

Research Article

Geometric Programming with Discrete Variables Subject to Max-Product Fuzzy Relation Constraints

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The problem of geometric programming subject to max-product fuzzy relation constraints with discrete variables is studied. The major difficulty in solving this problem comes from nonconvexity caused by these product terms in the general geometric function and the max-product relation constraints. We proposed a 0-1 mixed integer linear programming model and adopted the branch-and-bound scheme to solve the problem. Numerical experiments confirm that the proposed solution method is effective.

1. Introduction

Since fuzzy relation equations with max-min composition were firstly introduced by Sanchez [1–3], they have attracted much research attention. As an extension, fuzzy relation inequalities associated with max- t -norm were also studied. As demonstrated in [4], the complete solution set of max- t -norm fuzzy relation equations can be completely determined by a unique maximum solution and a finite number of minimal solutions. It is easy to compute the maximum solution, but finding all the minimal solutions is an NP-hard problem [4–6]. It is worth mentioning that Li and Fang [5] provided a complete survey and detailed discussion on fuzzy relational equations. They studied fuzzy relational equations in a general lattice-theoretic framework and introduced classification of basic fuzzy relational equations.

Meanwhile, optimization problems subject to fuzzy relation equations or inequalities were introduced and studied. Fang and Li [7] firstly investigated a linear optimization problem with a consistent system of max-min equations. They converted it into a 0-1 integer programming problem and solved this by the branch-and-bound method. Then it became a unified framework to deal with the resolution of linear optimization problem subject to a system of fuzzy

relational equations with max- t -norm composition. And linear programming problem with fuzzy relation constraints became a hot topic for further research [8–10]. Guo and Xia [11] proposed a method to accelerate the resolution of this problem. Zhang et al. [12] studied a linear objective optimization problem with max-min fuzzy relation inequalities.

For nonlinear programming with fuzzy relation constraints, many achievements were gained. Wang et al. [13] first studied latticized linear programming subject to max-min fuzzy relation inequalities. In [14] Li and Fang made a further study on the latticized linear optimization (LLO) problem and its variant, which are a special class of optimization problems constrained by fuzzy relational equations or inequalities. Yang and Cao [15] and Wu [16] considered geometric optimization problems with single-term exponents under fuzzy relation equation constraints with max-min composition, where the objective function is $Z = \bigvee_{i=1}^m (c_i \wedge x_i^{r_i})$. Zhou and Ahat [17] investigated similar problem where the composition was replaced by max-product. Yang et al. [18] investigated min-max programming problem subject to fuzzy relation inequality constraints with a special composition of addition-min. Yang et al. [19] studied the single-variable term semi-latticized geometric programming subject to max-product fuzzy relation equations. As important nonlinear

programming, geometric programming with fuzzy relational constraints attracted some researchers' attention. Yang and Cao [20] proposed a monomial geometric programming subject to max-min fuzzy relation equations with the objective function being $z = c \prod_{j=1}^n x_j^{r_j}$. Shivanian and Khorram [21] considered monomial geometric programming subject to fuzzy relation inequalities with max-product composition. Zhou et al. [22] investigated a special posynomial geometric programming problem subject to max-min fuzzy relation equations, in which the exponents of each variable are all nonpositive or nonnegative real numbers. Aliannezhadi et al. [23] investigated a monomial geometric programming objective function subject to bipolar max-product fuzzy relation constraints. All the objective functions of these optimization problems are special geometric functions. Because of the nonconvexity and complexity, problems with a general geometric objective function have not been studied. At the same time, the data in the real world could often be discrete instead of continuous. In particular, statistics data are often discrete numbers associated with real facts. Therefore, in this paper we study geometric programming subject to max-product fuzzy relational constraints, in which the objective function is a general geometric function and all the variables are of discrete values.

The rest of the paper is organized as follows. In Section 2, we introduce some basic concepts. A method to transform the original problem into a linear mix-integer programming model is proposed in Section 3. Numerical experiments are given to illustrate the effectiveness of the proposed solution method in Section 4. A simple conclusion is arranged in Section 5.

2. Problem

Consider the following geometric programming problem subject to max-product fuzzy relational constraints with discrete variables:

$$\begin{aligned} \min \quad & f(x) = \sum_{l \in L} c_l \prod_{j \in J} x_j^{r_{l,j}}, \\ \text{s.t.} \quad & \bigvee_{j \in J} a_{i,j} x_j \sim b_i, \quad \forall i \in I, \\ & x_j \in \{d_{j_1}, d_{j_2}, \dots, d_{j_h}\} \subset (0, 1] \quad \forall j \in J, \end{aligned} \quad (1)$$

where $I = \{1, 2, \dots, m\}$, $J = \{1, 2, \dots, n\}$, and $L = \{1, 2, \dots, p\}$ are three index sets, and $c_l, r_{l,j} \in \mathbf{R}$, $a_{i,j} \in (0, 1]$ for $i \in I$, $j \in J$, $l \in L$. The notation " \bigvee " denotes the maximum operator and " \sim " can be " \leq ", " \geq ", or " $=$ " for upper-bound constraints, lower-bound constraints, and equation constraints, respectively.

