underline{H}(g)(x) \vee \underline{H}(g)(y) = \underline{H}(g)(x)$ or $\underline{H}(g)(x) \vee \underline{H}(g)(y) = \underline{H}(g)(y)$. Hence $\underline{H}(g)(x *_1 y) = \underline{H}(g)(x)$ or $\underline{H}(g)(x *_1 y) = \underline{H}(g)(y) \quad \forall x, y \in Q_t$. This shows that $\underline{H}(g)$ is a f -prime ideal of Q_t . $\square$

Theorem 25. Let H be a SSV-Hom and let g be a f -prime ideal of Q_t . Then $\underline{H}(g)$ (respectively, $\overline{H}(g)$) is a f -prime ideal of Q_t if and only if, for each $\alpha \in [0, 1]$, $\underline{H}(g_\alpha)$ (respectively, $\overline{H}(g_\alpha)$), where $g_\alpha \neq \emptyset$, is a prime ideal of Q_t .

Proof. As g is a f -prime ideal of Q_t , therefore $g(a *_2 c) = g(a)$ or $g(a *_2 c) = g(c) \quad \forall a, c \in Q_t$. Suppose $\underline{H}(g)$ is a f -prime ideal of Q_t , then $\underline{H}(g)$ is a f -ideal of Q_t . By Theorem 18, $\underline{H}(g_\alpha)$ is an ideal of Q_t . In order to show that $\underline{H}(g_\alpha)$ is a prime ideal for all $\alpha \in [0, 1]$, we have to show that for $a *_1 c \in \underline{H}(g_\alpha)$ implies that $a \in \underline{H}(g_\alpha)$ or $c \in \underline{H}(g_\alpha)$. Let $a *_1 c \in \underline{H}(g_\alpha)$. Then $\underline{H}(g)(a) = \underline{H}(g)(a *_1 c) \geq \alpha$ or $\underline{H}(g)(a *_1 c) = \underline{H}(g)(c) \geq \alpha$. Thus, $a \in \underline{H}(g_\alpha)$ or $c \in \underline{H}(g_\alpha)$. Hence $\underline{H}(g_\alpha)$ is a prime ideal of Q_t .

Conversely, suppose that $\underline{H}(g_\alpha)$ is a prime ideal of Q_t , then $\underline{H}(g_\alpha)$ is an ideal of Q_t . By Theorem 18, $\underline{H}(g)$ is a f -ideal of Q_t .

Consider

$$\begin{aligned} \underline{H}(g)(x *_1 y) &= \inf_{d \in H(x *_1 y)} g(d) \\ &= \inf_{d \in H(x *_2 H(y))} g(d). \end{aligned} \quad (28)$$

Since H is a SSV-Hom, we have $a \in H(x)$, $c \in H(y)$ such that $d = a *_2 c$.

Hence,

$$\begin{aligned} \underline{H}(g)(x *_1 y) &= \inf_{a *_2 c \in H(x *_2 H(y))} g(a *_2 c) \\ &= \inf_{a \in H(x), c \in H(y)} g(a *_2 c) \\ &= \inf_{a \in H(x), c \in H(y)} [g(a) \text{ or } g(c)] \\ &= \inf_{a \in H(x)} g(a) \text{ or } \inf_{c \in H(y)} g(c) \\ &= \underline{H}(g)(x) \text{ or } \underline{H}(g)(y). \end{aligned} \quad (29)$$

Therefore $\underline{H}(g)(x *_1 y) = \underline{H}(g)(x)$ or $\underline{H}(g)(x *_1 y) = \underline{H}(g)(y) \quad \forall x, y \in Q_t$. Hence $\underline{H}(g)$ is a f -prime ideal of Q_t . $\square$

Theorem 26. Let H be a SSV-Hom and let g be a f -semiprime ideal of Q_t . Then $\underline{H}(g)$ is a f -semiprime ideal of Q_t .

Proof. As g is a f -semiprime ideal of Q_t , therefore $g(d^2) = g(d) \quad \forall d \in Q_t$ and g is a f -ideal of Q_t , so by Theorem 14, $\underline{H}(g)$ is a f -ideal of Q_t .

