Research Article

On a System of Volterra Type Hadamard Fractional Integral Equations in Fréchet Spaces

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In this article we present some results concerning the existence of solutions for a system of Hadamard integral equations. Our investigation is conducted with an application of an extension of the fixed point theorem of Burton-Kirk in Fréchet spaces.

1. Introduction

In recent years, fractional differential and integral equations are emerging as a useful tool in modeling of the dynamics of many physical systems and electrical phenomena, which has been demonstrated by many researchers in the fields of mathematics, science, and engineering; see monographs [1–7] and a series of papers [8–18] and the references cited therein. Some properties of the Hadamard fractional calculus were investigated by Butzer et al. [19, 20]. Additional papers can be found in [7, 21] and the references therein. Recently, some interesting results on the existence of the solutions of some classes of integral equations have been obtained by Abbas et al. [22–24], Banaś et al. [25, 26], and Pachpatte [27, 28] and the references therein.

This paper deals with the existence of solutions of the following system of Hadamard fractional integral equations of the form

$$\begin{aligned} u_1(t, x) &= \mu_1(t, x) + f_1\left(t, x, \left({}^H I_{\sigma}^r u_1\right)(t, x), u_1(t, x)\right) \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{y}\right)^{r_2-1} \times g_1(t, \\ &\quad x, s, y, u_1(s, y), u_2(s, y), \dots, u_n(s, y)) \frac{dy ds}{sy} \\ u_2(t, x) &= \mu_2(t, x) + f_2\left(t, x, \left({}^H I_{\sigma}^r u_2\right)(t, x), u_2(t, x)\right) \end{aligned}$$

$$\begin{aligned} &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{y}\right)^{r_2-1} \times g_2(t, \\ &\quad x, s, y, u_1(s, y), u_2(s, y), \dots, u_n(s, y)) \frac{dy ds}{sy} \\ &\dots\dots\dots \\ u_n(t, x) &= \mu_n(t, x) + f_n\left(t, x, \left({}^H I_{\sigma}^r u_n\right)(t, x), u_n(t, x)\right) \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{y}\right)^{r_2-1} \\ &\quad \times g_n(t, x, s, y, u_1(s, y), u_2(s, y), \dots, \\ &\quad u_n(s, y)) \frac{dy ds}{sy}; \\ &\quad (t, x) \in J := [1, +\infty) \times [1, b]. \end{aligned} \tag{1}$$

That is,

$$\begin{aligned} u_i(t, x) &= \mu_i(t, x) + f_i\left(t, x, \left({}^H I_{\sigma}^r u_i\right)(t, x), u_i(t, x)\right) \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{y}\right)^{r_2-1} \times g_i(t, \end{aligned}$$

$$x, s, y, u_1(s, y), u_2(s, y), \dots, u_n(s, y) \frac{dy ds}{sy};$$

$$(t, x) \in J, \quad i = 1, \dots, n, \quad (2)$$

where $b > 1$, $\sigma = (1, 1)$, $r = (r_1, r_2)$, $r_1, r_2 \in (0, \infty)$, ${}^H I_\sigma^r$ is the Hadamard integral of order r , $\mu_i : J \rightarrow \mathbb{R}$, $f_i : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $g_i : J' \times \mathbb{R}^n \rightarrow \mathbb{R}$ are given continuous functions, $J' = \{(t, x, s, y) \in J^2 : s \leq t, y \leq x\}$, and $\Gamma(\cdot)$ is the (Euler's) gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt; \quad \xi > 0. \quad (3)$$

2. Preliminaries

Let $L^1([1, +\infty) \times [1, b])$; for $b > 1$, be the space of Lebesgue-integrable functions $u : [1, +\infty) \times [1, b] \rightarrow \mathbb{R}$ with the standard norm

$$\|u\|_1 = \int_1^\infty \int_1^b |u(t, x)| dx dt. \quad (4)$$

Definition 1 (see [4]). The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1([1, a], \mathbb{R})$ is defined as

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\log \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds. \quad (5)$$

Example 2. The Hadamard fractional integral of order $q > 0$ for the function $w : [1, e] \rightarrow \mathbb{R}$, defined by $w(x) = (\log x)^{\beta-1}$ with $\beta > 0$, is

$$({}^H I_1^q w)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta + q)} (\log x)^{\beta+q-1}. \quad (6)$$

Definition 3. Let $r_1, r_2 \geq 0$, $\sigma = (1, 1)$, and $r = (r_1, r_2)$. For $w \in L^1(J, \mathbb{R})$, define the Hadamard partial fractional integral of order r by the expression

$$({}^H I_\sigma^r w)(x, y) = \frac{1}{\Gamma(r_1) \Gamma(r_2)} \cdot \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{w(s, t)}{st} dt ds. \quad (7)$$

Let X be a Fréchet space with a family of seminorms $\{\|\cdot\|_n\}_{n \in \mathbb{N}^*}$ such that

$$\|x\|_1 \leq \|x\|_2 \leq \|x\|_3 \leq \dots \quad \text{for every } x \in X. \quad (8)$$

A subset Y of X is bounded if, for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leq \overline{M}_n \quad \text{for all } y \in Y. \quad (9)$$

We associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ to X as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by $x \sim_n y$ if and only if $\|x - y\|_n = 0$ for $x, y \in X$.