We analyze the three kinds of max-product fuzzy relation constraints with discrete variables first.

(i) For an upper-bound constraint:

$$\bigvee_{j \in J} a_{i,j} x_j \leq b_i. \quad (2)$$

It can be written in n constraints:

$$a_{i,j} x_j \leq b_i, \quad \text{i.e., } x_j \leq \frac{b_i}{a_{i,j}}, \quad j \in J. \quad (3)$$

Hence $x_j \leq \bigwedge_{i \in I} (b_i/a_{i,j})$, where the notation " \bigwedge " denotes the minimum operator. Combining with the discrete requirements:

$$x_j \in \{d_{j_1}, d_{j_2}, \dots, d_{j_h}\}, \quad j \in J, \quad (4)$$

we have

$$x_j \in \{d_{j_1}, d_{j_2}, \dots, d_{j_h}\} \cap \left[0, \bigwedge_{i \in I} \left(\frac{b_i}{a_{i,j}} \right) \right], \quad j \in J. \quad (5)$$

By checking $\{d_{j_1}, d_{j_2}, \dots, d_{j_h}\}$ and $\bigwedge_{i \in I} (b_i/a_{i,j})$, we can reset the discrete values of each variable.

(ii) For an equation constraint:

$$\bigvee_{j \in J} a_{i,j} x_j = b_i. \quad (6)$$

It can be written as an upper-bound constraint and a lower-bound constraint, that is,

$$\begin{aligned} \bigvee_{j \in J} a_{i,j} x_j &\leq b_i, \\ \bigvee_{j \in J} a_{i,j} x_j &\geq b_i. \end{aligned} \quad (7)$$

From (i) we know that the upper-bound constraint can be included in a new set of discrete values, and the lower-bound constraint can be grouped into other lower-bound constraints. Consequently, the constraints can all be transformed into max-product lower-bound constraints with discrete variables. Without loss of generality, in this paper we consider the following geometric programming problem subject to max-product fuzzy relational inequalities with discrete variables:

$$\begin{aligned} \min \quad & f(x) = \sum_{l \in L} c_l \prod_{j \in J} x_j^{r_{l,j}}, \\ \text{s.t.} \quad & \bigvee_{j \in J} a_{i,j} x_j \geq b_i \quad i \in I, \\ & x_j \in \{d_{j_1}, d_{j_2}, \dots, d_{j_h}\} \subset (0, 1] \quad j \in J, \end{aligned} \quad (8)$$

where $a_{i,j} \in (0, 1]$, $b_i \in [0, 1]$, $c_l, r_{l,j} \in \mathbf{R}$, for $i \in I$, $j \in J$, $l \in L$.

The major difficulty in solving problem (8) comes from the nonconvexity caused by the cross-product terms in the objective function and the max operation in the fuzzy inequalities. We will introduce methods to transform the original problem into an equivalent 0-1 linear mixed integer programming problem and adopt the branch-and-bound scheme to find an optimal solution.

3. Linear Reformulation

3.1. *Treating Geometric Function with Multivalued Discrete Variables.* For solving problem (8), the first step is to convert each multivalued variable into 0-1 valued by considering

$$x_j = \sum_{k \in K} d_{jk} u_{jk}, \quad (9)$$

$$\sum_{k \in K} u_{jk} = 1,$$

where $K = \{1, 2, \dots, h\}$ is an index set and $u_{jk} \in \{0, 1\}$ for $j \in J$ and $k \in K$.

It is important to know that the nh binary variables of $\{u_{jk}\}$, $j \in J$ and $k \in K$, induced by (9), may cause heavy computational burden when n and h become large. To avoid this burden, Li et al. [24] proposed a logarithmic approach which needs only $n \lceil \log_2 h \rceil$ binary variables ($\lceil \cdot \rceil$ denotes the ceiling function), nh nonnegative variables, and $n \lceil \log_2 h \rceil$ linear equations to represent the n discrete variables with h values by considering

$$\lambda_{jw} = \sum_{k \in K} t_{w,k} u_{jk}, \quad j \in J, w \in W, \quad (10)$$

where $W = \{1, 2, \dots, \lceil \log_2 h \rceil\}$ is an index set; $\lambda_{jw} \in \{0, 1\}$ is a binary variable; $u_{jk} \geq 0$ is now a nonnegative variable, for $j \in J$, $k \in K$ and $w \in W$; and $t_{w,k}$ are binary variables obtained by solving the equations of $1 + \sum_{w \in W} 2^{w-1} t_{w,k} = k$ for $k \in K$.

The next step is to reformulate the objective function. Remembering that $u_{jk} \in \{0, 1\}$ and $\sum_{k \in K} u_{jk} = 1$, we denote $z_{l,1} = x_1^{r_{l,1}}$ for $l \in L$; then we have

$$z_{l,1} = \sum_{k=1}^h u_{1k} d_{1k}^{r_{l,1}}, \quad l \in L. \quad (11)$$

Moreover, we denote

$$z_{l,j} = z_{l,j-1} x_j^{r_{l,j}}, \quad l \in L, j \in J \setminus \{1\}. \quad (12)$$

Then the objective function can be expressed as

$$f(x) = \sum_{l \in L} c_l \prod_{j \in J} x_j^{r_{l,j}} = \sum_{l \in L} c_l z_{l,n}. \quad (13)$$

In this way we can express the objective function of problem (8) as a linear function in $\{z_{l,n} \mid l \in L\}$. Note that the expression (12) is nonlinear, but by adopting the approach in [25], we can express them by using linear inequalities in $\{u_{jk} \in \{0, 1\} \mid k \in K, j \in J \setminus \{1\}\}$ in the next result.