Hence consider

$$\begin{aligned} \underline{H}(g)(y) &= \inf_{d \in H(y)} g(d) = \inf_{d \in H(y)} g(d^2) \\ &= \inf_{d *_2 d \in H(y) *_2 H(y)} g(d^2) \\ &= \inf_{d *_2 d \in H(y *_1 y)} g(d^2) = \inf_{d^2 \in H(y^2)} g(d^2) \\ &= \underline{H}(g)(y^2). \end{aligned} \quad (30)$$

Thus $\underline{H}(g)(y) = \underline{H}(g)(y^2) \quad \forall y \in Q_t$. Therefore $\underline{H}(g)$ is a f -semiprime ideal of Q_t . $\square$

Theorem 27. Let H be a SSV-Hom and let g be a f -semiprime ideal of Q_t . Then $\overline{H}(g)$ is a f -semiprime ideal of Q_t .

Proof. Proof is similar as reported in Theorem 26. $\square$

Corollary 28. Let H be a SSV-Hom and let g be a f -semiprime ideal of Q_t . Then g is a GRF semiprime ideal of Q_t .

Theorem 29. Let g be a f -semiprime ideal of Q_t and let H be a SSV-Hom. Then $\underline{H}(g)$ (respectively, $\overline{H}(g)$) is a f -semiprime ideal of Q_t if and only if, for each $\alpha \in [0, 1]$, $\underline{H}(g_\alpha)$ (respectively, $\overline{H}(g_\alpha)$), where $g_\alpha \neq \emptyset$, is a semiprime ideal of Q_t .

Proof. Suppose $\underline{H}(g)$ is a f -semiprime ideal of Q_t , then $\underline{H}(g)$ is a f -ideal of Q_t . By Theorem 18, $\underline{H}(g_\alpha)$ is an ideal of Q_t . In order to show that $\underline{H}(g_\alpha)$ is a semiprime ideal $\forall \alpha \in [0, 1]$, we have to show that for $a *_1 a \in \underline{H}(g_\alpha)$ implies $a \in \underline{H}(g_\alpha)$. Let $a *_1 a \in \underline{H}(g_\alpha)$. Since $\underline{H}(g)$ is a f -semiprime ideal, we have $\underline{H}(g)(a) = \underline{H}(g)(a *_1 a) \geq \alpha$. Thus, we have $a \in \underline{H}(g_\alpha)$. Hence $\underline{H}(g_\alpha)$ is a semiprime ideal of Q_t .

Conversely, suppose that $\underline{H}(g_\alpha)$ is a semiprime ideal of Q_t . Then $\underline{H}(g_\alpha)$ is an ideal of Q_t . By Theorem 18, $\underline{H}(g)$ is a f -ideal.

For $\underline{H}(g)$ to be a f -semiprime ideal, we have to show that $\underline{H}(g)(z *_1 z) = \underline{H}(g)(z) \quad \forall z \in Q_t$. As H is a SSV-Hom and g is a f -semiprime ideal of Q_t , consider

$$\begin{aligned} \underline{H}(g)(z) &= \inf_{c \in H(z)} g(c) = \inf_{c \in H(z)} g(c^2) \\ &= \inf_{c *_2 c \in H(z) *_2 H(z)} g(c^2) = \inf_{c *_2 c \in H(z *_1 z)} g(c^2) \\ &= \inf_{c^2 \in H(z^2)} g(c^2) = \underline{H}(g)(z^2). \end{aligned} \quad (31)$$

Thus $\underline{H}(g)(z) = \underline{H}(g)(z^2) \quad \forall z \in Q_t$. Hence $\underline{H}(g)$ is a f -semiprime ideal of Q_t . $\square$

Example 30. Let $(Q_t, *_1)$ and $(Q'_t, *_2)$ be two quantales, where Q_t and Q'_t are depicted in Figures 1 and 2 and the binary operations $*_1$ and $*_2$ on both the quantales are the same as the meet operation in the lattices Q_t and Q'_t as shown in Tables 1 and 2.

Let $H : Q_t \rightarrow P^*(Q'_t)$ be a SSV-Hom as defined in Example 19.

Let λ be a fuzzy subset of Q'_t defined by $\lambda = 0.9/0' + 0.6/i + 0.9/j + 0.6/1'$. Then one can verify that λ is a f -prime ideal of Q'_t .

Hence GUA and GLA of the f -prime ideal λ are $\bar{H}(\lambda) = 0.9/0 + 0.9/a + 0.6/1$ and $\underline{H}(\lambda) = 0.9/0 + 0.6/a + 0.6/1$. It is observed that $\bar{H}(\lambda)$ and $\underline{H}(\lambda)$ are nonconstant f -prime ideals of Q_t .