We denote by $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $x \in X$, we denote by $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote by $\overline{Y^n}$, $\text{int}_n(Y^n)$, and $\partial_n Y^n$, respectively, the closure, the interior, and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n . For more details, we refer the reader to [29].

Definition 4. Let X be a Fréchet space. A function $N : X \rightarrow X$ is said to be a contraction if for each $n \in \mathbb{N} \setminus \{0\}$ there exists $k_n \in [0, 1)$ such that

$$\|N(u) - N(v)\|_n \leq k_n \|u - v\|_n \quad \text{for all } u, v \in X. \quad (10)$$

We need the following extension of the Burton-Kirk fixed point theorem in the case of a Fréchet space.

Theorem 5 (see [30]). *Let $(X, \|\cdot\|_n)$ be a Fréchet space and let $A, B : X \rightarrow X$ be two operators such that*

- (a) *A is a compact operator*
- (b) *B is a contraction operator with respect to a family of seminorms $\{\|\cdot\|_n\}$*
- (c) *the set $\{x \in X : x = \lambda A(x) + \lambda B(x/\lambda), \lambda \in (0, 1)\}$ is bounded*

Then the operator equation $A(u) + B(u) = u$ has a solution in X .

3. Existence Results

Set

$$C^n(J) = C(J) \times \dots \times C(J) \quad (n \text{ times}). \quad (11)$$

Definition 6. A function $(u_1, u_2, \dots, u_n) \in C^n(J)$ is said to be a solution of (2) if $(u_1, u_2, \dots, u_n)$ satisfies (2) on J .

For each $p \in \mathbb{N} \setminus \{0, 1\}$ we consider the set $C_p = C([1, p] \times [1, b])$, and we define in $C(J)$ the seminorms by

$$\|u\|_p = \sup_{(t,x) \in [1,p] \times [1,b]} |u(t, x)|. \quad (12)$$

Then C is a Fréchet space with the family of seminorms $\{\|u\|_p\}$.

Also, the product space $\mathcal{C} := C^n$ is a Fréchet space with the family of seminorms

$$\|(u_1, u_2, \dots, u_n)\|_p = \sum_{i=1}^n \|u_i\|_p. \quad (13)$$

Now, we are concerned with the existence of solutions for system (2). Let us introduce the following hypotheses:

(H₁) There exist continuous functions $l_i, k_i : J \rightarrow \mathbb{R}_+; i = 1, \dots, n$, with

$$\lim_{t \rightarrow \infty} l_i(t, x) = \lim_{t \rightarrow \infty} k_i(t, x) = 0; \quad \text{for } x \in [1, b], \quad (14)$$

such that

$$\begin{aligned} & |f_i(t, x, u_1, v_1) - f_i(t, x, u_2, v_2)| \\ & \leq \frac{l_i(t, x) |u_1 - u_2| + k_i(t, x) |v_1 - v_2|}{1 + |u_1 - u_2| + |v_1 - v_2|}, \end{aligned} \quad (15)$$

for each $(t, x) \in J$ and each $u_1, u_2, v_1, v_2 \in \mathbb{R}$

(H₂) There exist continuous functions $Q_{ij}, \varphi_i : J' \rightarrow \mathbb{R}_+$; $i = 1, \dots, n, j = 0, \dots, n$, and nondecreasing functions $\psi_i : [0, \infty) \rightarrow (0, \infty)$ such that

$$\begin{aligned} & |g_i(t, x, s, y, u_1, u_2, \dots, u_n)| \\ & \leq \frac{Q_{i0}(t, x, s, y) + \sum_{j=1}^n Q_{ij}(t, x, s, y) |u_j|}{1 + \sum_{j=1}^n |u_j|}, \end{aligned} \quad (16)$$

for $(t, x, s, y) \in J', u_i \in \mathbb{R}; i = 1, \dots, n$, and

$$\begin{aligned} & |g_i(t_1, x_1, s, y, u_1, \dots, u_n) - g_i(t_2, x_2, s, y, u_1, \dots, u_n)| \\ & \leq \varphi_i(s, y) (|t_1 - t_2| + |x_1 - x_2|) \times \sum_{j=1}^n \psi_i(|u_j|); \end{aligned} \quad (17)$$