Lemma 1. *Given $l \in L$, $k \in K$, $j \in J \setminus \{1\}$, the product term $z_{l,j} = z_{l,j-1} x_j^{r_{l,j}}$, with $x_j \in \{d_{j_1}, d_{j_2}, \dots, d_{j_h}\}$, can be expressed by the following linear inequalities in $\{u_{jk}\}$ and $\{z_{l,j}\}$:*

$$d_{jk}^{r_{l,j}} z_{l,j-1} - M_{l,j} (1 - u_{jk}) \leq z_{l,j} \leq d_{jk}^{r_{l,j}} z_{l,j-1} + M_{l,j} (1 - u_{jk}), \quad (14)$$

where u_{jk} is defined in (9), and $M_{l,j}$ is a sufficient large positive value such that

$$M_{l,1} = \bigvee_{k \in K} d_{1k}^{r_{l,1}}, \quad (15)$$

$$M_{l,j} = M_{l,j-1} \bigvee_{k \in K} d_{jk}^{r_{l,j}} \quad j \in J \setminus \{1\}, l \in L.$$

Proof. Assume that $x_j = d_{j_k^*}$ for a particular k^* . From (9), we have $u_{j_k^*} = 1$ and other $u_{jk} = 0$.

(i) For $k = k^*$, inequality (14) can be converted into the following equation:

$$z_{l,j} = z_{l,j-1} d_{j_k^*}^{r_{l,j}} = z_{l,j-1} x_j^{r_{l,j}}. \quad (16)$$

(ii) For $k \neq k^*$, inequality (14) can be transformed as

$$d_{jk}^{r_{l,j}} z_{l,j-1} - M_{l,j} \leq z_{l,j} \leq d_{jk}^{r_{l,j}} z_{l,j-1} + M_{l,j}. \quad (17)$$

Actually, from (i), we have

$$z_{l,j} = z_{l,j-1} x_j^{r_{l,j}} = \dots = z_{l,1} x_2^{r_{l,2}} \dots x_j^{r_{l,j}} = \prod_{s \leq j} x_s^{r_{l,s}}. \quad (18)$$

From (15), we know $M_{l,j} = \prod_{s \leq j} (\bigvee_{k \in K} d_{jk}^{r_{l,j}})$ and $x_j \in \{d_{j_1}, d_{j_2}, \dots, d_{j_h}\}$; therefore,

$$z_{l,j} \leq M_{l,j}. \quad (19)$$

Hence,

$$d_{jk}^{r_{l,j}} z_{l,j-1} - M_{l,j} \leq (z_{l,j-1} - M_{l,j-1}) d_{jk}^{r_{l,j}} \leq 0 \leq z_{l,j},$$

$$z_{l,j} \leq M_{l,j} \leq d_{jk}^{r_{l,j}} z_{l,j-1} + M_{l,j}, \quad (20)$$

$$d_{jk}^{r_{l,j}} z_{l,j-1} - M_{l,j} \leq z_{l,j} \leq d_{jk}^{r_{l,j}} z_{l,j-1} + M_{l,j}.$$

From the above, we can see, when $k = k^*$, (14) will be converted into (12); when $k \neq k^*$, (14) will be converted into the redundant constraints (17). Consequently (12) can be expressed by (14). \square

3.2. *Treating Max-Product Fuzzy Relation Inequalities.* Firstly we recall some basic concepts and important properties of the max-product fuzzy inequalities:

$$\bigvee_{j \in J} a_{i,j} x_j \geq b_i \quad i \in I. \quad (21)$$

Denote $A = (a_{i,j})_{m \times n} \in (0, 1)^{m \times n}$ and $x = (x_1, x_2, \dots, x_n)^T \in [0, 1]^n$, $b = (b_1, b_2, \dots, b_m)^T \in [0, 1]^m$; “ \circ ” represent the max-product composite operation; then a system of max-product fuzzy relation inequalities can be represented in the following matrix form:

$$A \circ x \geq b. \quad (22)$$

The set of all the solutions of system (22) is called its solution set, denoted by $X(A, b, \geq) = \{x \mid A \circ x \geq b\}$. If $X(A, b, \geq) \neq \emptyset$, we say the system (22) is consistent; otherwise it is inconsistent [4, 26].

Lemma 2 (see [5, 26, 27]). *Let $A \circ x \geq b$ be a system of consistent max-product fuzzy relation inequalities; if $x^1 \in X(A, b, \geq)$ and $x^1 \leq x^2 \in [0, 1]^n$, then we know $x^2 \in X(A, b, \geq)$.*

Definition 3 (see [26]). A solution $\hat{x} \in X(A, b, \geq)$ is said to be the maximum or greatest solution if and only if $x \leq \hat{x}$ for all $x \in X(A, b, \geq)$. A solution $\check{x} \in X(A, b, \geq)$ is said to be a minimal solution if and only if $x \leq \check{x}$ implies $x = \check{x}$ for any $x \in X(A, b, \geq)$.