Let g be a fuzzy subset of Q'_t defined by $g(x) = \{1, \text{ if } x = 0'; 0.6, \text{ if } x \neq 0'\} \forall x \in Q'_t$. Then g is a f -semiprime ideal of Q'_t . Hence GLA and GUA of f -semiprime ideal g are as follows: $\underline{H}(g) = 1/0 + 0.6/a + 0.6/1$ and $\bar{H}(g) = 1/0 + 0.6/a + 0.6/1$. It is clear that $\bar{H}(g)$ and $\underline{H}(g)$ are f -semiprime ideals of Q_t .

Theorem 31. Let g be a f -primary ideal of Q'_t and let H be a SSV-Hom. Then $\underline{H}(g)$ is a f -primary ideal of Q_t .

Proof. As g is a f -primary ideal of Q'_t , therefore $g(a *_2 b) = g(a)$ or $g(a *_2 b) = g(b^n) \forall a, b \in Q'_t$ and hence, g is a f -ideal of Q'_t , so by Theorem 14, $\underline{H}(g)$ is f -ideal of Q_t . Since H is given as SSV-Hom, consider

$$\begin{aligned} \underline{H}(g)(z *_1 y) &= \inf_{d \in H(z *_1 y)} g(d) \\ &= \inf_{a *_2 b \in H(z) *_2 H(y)} g(a *_2 b) \\ &= \inf_{a \in H(z), b \in H(y)} g(a *_2 b) \\ &= \inf_{a \in H(z), b \in H(y)} [g(a) \text{ or } g(b^n)] \quad (32) \\ &= \inf_{a \in H(z)} g(a) \text{ or } \inf_{b \in H(y)} g(b^n) \\ &= \inf_{a \in H(z)} g(a) \text{ or } \inf_{b^n \in H(y^n)} g(b^n) \\ &= \underline{H}(g)(z) \text{ or } \underline{H}(g)(y^n). \end{aligned}$$

Here $b^n = b *_2 b *_2, \dots, *_2 b \in H(y) *_2 H(y) *_2, \dots, *_2 H(y) = H(y *_1 y *_1 y *_1, \dots, *_1 y) = H(y^n)$ up to n times for some positive integer n . Thus $\underline{H}(g)(z *_1 y) = \underline{H}(g)(z)$ or $\underline{H}(g)(z *_1 y) = \underline{H}(g)(y^n) \forall z, y \in Q_t$. Therefore $\underline{H}(g)$ is a f -primary ideal of Q_t . $\square$

Theorem 32. Let g be a f -primary ideal of Q'_t and let H be a SSV-Hom. Then $\bar{H}(g)$ is a f -primary ideal of Q_t .

Proof. The proof is similar to the proof of Theorem 31. $\square$

Theorem 33. Let H be a SSV-Hom and let g be a nonconstant f -primary ideal of Q'_t . Then $\underline{H}(g)$ (respectively, $\bar{H}(g)$) is

a f -primary ideal of Q_t if and only if for each $\alpha \in [0, 1]$, $\underline{H}(g_\alpha)$ (respectively, $\bar{H}(g_\alpha)$), where $g_\alpha \neq \emptyset$, is a primary ideal of Q_t .

4. Homomorphic Images of Generalized Rough Ideals Based on Quantale Homomorphism

In this section, we will describe the images of generalized lower and upper approximations by using quantale homomorphism and set-valued homomorphism of quantales.

Definition 34 (see [47]). Let $(Q_t, *_1)$ and $(Q'_t, *_2)$ be two quantales. A map $f : Q_t \rightarrow Q'_t$ is called a quantale homomorphism if

- (1) $f(a *_1 b) = f(a) *_2 f(b)$;
- (2) $f(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} f(a_i) \forall a, b, a_i \in Q_t (i \in I)$.

A quantale homomorphism $f : Q_t \rightarrow Q'_t$ is called an *epimorphism* if f is onto Q'_t and f is called a *monomorphism* if f is one-one. If f is bijective, then it is called an *isomorphism*.

It is clear that if $x \leq y$, then $f(x) \leq f(y)$; that is, f is order-preserving.

Proposition 35. Let $(Q_t, *_1)$ and $(Q'_t, *_2)$ be two quantales, let $f : Q_t \rightarrow Q'_t$ be an epimorphism, and let $H_2 : Q'_t \rightarrow P^*(Q'_t)$ be a SV-Hom. Then one has the following:

- (1) If f is one to one and $H_1(x) = \{y \in Q_t \mid f(y) \in H_2(f(x))\} \forall x \in Q_t$, then H_1 is a SV-Hom from Q_t to $P^*(Q_t)$.
- (2) If H_2 is a SSV-Hom, then H_1 is a SSV-Hom.