$(t_1, x_1), (t_2, x_2), (s, y) \in J, u_i \in \mathbb{R}.$

Moreover, assume that

$$\lim_{t \rightarrow \infty} \int_1^t \left| \log \frac{t}{s} \right|^{r_1-1} \frac{Q_{ij}(t, x, s, y)}{s} ds = 0, \quad (18)$$

for each $(x, y) \in J$ with $y \leq x$

For any $p \in \mathbb{N} \setminus \{0, 1\}$, set

$$k_p^* = \max_{i=1, \dots, n} k_{ip} = \max_{i=1, \dots, n} \sup_{(t, x) \in [1, p] \times [1, b]} k_i(t, x), \quad (19)$$

and

$$l_p^* = \max_{i=1, \dots, n} l_{ip} = \max_{i=1, \dots, n} \sup_{(t, x) \in [1, p] \times [1, b]} l_i(t, x). \quad (20)$$

Theorem 7. Assume that hypotheses (H₁) and (H₂) hold. If

$$L_p := k_p^* + \frac{l_p^* (\log p)^{r_1} (\log b)^{r_2}}{\Gamma(1 + r_1) \Gamma(1 + r_2)} < 1, \quad (21)$$

then system (2) has at least one solution in the space $\mathcal{C}$.

Proof. Define the operators $A_i, B_i : C \rightarrow C; i = 1, \dots, n$, by

$$\begin{aligned} (A_i u_i)(t, x) &= \int_1^t \int_1^x \left(\log \frac{t}{s} \right)^{r_1-1} \left(\log \frac{x}{y} \right)^{r_2-1} \\ & \times \frac{g_i(t, x, s, y, u_1(s, y), \dots, u_n(s, y))}{sy \Gamma(r_1) \Gamma(r_2)} dy ds; \end{aligned} \quad (22)$$

$(t, x) \in J,$

$$\begin{aligned} (B_i u_i)(t, x) &= \mu_i(t, x) + f_i(t, x, {}^H I_\sigma^r u_i(t, x), u_i(t, x)); \\ & (t, x) \in J, \end{aligned} \quad (23)$$

and consider the continuous operators $A, B : \mathcal{C} \rightarrow \mathcal{C}$ defined by

$$A(u_1, \dots, u_n) = (A_1(u_1), \dots, A_n(u_n)), \quad (24)$$

$$B(u_1, \dots, u_n) = (B_1(u_1), \dots, B_n(u_n)). \quad (25)$$

We shall show that operators A and B satisfied all the conditions of Theorem 5.

The proof will be given in several steps.

Step 1 (A is compact). To this aim, we must prove that A is continuous and it transforms every bounded set into a relatively compact set. Recall that $M \subset \mathcal{C}$ is bounded if and only if

$$\begin{aligned} & \forall p \in \mathbb{N} \setminus \{0, 1\}, \\ & \exists \ell_p > 0 : \forall (u_1, \dots, u_n) \in M, \end{aligned} \quad (26)$$

$$\|(u_1, \dots, u_n)\|_p \leq \ell_p,$$

and $M = \{(u_1(t, x), \dots, u_n(t, x)); (t, x) \in J\} \subset \mathcal{C}$ is relatively compact if and only if, $\forall p \in \mathbb{N} \setminus \{0, 1\}$, the family $\{(u_1(t, x), \dots, u_n(t, x))|_{(t, x) \in [1, p] \times [1, b]}\}$ is equicontinuous and uniformly bounded on $[1, p] \times [1, b]$. The proof will be given in several claims.

Claim 1 (A is continuous). Let $\{(u_{1m}, \dots, u_{nm})\}_{m \in \mathbb{N} \setminus \{0, 1\}}$ be a sequence in M such that $(u_{1m}, \dots, u_{nm}) \rightarrow (u_1, \dots, u_n)$ in M . Then, for each $(t, x) \in J$ and $i = 1, \dots, n$, we have

$$\begin{aligned} & |(A_i(u_{im}))(t, x) - (A_i(u_i))(t, x)| \leq \frac{1}{\Gamma(r_1) \Gamma(r_2)} \\ & \cdot \int_1^t \int_1^x \left| \log \frac{t}{s} \right|^{r_1-1} \left| \log \frac{x}{y} \right|^{r_2-1} \\ & \times |g_i(t, x, s, y, u_{1m}(s, y), \dots, u_{nm}(s, y)) \\ & - g_i(t, x, s, y, u_1(s, y), \dots, u_n(s, y))| dy ds \\ & \leq \frac{1}{\Gamma(r_1) \Gamma(r_2)} \int_1^t \int_1^x \left| \log \frac{t}{s} \right|^{r_1-1} \left| \log \frac{x}{y} \right|^{r_2-1} \\ & \times |g_i(t, x, s, y, u_{1m}(s, y), \dots, u_{nm}(s, y)) \\ & - g_i(t, x, s, y, u_1(s, y), \dots, u_n(s, y))| dy ds. \end{aligned} \quad (27)$$