Obviously, if the maximum solution of (22) exists, it is unique and $\hat{x} = (1, 1, \dots, 1)$.

Theorem 4 (see [26, 27]). *Let $A \circ x \geq b$ be a system of max-product fuzzy relation inequalities; then it is consistent if and only if $\hat{x} = (1, 1, \dots, 1) \in X(A, b, \geq)$. Moreover, if the system is consistent, the solution set $X(A, b, \geq)$ can be fully determined by one maximum solution and a finite number of minimal solutions, that is,*

$$X(A, b, \geq) = \bigcup_{\check{x} \in \check{X}(A, b, \geq)} \{x \in [0, 1]^n \mid \check{x} \leq x \leq \hat{x}\}, \quad (23)$$

where $\check{X}(A, b, \geq)$ is the set of all minimal solutions of (22).

The feasible domain restricted by the max-product fuzzy relation inequalities (22) can be fully characterised by the solution set $X(A, b, \geq)$. Actually, if we know all the minimal solutions, then the solution set $X(A, b, \geq)$ can be expressed. The main problem we face is computing all the minimal solutions of system (22). However this is an NP-hard problem [5, 26]. Hence we introduce a method to find all the potential minimal solutions instead of minimal solutions.

Definition 5 (see [26]). Matrix $Q = (q_{i,j})_{m \times n}$ is called a discrimination matrix of (22) with

$$q_{ij} = \begin{cases} \frac{b_i}{a_{ij}}, & a_{ij} \geq b_i, \\ a_{ij} & a_{ij} < b_i. \end{cases} \quad (24)$$

Theorem 6 (see [26]). *Let matrix Q be the discrimination matrix of the system (22). The system is consistent if and only if, for any $i \in I$, there exists at least one $j_i \in J$ such that $q_{i,j_i} \neq 0$.*

Definition 7 (see [27]). Let Q be the discrimination matrix of the system (22) and matrix $S = (s_{i,j})_{m \times n}$, where $s_{i,j} \in \{0, q_{i,j}\}$. We call S a solution matrix if, for any $i \in I$, there exists a unique $j_i \in J$ such that $s_{i,j_i} \neq 0$.

Let S be a solution matrix of the system (22); denote $x^S = (x_1^S, x_2^S, \dots, x_n^S)$, where

$$x_j^S = \bigvee_{i \in I} s_{i,j}. \quad (25)$$

Theorem 8 (see [27]). *Let S be a solution matrix of the system (22) and x^S be defined by (25); then $x^S \in X(A, b, \geq)$. Moreover, for any $x \in X(A, b, \geq)$, there exists a solution matrix S with a corresponding x^S such that $x^S \leq x$.*

From Theorem 8, we know that, for any minimal solution \check{x} of (22), the solution matrix S and its corresponding x^S can be formed such that $x^S = \check{x}$. Then the feasible domain restricted by the max-product fuzzy relation inequalities constraints (22) can be expressed as

$$\begin{aligned} x \in X(A, b, \geq) &= \bigcup_{\check{x} \in \check{X}(A, b, \geq)} \{x \mid \check{x} \leq x \leq \hat{x}\} \\ &= \bigcup_{S \in S(Q)} \{x \mid x^S \leq x \leq \hat{x}\}, \end{aligned} \quad (26)$$

where $S(Q)$ denotes the set of all the solution matrix determined by discrimination matrix Q of system (22), and x^S is a solution corresponding to the solution matrix S .

Consider the definition of (25); we see that, for any solution matrix S and its corresponding x^S , the inequality $x \geq x^S$ is equivalent to the following constraints:

$$x_j \geq \bigvee_{i \in I} s_{i,j}, \quad j \in J, \text{ i.e., } x_j \geq s_{i,j}, \quad i \in I, \quad j \in J. \quad (27)$$

At the same time, we bring in mn binary variables $v_{i,j} \in \{0, 1\}$, for $i \in I$ and $j \in J$; the element of matrix S can be expressed as

$$\begin{aligned} s_{ij} &= v_{i,j} q_{i,j}, \quad i \in I, \quad j \in J, \\ \sum_{j \in J} p_{i,j} v_{i,j} &= 1, \quad i \in I, \quad j \in J, \end{aligned} \quad (28)$$

where $v_{i,j} \in \{0, 1\}$ and $p_{i,j} = \{1, q_{ij} \neq 0; 0, q_{ij} = 0\}$.

Therefore, the feasible domain restricted by (22) can be characterised by

$$\begin{aligned} \sum_{j \in J} p_{i,j} v_{i,j} &= 1, \quad i \in I, \\ x_j &\geq q_{i,j} v_{i,j}, \quad i \in I, \quad j \in J, \end{aligned} \quad (29)$$

where $v_{i,j} \in \{0, 1\}$ and $p_{i,j} = \{1, q_{ij} \neq 0; 0, q_{ij} = 0\}$.