Proof. (1) First of all, we show that H_1 is a well-defined mapping. Suppose $x_1 = x_2$, then we have $y_1 \in H_1(x_1) \Leftrightarrow f(y_1) \in H_2(f(x_1)) = H_2(f(x_2)) \Leftrightarrow y_1 \in H_1(x_2)$. Thus we have $H_1(x_1) = H_1(x_2)$. Now we show that H_1 is SV-Hom. Suppose $y \in H_1(x_1) *_1 H_1(x_2)$, then there exist $a \in H_1(x_1)$ and $b \in H_1(x_2)$ such that $y = a *_1 b$. Since H_2 is a SV-Hom and f is a quantale homomorphism, then $f(a) *_2 f(b) \in H_2(f(x_1)) *_2 H_2(f(x_2)) \subseteq H_2(f(x_1) *_2 f(x_2)) = H_2(f(x_1 *_1 x_2))$. Therefore, $f(a *_1 b) = f(a) *_2 f(b) \in H_2(f(x_1 *_1 x_2))$. Hence $y = a *_1 b \in H_1(x_1 *_1 x_2)$. Thus, we have $H_1(x_1) *_1 H_1(x_2) \subseteq H_1(x_1 *_1 x_2)$. Now we show that $\bigvee_{i \in I} H_1(x_i) \subseteq H_1(\bigvee_{i \in I} x_i) \forall x_i \in Q_t (i \in I)$. Let $y \in \bigvee_{i \in I} H_1(x_i)$, then there exists $a_i \in H_1(x_i) \forall i \in I$ such that $y = \bigvee_{i \in I} a_i$. Hence $f(y) = f(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} f(a_i) \in \bigvee_{i \in I} H_2(f(x_i)) \subseteq H_2(\bigvee_{i \in I} f(x_i)) = H_2(f(\bigvee_{i \in I} x_i))$. Finally, $y = \bigvee_{i \in I} a_i \in H_1(\bigvee_{i \in I} x_i)$. Hence $\bigvee_{i \in I} H_1(x_i) \subseteq H_1(\bigvee_{i \in I} x_i)$. So, H_1 is a SV-Hom from Q_t to $P^*(Q_t)$.

(2) It is similar to part(1). $\square$

Theorem 36. Let $f : Q_t \rightarrow Q'_t$ be a quantale isomorphism and let $H_2 : Q'_t \rightarrow P^*(Q'_t)$ be a SV-Hom. Set $H_1(m) = \{z \in Q_t \mid f(z) \in H_2(f(m))\} \forall m \in Q_t$ and $\forall \emptyset \neq A \subseteq Q'_t$, then

- (1) $f(\bar{H}_1(A)) = \bar{H}_2(f(A))$;
- (2) $f(\underline{H}_1(A)) = \underline{H}_2(f(A))$;
- (3) $f(x) \in f(\bar{H}_1(A)) \Leftrightarrow x \in \bar{H}_1(A)$.

Proof. (1) Let $z \in f(\overline{H}_1(A))$. Then there exists $x \in \overline{H}_1(A)$ such that $f(x) = z$. Since $x \in \overline{H}_1(A)$, $H_1(x) \cap A \neq \emptyset$. Suppose $z' \in H_1(x) \cap A$, then $f(z') \in f(A)$, and by the definition of $H_1(x)$, we obtain $f(z') \in H_2(f(x))$. Thus, $H_2(f(x)) \cap f(A) \neq \emptyset$, and hence $z = f(x) \in \overline{H}_2(f(A))$. Thus, we obtain $f(\overline{H}_1(A)) \subseteq \overline{H}_2(f(A))$.

Now we take $y \in \overline{H}_2(f(A))$, then there is $m \in Q_t$ such that $f(m) = y$. Hence $H_2(f(m)) \cap f(A) \neq \emptyset$. So there is $z_1 \in A$ such that $f(z_1) \in f(A)$ and $f(z_1) \in H_2(f(m))$. By the definition of $H_1(m)$, we have $z_1 \in H_1(m)$. Thus $H_1(m) \cap A \neq \emptyset$. This gives $m \in \overline{H}_1(A)$. Hence, $y = f(m) \in f(\overline{H}_1(A))$. Thus $\overline{H}_2(f(A)) \subseteq f(\overline{H}_1(A))$. Finally, we obtain $f(\overline{H}_1(A)) = \overline{H}_2(f(A))$.