Since $u_{im} \rightarrow u_i$ as $m \rightarrow \infty$ and g_i is continuous, then for each $(t, x) \in [1, p] \times [1, b]; p \in \mathbb{N} \setminus \{0, 1\}$, and $i = 1, \dots, n$, (27) gives

$$\begin{aligned} & \|A(u_{1m}, \dots, u_{nm}) - A(u_1, \dots, u_n)\|_p \\ & = \sum_{i=1}^n \|A_i(u_{im}) - A_i(u_i)\|_p \rightarrow 0 \quad \text{as } m \rightarrow \infty. \end{aligned} \quad (28)$$

Claim 2 (A maps bounded sets into bounded sets in C). Let M be the bounded set in $\mathcal{C}$ as in Claim 1. Then, for each

$p \in \mathbb{N} \setminus \{0, 1\}$, there exists $\ell_p > 0$, such that for all $(u_1, \dots, u_n) \in \mathcal{C}$ we have $\|(u_1, \dots, u_n)\|_p \leq \ell_p$.

For arbitrarily fixed $(t, x) \in [1, p] \times [1, b]$ and each $i = 1, \dots, n$, we have

$$\begin{aligned} |(A_i u_i)(t, x)| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left| \log \frac{t}{s} \right|^{r_1-1} \left| \log \frac{x}{y} \right|^{r_2-1} \\ &\quad \times |g_i(t, x, s, y, u_1(s, y), \dots, u_n(s, y))| dy ds \\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left| \log \frac{t}{s} \right|^{r_1-1} \left| \log \frac{x}{y} \right|^{r_2-1} \\ &\quad \times \frac{Q_{i0}(t, x, s, y) + \sum_{j=1}^n Q_{ij}(t, x, s, y) |u_j(s, y)|}{1 + \sum_{j=1}^n |u_j(s, y)|} dy ds \quad (29) \\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left| \log \frac{t}{s} \right|^{r_1-1} \left| \log \frac{x}{y} \right|^{r_2-1} \\ &\quad \times \left(\sum_{j=0}^n Q_{ij}(t, x, s, y) \right) dy ds \leq Q_{ip}, \end{aligned}$$

$$\begin{aligned} Q_{ip} &= \sup_{(t,x) \in [1,p] \times [1,b]} \int_1^t \int_1^x \left| \log \frac{t}{s} \right|^{r_1-1} \left| \log \frac{x}{y} \right|^{r_2-1} \\ &\quad \cdot \frac{\sum_{j=0}^n Q_{ij}(t, x, s, y)}{\Gamma(r_1)\Gamma(r_2)} dy ds. \end{aligned} \quad (30)$$

Hence

$$\|A(u_1, \dots, u_n)\|_p \leq \sum_{i=0}^n Q_{ip} := Q_p^*. \quad (31)$$

Claim 3 (A maps bounded sets into equicontinuous sets in C).
Let $(t_1, x_1), (t_2, x_2) \in [1, p] \times [1, b]$, $t_1 < t_2$, $x_1 < x_2$, and let $u \in M$; thus for each $i = 1, \dots, n$, we have