Adopting the method introduced above, we have a linear reformulation of the geometric objective function and max-product fuzzy relation inequality constraints. The original problem (8) can then be converted into the following 0-1 linear mixed integer programming model:

$$\begin{aligned} \min \quad & f = \sum_{l \in L} c_l z_{l,n} \\ \text{s.t.} \quad & (9), (10), (11), (14), (29), (30) \\ & x_j, z_{l,j}, u_{j,k} \geq 0, \quad \lambda_{j,w}, v_{i,j} \in \{0, 1\} \\ & i \in I, \quad j \in J, \quad k \in K, \quad w \in W. \end{aligned} \quad (\text{LMIP})$$

Note that, in this model, there are $n \lceil \log_2 h \rceil$ binary variables of $\lambda_{j,w}$; mn binary variables of $v_{i,j}$; n nonnegative variables of x_j ; np nonnegative variables of $z_{l,j}$; nh nonnegative variables of $u_{j,k}$; $2n + n \lceil \log_2 h \rceil + p + m$ linear equations (i.e., (9) + (10) + (11) + (29)); and $2(n-1)ph + mn$ linear inequality constraints (i.e., (14) + (30)) involved.

Now we illustrate this model by a simple example.

Example 9. Consider the following geometric programming problem in discrete variables with max-product fuzzy relation inequality constraints:

$$\begin{aligned} \min \quad & f(x) = x_1^2 x_2 x_3 - x_1 x_2 x_3^{-1}, \\ \text{s.t.} \quad & A \circ x = \begin{pmatrix} 0.8 & 0.9 & 0.3 \\ 0.9 & 0.6 & 0.5 \\ 0.4 & 0.1 & 0.9 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq b \\ & = \begin{pmatrix} 0.72 \\ 0.27 \\ 0.36 \end{pmatrix}, \end{aligned} \tag{31}$$

$$x_1, x_2, x_3 \in \{0.3, 0.5, 0.7, 0.9\}.$$

In this case, $h = 4$, $\lceil \log_2 h \rceil = 2$, adopting the logarithmic approach (10) and solving the equations of $1 + t_{1,k} + 2t_{2,k} = k$ for $k = 1, 2, 3, 4$; we have

$$(t_{w,k})_{2 \times 4} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{32}$$

Using (9) and (10), we have

$$\begin{aligned} 0.3u_{j_1} + 0.5u_{j_2} + 0.7u_{j_3} + 0.9u_{j_4} &= x_j, \quad j = 1, 2, 3, \\ u_{j_1} + u_{j_2} + u_{j_3} + u_{j_4} &= 1, \quad j = 1, 2, 3, \\ u_{j_2} + u_{j_4} &= \lambda_{j_1}, \\ u_{j_3} + u_{j_4} &= \lambda_{j_2}, \\ & j = 1, 2, 3. \end{aligned} \tag{33}$$

By (24) and (30), we have

$$\begin{aligned} Q &= \begin{pmatrix} 0.9 & 0.8 & 0 \\ 0.3 & 0.45 & 0.54 \\ 0.9 & 0 & 0.4 \end{pmatrix}, \\ (P_{i,j})_{3 \times 4} &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{34}$$

Then by applying (29) and (30), we have

$$\begin{aligned} v_{1,1} + v_{1,2} &= 1, \\ v_{2,1} + v_{2,2} + v_{2,3} &= 1, \\ v_{3,1} + v_{3,3} &= 1, \end{aligned}$$

$$\begin{aligned} x_1 &\geq 0.9v_{1,1}, \\ x_1 &\geq 0.3v_{2,1}, \\ x_1 &\geq 0.9v_{3,1}, \\ x_2 &\geq 0.8v_{1,2}, \\ x_2 &\geq 0.45v_{2,2}, \\ x_3 &\geq 0.54v_{2,3}, \\ x_3 &\geq 0.4v_{3,3}. \end{aligned} \tag{35}$$

Finally, adding (11) and (14), problem (31) can be converted into the following model:

$$\begin{aligned} \min \quad & f = z_{1,3} + z_{2,3}, \\ \text{s.t.} \quad & 0.3u_{j_1} + 0.5u_{j_2} + 0.7u_{j_3} + 0.9u_{j_4} = x_j, \\ & j = 1, 2, 3, \\ & u_{j_1} + u_{j_2} + u_{j_3} + u_{j_4} = 1, \quad j = 1, 2, 3, \\ & u_{j_2} + u_{j_4} = \lambda_{j_1}, \\ & u_{j_3} + u_{j_4} = \lambda_{j_2}, \\ & j = 1, 2, 3, \end{aligned}$$