(2) Suppose $z \in f(\underline{H}_1(A))$, then there exists $m \in \underline{H}_1(A)$ such that $f(m) = z$ and $H_1(m) \subseteq A$. Suppose $z' \in H_2(f(m))$, then there is $n' \in Q_t$ such that $f(n') = z'$; hence $f(n') \in H_2(f(m))$. Thus $n' \in H_1(m) \subseteq A$, so $z' = f(n') \in f(A)$. Hence, $H_2(f(m)) \subseteq f(A)$. Thus $z = f(m) \in \underline{H}_2(f(A))$, so we have $f(\underline{H}_1(A)) \subseteq \underline{H}_2(f(A))$.

Now let $y \in \underline{H}_2(f(A))$. Then there exists $n \in Q_t$ such that $f(n) = y$ and $H_2(f(n)) \subseteq f(A)$. Suppose $n' \in H_1(n)$, then $f(n') \in H_2(f(n)) \subseteq f(A)$ and hence $n' \in A$. Thus $H_1(n) \subseteq A$ and we obtain $n \in \underline{H}_1(A)$. Hence $f(n) = y \in f(\underline{H}_1(A))$ and thus, $\underline{H}_2(f(A)) \subseteq f(\underline{H}_1(A))$. Hence finally, we have $f(\underline{H}_1(A)) = \underline{H}_2(f(A))$.

(3) Let $x \in \overline{H}_1(A)$. Then $f(x) \in f(\overline{H}_1(A))$. Conversely, suppose that $f(x) \in f(\overline{H}_1(A))$, then there is $y \in \overline{H}_1(A)$ such that $f(x) = f(y)$. Since f is one-one, hence $x = y \in \overline{H}_1(A)$. $\square$

Remark 37. From Theorem 36(3), it is easily obtained that $f(x) \in f(\underline{H}_1(A)) \Leftrightarrow x \in \underline{H}_1(A) \forall \emptyset \neq A \subseteq Q_t'$.

Theorem 38. Let $f : Q_t \rightarrow Q_t'$ be a surjective quantale homomorphism and let $H_2 : Q_t' \rightarrow P^*(Q_t')$ be a SV-Hom. Let $H_1(x) = \{y \in Q_t \mid f(y) \in H_2(f(x))\} \forall x \in Q_t$ and $\forall \emptyset \neq A \subseteq Q_t'$. Then,

- (1) $\overline{H}_1(A)$ is an ideal of Q_t iff $\overline{H}_2(f(A))$ is an ideal of Q_t' ;
- (2) $\overline{H}_1(A)$ is a prime ideal of Q_t iff $\overline{H}_2(f(A))$ is a prime ideal of Q_t' ;
- (3) $\overline{H}_1(A)$ is a semiprime ideal of Q_t iff $\overline{H}_2(f(A))$ is a semiprime ideal of Q_t' ;
- (4) $\overline{H}_1(A)$ is a primary ideal of Q_t iff $\overline{H}_2(f(A))$ is a primary ideal of Q_t' .

Proof. (1) Suppose $\overline{H}_1(A)$ is an ideal of Q_t . We show that $\overline{H}_2(f(A))$ is an ideal of Q_t' , where $f(\overline{H}_1(A)) = \overline{H}_2(f(A))$ by Theorem 36(1).

(i) Let $x, z \in f(\overline{H}_1(A))$. Then there exists $x_1, z_1 \in \overline{H}_1(A)$ such that $f(x_1) = x$ and $f(z_1) = z$. Since f is a surjective quantale homomorphism and $\overline{H}_1(A)$ is an ideal of Q_t , we have $x \vee z = f(x_1 \vee f(z_1)) = f(x_1 \vee z_1) \in f(\overline{H}_1(A))$. Therefore $x \vee z \in f(\overline{H}_1(A)) \forall x, z \in f(\overline{H}_1(A))$.

(ii) Let $z \leq x \in f(\overline{H}_1(A))$. Then we obtain $x_1 \in \overline{H}_1(A)$ and $z_1 \in Q_t$ such that $f(x_1) = x$ and $f(z_1) = z$. Since

$f(z_1) \leq f(x_1)$, we have $f(x_1 \vee z_1) = f(x_1) \vee f(z_1) = f(x_1) \in f(\overline{H}_1(A))$. But $\overline{H}_1(A)$ is a lower set and $z_1 \leq x_1 \vee z_1$, and we have $z_1 \in \overline{H}_1(A)$. Thus $z = f(z_1) \in f(\overline{H}_1(A))$.