$$\begin{aligned} |(A_i u_i)(t_2, x_2) - (A_i u_i)(t_1, x_1)| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \left| \int_1^{t_2} \int_1^{x_2} \left| \log \frac{t_2}{s} \right|^{r_1-1} \left| \log \frac{x_2}{y} \right|^{r_2-1} \right. \\ &\quad \times [g_i(t_2, x_2, s, y, u_1(s, y), \dots, u_n(s, y)) - g_i(t_1, x_1, s, y, u_1(s, y), \dots, u_n(s, y))] dy ds \Big| \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \left| \int_1^{t_2} \int_1^{x_2} \left| \log \frac{t_2}{s} \right|^{r_1-1} \left| \log \frac{x_2}{y} \right|^{r_2-1} g_i(t_1, x_1, s, y, u_1(s, y), \dots, u_n(s, y)) dy ds \right. \\ &\quad \left. - \int_1^{t_1} \int_1^{x_1} \left| \log \frac{t_1}{s} \right|^{r_1-1} \left| \log \frac{x_1}{y} \right|^{r_2-1} g_i(t_1, x_1, s, y, u_1(s, y), \dots, u_n(s, y)) dy ds \right| \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \left| \int_1^{t_2} \int_1^{x_2} \left| \log \frac{t_1}{s} \right|^{r_1-1} \left| \log \frac{x_1}{y} \right|^{r_2-1} g_i(t_1, x_1, s, y, u_1(s, y), \dots, u_n(s, y)) dy ds \right. \\ &\quad \left. - \int_1^{t_1} \int_1^{x_1} \left| \log \frac{t_1}{s} \right|^{r_1-1} \left| \log \frac{x_1}{y} \right|^{r_2-1} g_i(t_1, x_1, s, y, u_1(s, y), \dots, u_n(s, y)) dy ds \right| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_2} \int_1^{x_2} \left| \log \frac{t_2}{s} \right|^{r_1-1} \\ &\quad \cdot \left| \log \frac{x_2}{y} \right|^{r_2-1} \times |g_i(t_2, x_2, s, y, u_1(s, y), \dots, u_n(s, y)) - g_i(t_1, x_1, s, y, u_1(s, y), \dots, u_n(s, y))| dy ds \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_1} \int_1^{x_1} \left| \left(\log \frac{t_2}{s} \right)^{r_1-1} \left(\log \frac{x_2}{y} \right)^{r_2-1} - \left(\log \frac{t_1}{s} \right)^{r_1-1} \left(\log \frac{x_1}{y} \right)^{r_2-1} \right| \\ &\quad \cdot |g_i(t_1, x_1, s, y, u_1(s, y), \dots, u_n(s, y))| dy ds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_1} \int_{x_1}^{x_2} \left| \log \frac{t_2}{s} \right|^{r_1-1} \left| \log \frac{x_2}{y} \right|^{r_2-1} \\ &\quad \cdot |g_i(t_1, x_1, s, y, u_1(s, y), \dots, u_n(s, y))| dy ds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_1^{x_1} \left| \log \frac{t_2}{s} \right|^{r_1-1} \left| \log \frac{x_2}{y} \right|^{r_2-1} \\ &\quad \cdot |g_i(t_1, x_1, s, y, u_1(s, y), \dots, u_n(s, y))| dy ds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_{x_1}^{x_2} \left| \log \frac{t_2}{s} \right|^{r_1-1} \left| \log \frac{x_2}{y} \right|^{r_2-1} \\ &\quad \cdot |g_i(t_1, x_1, s, y, u_1(s, y), \dots, u_n(s, y))| dy ds \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_2} \int_1^{x_2} \left| \log \frac{t_2}{s} \right|^{r_1-1} \left| \log \frac{x_2}{y} \right|^{r_2-1} \times \varphi_i(s, y) \end{aligned}$$

$$\begin{aligned}
& \cdot (|t_1 - t_2| + |x_1 - x_2|) \psi_i(\ell_p) dy ds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_1} \int_1^{x_1} \left| \left(\log \frac{t_2}{s} \right)^{r_1-1} \left(\log \frac{x_2}{y} \right)^{r_2-1} \right. \\
& - \left. \left(\log \frac{t_1}{s} \right)^{r_1-1} \left(\log \frac{x_1}{y} \right)^{r_2-1} \right| \left(\sum_{j=0}^n Q_{ij}(t_1, x_1, s, y) \right) dy ds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_2}^{t_1} \int_1^{x_2} \left| \log \frac{t_2}{s} \right|^{r_1-1} \left| \log \frac{x_2}{y} \right|^{r_2-1} \\
& \cdot \left(\sum_{j=0}^n Q_{ij}(t_1, x_1, s, y) \right) dy ds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_1} \int_{x_1}^{x_2} \left| \log \frac{t_2}{s} \right|^{r_1-1} \left| \log \frac{x_2}{y} \right|^{r_2-1} \left(\sum_{j=0}^n Q_{ij}(t_1, x_1, s, y) \right) dy ds \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_{x_1}^{x_2} \left| \log \frac{t_2}{s} \right|^{r_1-1} \left| \log \frac{x_2}{y} \right|^{r_2-1} \left(\sum_{j=0}^n Q_{ij}(t_1, x_1, s, y) \right) dy ds.
\end{aligned} \tag{32}$$

From the continuity of functions Q_{ij}, φ_i and as $t_1 \rightarrow t_2$ and $x_1 \rightarrow x_2$, the right-hand side of the above inequality tends to zero. As a consequence of Claims 1–3 and from the Arzelà-Ascoli theorem, we can conclude that A is continuous and compact.