$$\begin{aligned} z_{1,1} &= 0.09u_{1_1} + 0.25u_{1_2} + 0.49u_{1_3} + 0.81u_{1_4}, \\ z_{2,1} &= 0.3u_{1_1} + 0.5u_{1_2} + 0.7u_{1_3} + 0.9u_{1_4}, \\ 0.3z_{1,1} - 0.729(1 - u_{2_1}) &\leq z_{1,2} \\ &\leq 0.3z_{1,1} + 0.729(1 - u_{2_1}), \\ 0.5z_{1,1} - 0.729(1 - u_{2_2}) &\leq z_{1,2} \\ &\leq 0.5z_{1,1} + 0.729(1 - u_{2_2}), \\ 0.7z_{1,1} - 0.729(1 - u_{2_3}) &\leq z_{1,2} \\ &\leq 0.7z_{1,1} + 0.729(1 - u_{2_3}), \\ 0.9z_{1,1} - 0.729(1 - u_{2_4}) &\leq z_{1,2} \\ &\leq 0.9z_{1,1} + 0.729(1 - u_{2_4}), \\ 0.3z_{1,2} - 0.6561(1 - u_{3_1}) &\leq z_{1,3} \\ &\leq 0.3z_{1,2} + 0.6561(1 - u_{3_1}), \\ 0.5z_{1,2} - 0.6561(1 - u_{3_2}) &\leq z_{1,3} \\ &\leq 0.5z_{1,2} + 0.6561(1 - u_{3_2}), \\ 0.7z_{1,2} - 0.6561(1 - u_{3_3}) &\leq z_{1,3} \\ &\leq 0.7z_{1,2} + 0.6561(1 - u_{3_3}), \end{aligned}$$

$$\begin{aligned}
0.9z_{1,2} - 0.6561(1 - u_{3_4}) &\leq z_{1,3} \\
&\leq 0.9z_{1,2} + 0.6561(1 - u_{3_4}), \\
0.3z_{2,1} - 0.81(1 - u_{2_1}) &\leq z_{2,2} \\
&\leq 0.3z_{2,1} + 0.81(1 - u_{2_1}), \\
0.5z_{2,1} - 0.81(1 - u_{2_2}) &\leq z_{2,2} \\
&\leq 0.5z_{2,1} + 0.81(1 - u_{2_2}), \\
0.7z_{2,1} - 0.81(1 - u_{2_3}) &\leq z_{2,2} \\
&\leq 0.7z_{2,1} + 0.81(1 - u_{2_3}), \\
0.9z_{2,1} - 0.81(1 - u_{2_4}) &\leq z_{2,2} \\
&\leq 0.9z_{2,1} + 0.81(1 - u_{2_4}), \\
0.3z_{2,2} - 2.7(1 - u_{3_1}) &\leq z_{2,3} \\
&\leq 0.3z_{2,2} + 2.7(1 - u_{3_1}), \\
0.5z_{2,2} - 2.7(1 - u_{3_2}) &\leq z_{2,3} \\
&\leq 0.5z_{2,2} + 2.7(1 - u_{3_2}), \\
0.7z_{2,2} - 2.7(1 - u_{3_3}) &\leq z_{2,3} \\
&\leq 0.7z_{2,2} + 2.7(1 - u_{3_3}), \\
0.9z_{2,2} - 2.7(1 - u_{3_4}) &\leq z_{2,3} \\
&\leq 0.9z_{2,2} + 2.7(1 - u_{3_4}), \\
v_{1,1} + v_{1,2} &= 1, \\
v_{2,1} + v_{2,2} + v_{2,3} &= 1, \\
v_{3,1} + v_{3,3} &= 1, \\
x_1 &\geq 0.9v_{1,1}, \\
x_1 &\geq 0.3v_{2,1}, \\
x_1 &\geq 0.9v_{3,1}, \\
x_2 &\geq 0.8v_{1,2}, \\
x_2 &\geq 0.45v_{2,2}, \\
x_3 &\geq 0.54v_{2,3}, \\
x_3 &\geq 0.4v_{3,3}, \\
\lambda_{j_w}, v_{i,j} &\in \{0, 1\}, u_{j_k} \geq 0, \\
i = 1, 2, 3, w = 1, 2, j = 1, 2, 3, k = 1, 2, 3, 4.
\end{aligned}
\tag{36}$$

Solving problem (36) leads to a solution $(x_1^*, x_2^*, x_3^*) = (0.9, 0.9, 0.3)$ with $u_{1_4}^* = u_{2_2}^* = u_{3_1}^* = 1$, $\lambda_{1_1}^* = \lambda_{1_2}^* = \lambda_{2_1}^* = \lambda_{2_2}^* = 1$, $z_{1,1}^* = 0.81$, $z_{1,2}^* = 0.729$, $z_{1,3}^* = 0.2178$, $z_{2,1}^* = 0.9$, $z_{2,2}^* = 0.81$, $z_{2,3}^* = 2.7$, $v_{1,1}^* = v_{2,2}^* = v_{3,1}^* = 1$ and 0 for other variables. The corresponding optimal objective value is -2.4813 .

4. Computational Experiments

To illustrate the effectiveness of the proposed method, we conduct some computational experiments. All the experiments have been run on a PC equipped with the Intel Core I7-6700HQ CPU, 8 GB RAM, and Windows 10 (64 bit) operating system. GUROBI (2016) is the chosen MIP solver for solving all the instances. To ensure the existence of an optimal solution in each case, we apply the following rules to randomly generate its coefficients:

(1) $a_{i,j}$ are generated randomly by the uniform distribution over $(0, 1]$, and c_j are taken randomly from $[-5, 5]$ for $i \in I$ and $j \in$.