(iii) Let $x \in f(\overline{H}_1(A))$ and $z \in Q_t'$. Then there exist $x_1 \in \overline{H}_1(A)$ and $z_1 \in Q_t$ such that $f(x_1) = x$ and $f(z_1) = z$. Since $\overline{H}_1(A)$ is an ideal and f is a quantale homomorphism, we have $x_1 * z_1 \in \overline{H}_1(A)$. Hence $x * z = f(x_1 * z_1) = f(x_1 * z_1) \in f(\overline{H}_1(A))$. In the same way, we have $z * x \in f(\overline{H}_1(A))$. Hence, $f(\overline{H}_1(A))$ is an ideal of Q_t' . But $f(\overline{H}_1(A)) = \overline{H}_2(f(A))$. So $\overline{H}_2(f(A))$ is an ideal of Q_t' .

Conversely, suppose $f(\overline{H}_1(A)) = \overline{H}_2(f(A))$ is an ideal of Q_t' .

(i) Let $z_1, z_2 \in \overline{H}_1(A)$. Then $f(z_1), f(z_2) \in f(\overline{H}_1(A))$. Since $f(\overline{H}_1(A))$ is directed, $f(z_1 \vee z_2) = f(z_1) \vee f(z_2) \in f(\overline{H}_1(A))$. So by Theorem 36(3), we have $z_1 \vee z_2 \in \overline{H}_1(A)$. Hence $\overline{H}_1(A)$ is directed.

(ii) Let $z_1 \leq z_2 \in \overline{H}_1(A)$. Then $f(z_1) \leq f(z_2) \in f(\overline{H}_1(A))$. Since $f(\overline{H}_1(A))$ is a lower set, then $f(z_1) \in f(\overline{H}_1(A))$. By Theorem 36(3), we obtain $z_1 \in \overline{H}_1(A)$. So $\overline{H}_1(A)$ is a lower set.

(iii) Suppose $y' \in Q_t$ and $y \in \overline{H}_1(A)$, then $f(y') \in Q_t'$ and $f(y) \in f(\overline{H}_1(A))$. But $f(\overline{H}_1(A))$ is an ideal of Q_t' , and we have $f(y * y') = f(y) * f(y') \in f(\overline{H}_1(A))$. Thus by Theorem 36(3), we have $y * y' \in \overline{H}_1(A)$. Similarly, $y' * y \in \overline{H}_1(A)$. So, $\overline{H}_1(A)$ is an ideal of Q_t .

(2) Let $\overline{H}_1(A)$ be a prime ideal of Q_t . Then $\overline{H}_1(A)$ is obviously an ideal of Q_t and $\overline{H}_1(A) \neq Q_t$. By part (1), $\overline{H}_2(f(A))$ is an ideal of Q_t' . We also have that $\overline{H}_2(f(A)) = f(\overline{H}_1(A)) \neq Q_t'$. Now suppose $y_1, y_2 \in Q_t'$ and $y_1 * y_2 \in \overline{H}_2(f(A))$. Since f is surjective, there are $z_1, z_2 \in Q_t$ such that $y_1 = f(z_1)$, $y_2 = f(z_2)$. Then $f(z_1 * z_2) = f(z_1) * f(z_2) = y_1 * y_2 \in \overline{H}_2(f(A))$. By Theorem 36(3), we obtain $z_1 * z_2 \in \overline{H}_1(A)$. But $\overline{H}_1(A)$ is prime, and we have $z_1 \in \overline{H}_1(A)$ or $z_2 \in \overline{H}_1(A)$. Thus $y_1 \in f(\overline{H}_1(A)) = \overline{H}_2(f(A))$ or $y_2 \in f(\overline{H}_1(A)) = \overline{H}_2(f(A))$. So $\overline{H}_2(f(A))$ is a prime ideal of Q_t' .

Conversely, let $\overline{H}_2(f(A))$ be a prime ideal of Q_t' . Then $\overline{H}_2(f(A))$ is an ideal of Q_t' . Since $f(\overline{H}_1(A)) = \overline{H}_2(f(A)) \neq Q_t'$, thus $\overline{H}_1(A) \neq Q_t$. By part (1), $\overline{H}_1(A)$ is an ideal of Q_t . Now suppose $z_1, z_2 \in Q_t$ and $z_1 * z_2 \in \overline{H}_1(A)$. So $f(z_1) * f(z_2) = f(z_1 * z_2) \in f(\overline{H}_1(A))$. Since $f(\overline{H}_1(A)) = \overline{H}_2(f(A))$ is prime, we have $f(z_1) \in f(\overline{H}_1(A))$ or $f(z_2) \in f(\overline{H}_1(A))$. So by Theorem 36(3), we have $z_1 \in \overline{H}_1(A)$ or $z_2 \in \overline{H}_1(A)$. Thus $\overline{H}_1(A)$ is a prime ideal of Q_t .