Step 2 (B is a contraction). Consider $(v_1, \dots, v_n), (w_1, \dots, w_n) \in \mathcal{C}$. Then, by (H_1) , for any $p \in \mathbb{N} \setminus \{0, 1\}$ and each $(t, x) \in [1, p] \times [1, b]$ and $i = 1, \dots, n$, we have

$$\begin{aligned}
& |(B_i v_i)(t, x) - (B_i w_i)(t, x)| \\
& \leq l_i(t, x) \left| {}^H I_\sigma^r (v_i - w_i)(t, x) \right| \\
& \quad + k_i(t, x) |(v_i - w_i)(t, x)| \\
& \leq \left(k_{ip} + \frac{l_{ip} (\log p)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right) \|v_i - w_i\|_p \\
& \leq \left(k_p^* + \frac{l_p^* (\log p)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right) \|v_i - w_i\|_p.
\end{aligned} \tag{33}$$

Thus,

$$\begin{aligned}
& \sum_{i=1}^n |(B_i v_i)(t, x) - (B_i w_i)(t, x)| \\
& \leq \left(k_p^* + \frac{l_p^* (\log p)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right) \sum_{i=1}^n \|v_i - w_i\|_p.
\end{aligned} \tag{34}$$

Hence,

$$\begin{aligned}
& \|B(v_1, \dots, v_n) - B(w_1, \dots, w_n)\|_p \\
& \leq L_p \|(v_1, \dots, v_n) - (w_1, \dots, w_n)\|_p.
\end{aligned} \tag{35}$$

Inequality (21) implies that B is a contraction operator.

Step 3 (the set $\mathcal{E} := \{u = (u_1, \dots, u_n) \in \mathcal{C} : u = \lambda A(u) + \lambda B(u/\lambda), \lambda \in (0, 1)\}$ is bounded). Let $(u_1, \dots, u_n) \in \mathcal{C}$, such that $(u_1, \dots, u_n) = \lambda A(u_1, \dots, u_n) + \lambda B(u_1/\lambda, \dots, u_n/\lambda)$, for some $\lambda \in (0, 1)$. Then, for any $p \in \mathbb{N} \setminus \{0, 1\}$ and each $(t, x) \in [1, p] \times [1, b]$ and $i = 1, \dots, n$, we have

$$\begin{aligned}
& |u_i(t, x)| \leq \lambda |A_i(u_i)| + \lambda \left| B_i \left(\frac{u_i}{\lambda} \right) \right| \leq |\mu_i(t, x)| \\
& \quad + |f_i(t, x, 0, 0)| + k_i(t, x) + l_i(t, x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
& \quad \cdot \int_1^t \int_1^x \left| \log \frac{t}{s} \right|^{r_1-1} \left| \log \frac{x}{y} \right|^{r_2-1} \\
& \quad \times \left(\sum_{j=0}^n Q_{ij}(t, x, s, y) \right) \frac{dy ds}{sy} \leq \mu_{ip} + f_{ip} \\
& \quad + k_{ip} + l_{ip} + \sum_{j=0}^n Q_{ijp},
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
\mu_{ip} &= \sup_{(t,x) \in [1,p] \times [1,b]} |\mu_i(t, x)|, \\
f_{ip} &= \sup_{(t,x) \in [1,p] \times [1,b]} |f(t, x, 0, 0)|; \\
p &\in \mathbb{N} \setminus \{0, 1\}.
\end{aligned} \tag{37}$$

Thus,

$$\|(u_1, \dots, u_n)\|_p \leq \mu_p^* + f_p^* + k_p^* + l_p^* + Q_p^* =: \ell_p^*. \tag{38}$$

Hence, the set $\mathcal{E}$ is bounded. As a consequence of Steps 1–3 and from Theorem 5, we deduce that N has a fixed point $(u_1, \dots, u_n)$ which is a solution of system (2). $\square$

4. An Example

Consider the following coupled system of Hadamard fractional integral equations of the form

$$\begin{aligned}
 u(t, x) &= \frac{xe^{3-2t}}{1+t+x^2} \\
 &+ \frac{xe^{-t-2}}{c_p(1+e^{-2p}|(H I_{\sigma}^r u)(t, x)| + e^{-p}|u(t, x)|)} \\
 &+ \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{y}\right)^{r_2-1} \\
 &\cdot \frac{g_1(t, x, s, y, u(s, y), v(s, y))}{\Gamma(r_1)\Gamma(r_2)} dy ds \\
 v(t, x) &= \frac{1}{1+e^{t+x}} \\
 &+ \frac{xe^{-t-2}}{c_p(1+e^{-2p}|(H I_{\sigma}^r v)(t, x)| + e^{-p}|v(t, x)|)} \\
 &+ \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{y}\right)^{r_2-1} \\
 &\cdot \frac{g_2(t, x, s, y, u(s, y), v(s, y))}{\Gamma(r_1)\Gamma(r_2)} dy ds; \\
 &(t, x) \in [1, +\infty) \times [1, e],
 \end{aligned} \tag{39}$$

where $c_p = e^{-p} + e^{-2p} p^{r_1} / \Gamma(1+r_1)\Gamma(1+r_2)$; $p \in \mathbb{N} \setminus \{0, 1\}$, $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, and