(2) d_{j_k} are generated randomly by the uniform distribution over $(0, 1]$ such that $d_{j_1} < d_{j_2} < \dots < d_{j_n}$ for $j \in J$ and $k \in K$.

(3) When $r_{i,j} \leq 0$, the elements $d_{j_k}^{r_{i,j}}$ and M_{l_n} may go to infinite; in order to avoid this, we generate $r_{l,j}$ randomly in the interval $[-3, 3]$ such that $\sum_{j \in J} r_{l,j} \in [-3, 3]$ for $l \in L$ and $j \in J$.

(4) To make sure the test instance is feasible, each b_i is generated randomly by the uniform distribution over the interval of $[(\bigvee_{j \in J} a_{i,j} d_{j_n}) \times 0.1, (\bigvee_{j \in J} a_{i,j} d_{j_n}) \times 0.7]$ for $i \in I$. Our computational experiments consist of 7 cases with $\{m = 10, n = 8, p = 6\}$, $\{m = 12, n = 10, p = 6\}$ and $h \in \{10, 20, 30, 40\}$. Each case includes 20 test instances which are executed with the coefficients generated by the rules mentioned above.

The computational results are shown in Table 1. In the Table, V-Bin denotes the number of 0-1 variables, V-Con denotes the number of continuous variables, L-Eq denotes the number of linear equation constraints, L-Ine denotes the number of linear inequality constraints, $T_{ave}(s)$ denotes the average CPU time in seconds of solving 20 randomly generated instances of each case, and $T_{max}(s)$ means the longest CPU time in seconds we meet in each case.

From Table 1, we can see that there are hundreds of binary variables and thousands of linear inequality constraints in each case. All the problem can be solved in 4 minutes in average. Even in the worst condition, we can also find the optimal solution in 1 hour. Therefore, our computational experiments support the efficiency of the proposed algorithm.

Because of the nonconvexity of the geometric function and fuzzy relation equations (inequalities), the optimization problem of geometric programming with fuzzy relational constraints is difficult to solve. To make problem easier to solve, researchers chose special geometric functions as objective functions, such as monomials in [20, 21, 23] and posynomial function with special indexes in [17]. In our problem, the objective function is a general geometric function. At the same time, we add the restriction of discrete values for

TABLE 1

m	n	p	h	V-Bin	V-Cont	L-Eq	L-Ine	T_{ave} (s)	T_{max} (s)
10	8	6	10	112	136	64	920	4.56	24.92
10	8	6	20	120	216	72	1760	25.67	210.85
10	8	6	30	120	296	72	2600	61.31	588.89
10	8	6	40	128	376	80	3440	238.77	2172.66
12	10	6	10	160	170	78	1200	30.30	141.03
12	10	6	20	170	270	88	2280	66.82	592.63
12	10	6	30	170	370	88	3360	202.51	1587.88

each variable. We transformed the problem into a linear mixed-integer programming model and adopt the branch-bound method to develop an effective solution method.

5. Conclusion

In this paper, a geometric optimization problem subject to max-product fuzzy relation constraints with discrete variables is studied. For solving the problem, we replace n discrete variables in h values with binary variables by the logarithmic method. Then we transform the objective function into a linear function of p continuous variables subject to p equations and $2(n - 1)ph$ inequalities. By checking all the potential minimal solutions, we further transform the max-product fuzzy relation inequality constraints into linear equality and inequality constraints with mn 0-1 binary variables. Finally, we built a 0-1 mixed integer linear programming model for solving the original problem. Numerical experiments support the efficiency of the proposed method. To the best of our knowledge, this is the first fuzzy optimization model including discrete variables.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