The remaining parts (3) and (4) are similar to the proof (2). $\square$

Theorem 39. Let $f : Q_t \rightarrow Q_t'$ be a surjective quantale homomorphism and let $H_2 : Q_t' \rightarrow P^*(Q_t')$ be a SV-Hom. Set $H_1(x) = \{y \in Q_t \mid f(y) \in H_2(f(x))\} \forall x \in Q_t$ and $\forall \emptyset \neq B \subseteq Q_t'$. Then the following hold:

- (1) $\underline{H}_1(B)$ is an ideal of Q_t iff $\underline{H}_2(f(B))$ is an ideal of Q_t' ;
- (2) $\underline{H}_1(B)$ is a prime ideal of Q_t iff $\underline{H}_2(f(B))$ is a prime ideal of Q_t' ;

- (3) $\underline{H}_1(B)$ is a semiprime ideal of Q_t iff $\underline{H}_2(f(B))$ is a semiprime ideal of Q'_t ;
 (4) $\underline{H}_1(B)$ is a primary ideal of Q_t iff $\underline{H}_2(f(B))$ is a primary ideal of Q'_t .

Proof. The proofs of all the parts can be obtained by Theorem 38. $\square$

5. Generalized Rough Fuzzy

Prime (Primary) Ideals Induced by Quantale Homomorphism

Theorem 40. Let $f : Q_t \rightarrow Q'_t$ be a surjective quantale homomorphism, let $H_2 : Q'_t \rightarrow P^*(Q'_t)$ be a SV-Hom, and let λ be a fuzzy subset of Q'_t . If $H_1(x) = \{y \in Q_t \mid f(y) \in H_2(f(x))\} \forall x \in Q_t$, then

- (1) $\overline{H}_1(\lambda)$ is a f -ideal of Q_t iff $\overline{H}_2(f(\lambda))$ is a f -ideal of Q'_t ;
 (2) $\overline{H}_1(\lambda)$ is a f -prime ideal of Q_t iff $\overline{H}_2(f(\lambda))$ is a f -prime ideal of Q'_t ;
 (3) $\overline{H}_1(\lambda)$ is a f -semiprime ideal of Q_t iff $\overline{H}_2(f(\lambda))$ is a f -semiprime ideal of Q'_t ;
 (4) $\overline{H}_1(\lambda)$ is a f -primary ideal of Q_t iff $\overline{H}_2(f(\lambda))$ is a f -primary ideal of Q'_t .

In the above, $f(\lambda)(y) = \bigvee \{\lambda(x) \mid f(x) = y, x \in Q_t\}$, $y \in Q'_t$; that is, $f(\lambda)$ is the standard Zadeh image of the fuzzy subset λ under the mapping f .

Proof. (1) We first point out that, for each $\alpha \in [0, 1]$, $(f(\lambda))_{\alpha^+} = f(\lambda_{\alpha^+})$ and $(\overline{H}_1(\lambda))_{\alpha^+} \neq \emptyset$ if and only if $(\overline{H}_2(f(\lambda)))_{\alpha^+} \neq \emptyset$.

Let $\overline{H}_1(\lambda)$ be a f -ideal of Q_t . Then for all $\alpha \in (0, 1]$, if $(\overline{H}_2(f(\lambda)))_{\alpha^+} \neq \emptyset$, then $(\overline{H}_1(\lambda))_{\alpha^+} \neq \emptyset$. By Theorem 18, we have $(\overline{H}_1(\lambda))_{\alpha^+}$ is an ideal of Q_t . Also by using Proposition 12, we obtain that $\overline{H}_1(\lambda_{\alpha^+})$ is an ideal of Q_t . Now, by Theorem 38(1), we have that $(\overline{H}_2(f(\lambda)))_{\alpha^+} = \overline{H}_2(f(\lambda))_{\alpha^+} = \overline{H}_2(f(\lambda_{\alpha^+}))$ is an ideal of Q'_t . Thus, by Theorem 18, we have that $\overline{H}_2(f(\lambda))$ is a f -ideal of Q'_t .

Conversely, suppose $\overline{H}_2(f(\lambda))$ is a f -ideal of Q'_t . We have that $(\overline{H}_2(f(\lambda)))_{\alpha^+} = \overline{H}_2(f(\lambda))_{\alpha^+} = \overline{H}_2(f(\lambda_{\alpha^+}))$ is an ideal of Q'_t by utilizing Theorem 18. It is obtained from Theorem 38(1) that $\overline{H}_1(\lambda_{\alpha^+})$ is an ideal of Q_t . Hence by Theorem 18, $\overline{H}_1(\lambda)$ is a f -ideal of Q_t .