$$\begin{aligned}
 g_1(t, x, s, y, u, v) &= \frac{xs^{-3/4} \sin \sqrt{t} \sin s}{(1+y^2+t^2)(1+|u|+|v|)}; \\
 &\text{if } (t, x, s, y) \in J', \text{ and } u \in \mathbb{R}, \\
 g_2(t, x, s, y, u, v) &= \frac{s^{-3/4} \ln \sqrt{t}}{(1+t^2)(1+|u|+|v|)}; \\
 &\text{if } (t, x, s, y) \in J', \text{ and } u \in \mathbb{R},
 \end{aligned} \tag{40}$$

and

$$J' = \{(t, x, s, y) : 1 \leq s \leq t \text{ and } 1 \leq x \leq y \leq e\}. \tag{41}$$

Set

$$\begin{aligned}
 \mu_1(t, x) &= \frac{xe^{3-2t}}{1+t+x^2}, \\
 \mu_2(t, x) &= \frac{1}{1+e^{t+x}}, \\
 f_1(t, x, u, v) &= \frac{xe^{-t-2}}{c_p(1+e^{-2p}|u|+e^{-p}|u|)}, \\
 f_2(t, x, u, v) &= \frac{xe^{-t-2}}{c_p(1+e^{-2p}|v|+e^{-p}|v|)}; \\
 &p \in \mathbb{N} \setminus \{0, 1\}.
 \end{aligned} \tag{42}$$

The functions f_i ; $i = 1, 2$ are continuous and satisfy assumption (H_1) , with

$$\begin{aligned}
 k_1(t, x) &= k_2(t, x) = \frac{xe^{-t-2-p}}{c_p}, \\
 l_1(t, x) &= l_2(t, x) = \frac{xe^{-t-2-2p}}{c_p}, \\
 k_p &= \frac{e^{-2-p}}{c_p}, \\
 l_p &= \frac{e^{-2-2p}}{c_p}.
 \end{aligned} \tag{43}$$

Also, the functions g_i , $i = 1, 2$ are continuous and satisfy assumption (H_2) , with

$$\begin{aligned}
 Q_{i0}(t, x, s, y) &= Q_{i1}(t, x, s, y) = \frac{xs^{-3/4} \sin \sqrt{t} \sin s}{1+y^2+t^2}; \\
 &(t, x, s, y) \in J'; \quad i = 1, 2.
 \end{aligned} \tag{44}$$

Then,

$$\begin{aligned}
 &\left| \int_1^t \left| \log \frac{t}{s} \right|^{r_1-1} \frac{Q_{01}(t, x, s, y)}{s} ds \right| \\
 &\leq \int_1^t \left| \log \frac{t}{s} \right|^{r_1-1} xs^{-1} |\sin \sqrt{t} \sin s| ds \\
 &\leq x \left| \sin \sqrt{t} \int_1^t \left| \log \frac{t}{s} \right|^{r_1-1} s^{-1} ds \right| \leq e \left| \frac{(\log t)^{r_1}}{\sqrt{t}} \right| \\
 &\rightarrow 0 \quad \text{as } t \rightarrow \infty.
 \end{aligned} \tag{45}$$

Finally, we shall show that condition (21) holds with $b = e$. Indeed, for each $p \in \mathbb{N} \setminus \{0, 1\}$, we get

$$\begin{aligned}
 L_p &= k_p^* + \frac{l_p^* (\log p)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
 &= \frac{1}{c_p} \left(e^{-2-p} + \frac{e^{-2-2p} p^{r_1}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right) = e^{-2} < 1.
 \end{aligned} \tag{46}$$

Hence by Theorem 7, system (39) has a solution defined on $[1, +\infty) \times [1, e]$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