- [1] E. Sanchez, "Equations de Relations Flous," in *These Biologie Humaine*, France, 1972.
- [2] E. Sanchez, "Resolution of composite fuzzy relation equations," *Information and Control*, vol. 30, no. 1, pp. 38–48, 1976.
- [3] E. Sanchez, "Solution in composite fuzzy relation equations: Application to medical diagnosis in Brouwerian logic," in *Fuzzy Automata and Decision Processes*, M. M. Gupta, G. N. Saridis, and B. R. Gaines, Eds., pp. 221–234, Amsterdam, Netherlands, 1977.
- [4] A. Di Nola, S. Sessa, W. Pedrycz, and E. Sanchez, *Fuzzy relation equations and their applications to knowledge engineering*, vol. 3 of *Theory and Decision Library. Series D: System Theory, Knowledge Engineering and Problem Solving*, Kluwer Academic Publishers Group, Dordrecht, 1989.
- [5] P. Li and S.-C. Fang, "A survey on fuzzy relational equations. I. Classification and solvability," *Fuzzy Optimization and Decision Making. A Journal of Modeling and Computation Under Uncertainty*, vol. 8, no. 2, pp. 179–229, 2009.
- [6] P. Li and S.-C. Fang, "On the resolution and optimization of a system of fuzzy relational equations with sup- T composition," *Fuzzy Optimization and Decision Making. A Journal of Modeling and Computation Under Uncertainty*, vol. 7, no. 2, pp. 169–214, 2008.
- [7] S.-C. Fang and G. Li, "Solving fuzzy relation equations with a linear objective function," *Fuzzy Sets and Systems*, vol. 103, no. 1, pp. 107–113, 1999.
- [8] S.-M. Guu and Y.-K. Wu, "Minimizing a linear objective function with fuzzy relation equation constraints," *Fuzzy Optimization and Decision Making. A Journal of Modeling and Computation Under Uncertainty*, vol. 1, no. 4, pp. 347–360, 2002.
- [9] S.-M. Guu and Y.-K. Wu, "Minimizing a linear objective function under a max-t-norm fuzzy relational equation constraint," *Fuzzy Sets and Systems*, vol. 161, no. 2, pp. 285–297, 2010.
- [10] Y.-K. Wu, S.-M. Guu, and J. Y.-C. Liu, "An accelerated approach for solving fuzzy relation equations with a linear objective function," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 4, pp. 552–558, 2002.
- [11] F.-F. Guo and Z.-Q. Xia, "An algorithm for solving optimization problems with one linear objective function and finitely many constraints of fuzzy relation inequalities," *Fuzzy Optimization and Decision Making. A Journal of Modeling and Computation Under Uncertainty*, vol. 5, no. 1, pp. 33–47, 2006.
- [12] H. T. Zhang, X. M. Dong, and Y. H. Ren, "A programming problem with relational inequality constraints," *Journal of Liaoning Normal University. Natural Science Edition. Liaoning Shifan Daxue Xuebao. Ziran Kexue Ban*, vol. 26, no. 3, pp. 231–233, 2003.
- [13] P. Z. Wang, D. Z. Zhang, E. Sanchez, and E. S. Lee, "Latticized linear programming and fuzzy relation inequalities," *Journal of Mathematical Analysis and Applications*, vol. 159, no. 1, pp. 72–87, 1991.
- [14] P. Li and S.-C. Fang, "Latticized linear optimization on the unit interval," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 6, pp. 1353–1365, 2009.
- [15] J. H. Yang and B. Y. Cao, "Geometric programming with fuzzy relation equation constraints," in *Proceedings of the IEEE International Conference on Fuzzy Systems*, vol. 2005, pp. 557–560.
- [16] Y.-K. Wu, "Optimizing the geometric programming problem with single-term exponents subject to max-min fuzzy relational

- equation constraints," *Mathematical and Computer Modelling*, vol. 47, no. 3-4, pp. 352-362, 2008.
- [17] X. Zhou and R. Ahat, "Geometric programming problem with single-term exponents subject to max-product fuzzy relational equations," *Mathematical and Computer Modelling*, vol. 53, no. 1-2, pp. 55-62, 2011.
- [18] X.-P. Yang, X.-G. Zhou, and B.-Y. Cao, "Min-max programming problem subject to addition-min fuzzy relation inequalities," *IEEE Transactions on Fuzzy Systems*, vol. 24, no. 1, pp. 111-119, 2016.
- [19] X.-P. Yang, X.-G. Zhou, and B.-Y. Cao, "Single-variable term semi-latticized fuzzy relation geometric programming with max-product operator," *Information Sciences*, vol. 325, pp. 271-287, 2015.
- [20] J. Yang and B. Cao, "Monomial geometric programming with fuzzy relation equation constraints," *Fuzzy Optimization and Decision Making. A Journal of Modeling and Computation Under Uncertainty*, vol. 6, no. 4, pp. 337-349, 2007.
- [21] E. Shivanian and E. Khorram, "Monomial geometric programming with fuzzy relation inequality constraints with max-product composition," *Computers & Industrial Engineering*, vol. 56, no. 4, pp. 1386-1392, 2009.
- [22] X.-G. Zhou, X.-P. Yang, and B.-Y. Cao, "Posynomial geometric programming problem subject to max-min fuzzy relation equations," *Information Sciences*, vol. 328, pp. 15-25, 2016.
- [23] S. Aliannezhadi, S. S. Ardalan, and A. A. Molai, "Maximizing a monomial geometric objective function subject to bipolar max-product fuzzy relation constraints," *Journal of Intelligent & Fuzzy Systems: Applications in Engineering and Technology*, vol. 32, no. 1, pp. 337-350, 2017.
- [24] H.-L. Li, Y.-H. Huang, and S.-C. Fang, "A logarithmic method for reducing binary variables and inequality constraints in solving task assignment problems," *INFORMS Journal on Computing*, vol. 25, no. 4, pp. 643-653, 2013.
- [25] H.-L. Li and H.-C. Lu, "Global optimization for generalized geometric programs with mixed free-sign variables," *Operations Research*, vol. 57, no. 3, pp. 701-713, 2009.
- [26] B.-Y. Cao, *Optimal Models and Methods with Fuzzy Quantities*, Springer-Verlag, Berlin, Germany, 2010.
- [27] X.-P. Yang, X.-G. Zhou, and B.-Y. Cao, "Latticized linear programming subject to max-product fuzzy relation inequalities with application in wireless communication," *Information Sciences*, vol. 358-359, pp. 44-55, 2016.