(2) Let $\overline{H}_1(\lambda)$ be a f -prime ideal of Q_t . Now for $\overline{H}_2(f(\lambda))_{\alpha^+} \neq \emptyset$, then $(\overline{H}_1(\lambda))_{\alpha^+} \neq \emptyset$ for any $\alpha \in [0, 1]$. Since $\overline{H}_1(\lambda)$ is a f -prime ideal of Q_t , then by Theorem 25, we have that $(\overline{H}_1(\lambda))_{\alpha^+}$ is a prime ideal of Q_t . It is also obtained from Proposition 12 that $\overline{H}_1(\lambda_{\alpha^+})$ is a prime ideal of Q_t . Hence $(\overline{H}_2(f(\lambda)))_{\alpha^+} = \overline{H}_2(f(\lambda))_{\alpha^+} = \overline{H}_2(f(\lambda_{\alpha^+}))$ is a prime ideal of Q'_t , by Theorem 38(2). Thus, by Theorem 25, we have that $\overline{H}_2(f(\lambda))$ is a f -prime ideal of Q'_t .

Conversely, suppose $\overline{H}_2(f(\lambda))$ is a f -prime ideal of Q'_t . By Theorem 25, we obtain that $(\overline{H}_2(f(\lambda)))_{\alpha^+} = \overline{H}_2(f(\lambda))_{\alpha^+} =$

$\overline{H}_2(f(\lambda_{\alpha^+}))$ is a prime ideal of Q'_t . Thus it is obtained, from Theorem 38(2), that $\overline{H}_1(\lambda_{\alpha^+})$ is a prime ideal of Q_t . Hence $\overline{H}_1(\lambda)$ is a f -prime ideal of Q_t by Theorem 25.

Axioms (3) and (4) can be obtained in a similar way. $\square$

Theorem 41. Let f be a surjective quantale homomorphism from a quantale $(Q_t, *_1)$ onto a quantale $(Q'_t, *_2)$. Let $H_2 : Q'_t \rightarrow P^*(Q'_t)$ be a SV-Hom and let λ be a fuzzy subset of Q'_t . If $H_1(x) = \{y \in Q_t \mid f(y) \in H_2(f(x))\} \forall x \in Q_t$, then

- (1) $\underline{H}_1(\lambda)$ is a f -ideal of Q_t iff $\underline{H}_2(f(\lambda))$ is a f -ideal of Q'_t ;
 (2) $\underline{H}_1(\lambda)$ is a f -prime ideal of Q_t iff $\underline{H}_2(f(\lambda))$ is a f -prime ideal of Q'_t ;
 (3) $\underline{H}_1(\lambda)$ is a f -semiprime ideal of Q_t iff $\underline{H}_2(f(\lambda))$ is a f -semiprime ideal of Q'_t ;
 (4) $\underline{H}_1(\lambda)$ is a f -primary ideal of Q_t iff $\underline{H}_2(f(\lambda))$ is a f -primary ideal of Q'_t .

Proof. The proof is similar as reported in Theorem 40. $\square$

6. Conclusion

Pure and applied mathematics are two important branches of mathematics and rough set theory has its own importance in both the branches. When we combine rough set theory with algebraic structures, we obtain new interesting results and research topics. These research topics are attracted by computer scientists and mathematicians. Researchers apply roughness into the algebraic system and find interesting algebraic properties of them. The combination of fuzzy set and rough set theory leads to various models. The relations between fuzzy sets, rough sets, and quantale theory have been already considered in [46]. We have examined the generalized rough fuzzy set theory and its properties in quantale.

In the present paper, we substituted a universe set by a quantale and introduced the notions of generalized rough fuzzy prime (semiprime, primary) ideals in quantale. We see that the lower and upper approximations of fuzzy ideals, using SSV-Hom, are fuzzy ideals, respectively. It is also seen that the approximations of fuzzy prime (semiprime, primary) ideals using SSV-Hom are fuzzy prime (semiprime, primary) ideals, respectively. We have discussed the relation between upper (lower) generalized rough fuzzy (prime, semiprime, primary) ideals and upper (lower) generalized rough fuzzy approximations of their homomorphic images.

We believe that in the near future the idea of generalized roughness will be extended to other algebraic structures.

Conflicts of Interest

There are no conflicts of interest related to this paper.

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