- [1] S. Abbas, M. Benchohra, and G. M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, NY, USA, 2012.
- [2] S. Abbas, M. Benchohra, and G. M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, NY, USA, 2015.
- [3] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, vol. 3 of *Series on Complexity, Nonlinearity and Chaos*, World Scientific, 2012.
- [4] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, New York, NY, USA, Elsevier, 2006.
- [5] V. Lakshmikantham, S. Leela, and J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.
- [6] F. H. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley and Sons Inc., New York, NY, USA, 1993.
- [7] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, Switzerland, 1993.
- [8] B. Ahmad, A. Alsaedi, and M. Kirane, "Nonexistence results for the Cauchy problem of time fractional nonlinear systems of thermo-elasticity," *Mathematical Methods in the Applied Sciences*, vol. 40, no. 12, pp. 4272–4279, 2017.
- [9] J. Wang, M. Feckan, and Y. Zhou, "A survey on impulsive fractional differential equations," *Fractional Calculus and Applied Analysis*, vol. 19, no. 4, pp. 806–831, 2016.
- [10] J. Wang, M. Feckan, and Y. Zhou, "Center stable manifold for planar fractional damped equations," *Applied Mathematics and Computation*, vol. 296, pp. 257–269, 2017.
- [11] Y. Zhou, "Attractivity for fractional differential equations in Banach space," *Applied Mathematics Letters*, vol. 75, pp. 1–6, 2018.
- [12] Y. Zhou, "Attractivity for fractional evolution equations with almost sectorial operators," *Fractional Calculus and Applied Analysis*, vol. 21, no. 3, pp. 786–800, 2018.
- [13] Y. Zhou, B. Ahmad, and A. Alsaedi, "Existence of nonoscillatory solutions for fractional neutral differential equations," *Applied Mathematics Letters*, vol. 72, pp. 70–74, 2017.
- [14] Y. Zhou, L. Shangerganesh, J. Manimaran, and A. Debbouche, "A class of time-fractional reaction-diffusion equation with nonlocal boundary condition," *Mathematical Methods in the Applied Sciences*, vol. 41, no. 8, pp. 2987–2999, 2018.
- [15] Y. Zhou, L. Peng, B. Ahmad, and A. Alsaedi, "Topological properties of solution sets of fractional stochastic evolution inclusions," *Advances in Difference Equations*, p. 90, 2017.
- [16] Y. Zhou, V. Vijayakumar, and R. Murugesu, "Controllability for fractional evolution inclusions without compactness," *Evolution Equations and Control Theory*, vol. 4, no. 4, pp. 507–524, 2015.
- [17] Y. Zhou and L. Zhang, "Existence and multiplicity results of homoclinic solutions for fractional Hamiltonian systems," *Computers & Mathematics with Applications*, vol. 73, no. 6, pp. 1325–1345, 2017.
- [18] Y. Zhou, L. Zhang, and X. H. Shen, "Existence of mild solutions for fractional evolution equations," *Journal of Integral Equations and Applications*, vol. 25, no. 4, pp. 557–586, 2013.
- [19] P. L. Butzer, A. A. Kilbas, and J. J. Trujillo, "Fractional calculus in the Mellin setting and Hadamard-type fractional integrals," *Journal of Mathematical Analysis and Applications*, vol. 269, no. 1, pp. 1–27, 2002.
- [20] P. L. Butzer, A. A. Kilbas, and J. J. Trujillo, "Mellin transform analysis and integration by parts for Hadamard-type fractional integrals," *Journal of Mathematical Analysis and Applications*, vol. 270, no. 1, pp. 1–15, 2002.
- [21] S. Pooseh, R. Almeida, and D. F. Torres, "Expansion formulas in terms of integer-order derivatives for the Hadamard fractional integral and derivative," *Numerical Functional Analysis and Optimization*, vol. 33, no. 3, pp. 301–319, 2012.
- [22] S. Abbas, W. Albarakati, and M. Benchohra, "Successive approximations for functional evolution equations and inclusions," *Journal of Nonlinear Functional Analysis*, vol. 2017, no. 39, pp. 1–13, 2017.
- [23] S. Abbas and M. Benchohra, "Fractional order Riemann-Liouville integral equations with multiple time delays," *Applied Mathematics E-Notes*, vol. 12, pp. 79–87, 2012.
- [24] S. Abbas, M. Benchohra, J.-E. Lazreg, and Y. Zhou, "A survey on Hadamard and Hilfer fractional differential equations: analysis and stability," *Chaos, Solitons & Fractals*, vol. 102, pp. 47–71, 2017.
- [25] J. Banaś and T. Zając, "Solvability of a functional integral equation of fractional order in the class of functions having limits at infinity," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 11, pp. 5491–5500, 2009.
- [26] J. Banaś and T. Zając, "A new approach to the theory of functional integral equations of fractional order," *Journal of Mathematical Analysis and Applications*, vol. 375, no. 2, pp. 375–387, 2011.
- [27] B. G. Pachpatte, "On Volterra-Fredholm integral equation in two variables," *Demonstratio Mathematica*, vol. 40, no. 4, pp. 839–852, 2007.
- [28] B. G. Pachpatte, "On Fredholm type integral equation in two variables," *Differential Equations & Applications*, vol. 1, no. 1, pp. 27–39, 2009.
- [29] M. Frigon and A. Granas, "Théorèmes d'existence pour des inclusions différentielles sans convexité," *Comptes rendus de l'Académie des Sciences*, vol. 310, no. 12, pp. 819–822, 1990.
- [30] C. Avramescu, "Some remarks on a fixed point theorem of Krasnoselskii," *Electronic Journal of Qualitative Theory of Differential Equations*, No. 5, 15 pages, 2003.

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