Research Article

On the Solutions of a System of Third-Order Rational Difference Equations

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The structure of the solutions for the system nonlinear difference equations $x_{n+1} = y_n y_{n-2} / (x_{n-1} + y_{n-2})$, $y_{n+1} = x_n x_{n-2} / (\pm y_{n-1} \pm x_{n-2})$, $n = 0, 1, \dots$, is clarified in which the initial conditions x_{-2} , x_{-1} , x_0 , y_{-2} , y_{-1} , y_0 are considered as arbitrary positive real numbers. To exemplify the theoretical discussion, some numerical examples are presented.

1. Introduction

The prime objective of this paper is to analyze the forms of the solution of the following difference equation systems:

$$\begin{aligned} x_{n+1} &= \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}}, \\ y_{n+1} &= \frac{x_n x_{n-2}}{\pm y_{n-1} \pm x_{n-2}}, \end{aligned} \quad (1)$$

$$n = 0, 1, \dots,$$

in which the initial conditions x_{-2} , x_{-1} , x_0 , y_{-2} , y_{-1} , y_0 are arbitrary positive real numbers.

Difference equations appear as a natural model of evolution phenomena. Furthermore, considerable findings in difference equation theory have been retrieved as discrete analogues and as numerical solutions of differential equations. This is notably true in the case of Lyapunov theory of stability. What is more, it has applications in biology, ecology, economy, physics, and so on. Due to this matter, there has been a rise in the interest in the study of qualitative analysis of scalar rational difference equations and rational system of difference equations. Although difference equations look simple in form, it is quite difficult to understand thoroughly the behaviors of their solutions because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equation come from the results of

rational difference equations (see [1–6]) and the references cited therein.

El-Dessoky and Elsayed [7] have analyzed the form of the solutions and the periodicity character of the following systems of rational difference equations:

$$\begin{aligned} x_{n+1} &= \frac{x_n y_{n-1}}{y_{n-1} \pm y_n}, \\ y_{n+1} &= \frac{y_n x_{n-1}}{x_{n-1} \pm x_n}. \end{aligned} \quad (2)$$

Elsayed and Alghamdi [8] have examined the form of the solutions of the following nonlinear difference equations systems:

$$\begin{aligned} x_{n+1} &= \frac{x_{n-7}}{1 + x_{n-7} y_{n-3}}, \\ y_{n+1} &= \frac{y_{n-7}}{\pm 1 \pm x_{n-3} y_{n-7}}. \end{aligned} \quad (3)$$

Elsayed et al. [9] have investigated the form of the solutions of the following difference equations systems:

$$\begin{aligned} x_{n+1} &= \frac{y_n x_{n-2}}{y_n + y_{n-3}}, \\ y_{n+1} &= \frac{x_n y_{n-2}}{\pm x_n \pm x_{n-3}}. \end{aligned} \quad (4)$$

Kurbanli [10] has explored the following system of difference equations:

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}}{y_n x_{n-1} + 1}, \\ y_{n+1} &= \frac{y_{n-1}}{x_n y_{n-1} + 1}. \end{aligned} \quad (5)$$

Mansour et al. [11] have got the form of the solutions of some systems of the following rational difference equations:

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}}{\alpha - x_{n-1} y_n}, \\ y_{n+1} &= \frac{y_{n-1}}{\beta + \gamma y_{n-1} x_n}. \end{aligned} \quad (6)$$

Touafek and Elsayed [12] have studied the periodicity and gave the form of the solutions of the following systems of difference equations of order two:

$$\begin{aligned} x_{n+1} &= \frac{y_n x_{n-1}}{\pm x_{n-1} \pm y_n}, \\ y_{n+1} &= \frac{x_n y_{n-1}}{\pm x_n \pm x_{n-1}}. \end{aligned} \quad (7)$$

Equations similar to difference equations and nonlinear systems of rational difference equations were investigated; see [13–20].

2. Main Results

2.1. The First System: $x_{n+1} = y_n y_{n-2} / (x_{n-1} + y_{n-2})$, $y_{n+1} = x_n x_{n-2} / (y_{n-1} + x_{n-2})$. The solutions of the system of two difference equations are analyzed in this subsection.

$$\begin{aligned} x_{n+1} &= \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}}, \\ y_{n+1} &= \frac{x_n x_{n-2}}{y_{n-1} + x_{n-2}}, \end{aligned} \quad (8)$$

$$n = 0, 1, \dots,$$

where the initial conditions $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are arbitrary positive real numbers.

Theorem 1. Suppose that $\{x_n, y_n\}$ are solutions of system (8). Then for $n = 0, 1, 2, \dots$, one has

$$\begin{aligned} x_{2n-1} &= \frac{bdf}{(f_{n-2}d + f_{n-1}b)(f_{n-1}b + f_n f)}, \\ x_{2n} &= \frac{ace}{(f_{n-1}a + f_n e)(f_{n-1}e + f_n c)}, \\ y_{2n-1} &= \frac{ace}{(f_{n-2}a + f_{n-1}e)(f_{n-1}e + f_n c)}, \\ y_{2n} &= \frac{bdf}{(f_{n-1}b + f_n f)(f_{n-1}d + f_n b)}, \end{aligned} \quad (9)$$

where $x_0 = a, x_{-1} = b, x_{-2} = c, y_0 = d, y_{-1} = e, y_{-2} = f$ and where f_n is the Fibonacci Sequence with $f_{-2} = 1, f_{-1} = 0$.

Proof. For $n = 0$, the result holds. By using mathematical induction now suppose that $n > 0$ and that our assumption holds for $-1, n-2$. That is,

$$\begin{aligned} x_{2n-4} &= \frac{ace}{(f_{n-3}a + f_{n-2}e)(f_{n-3}e + f_{n-2}c)}, \\ x_{2n-3} &= \frac{bdf}{(f_{n-3}d + f_{n-2}b)(f_{n-2}b + f_{n-1}f)}, \\ x_{2n-2} &= \frac{ace}{(f_{n-2}a + f_{n-1}e)(f_{n-2}e + f_{n-1}c)}, \\ y_{2n-4} &= \frac{bdf}{(f_{n-3}b + f_{n-2}f)(f_{n-3}d + f_{n-2}b)}, \\ y_{2n-3} &= \frac{ace}{(f_{n-3}a + f_{n-2}e)(f_{n-2}e + f_{n-1}c)}, \\ y_{2n-2} &= \frac{bdf}{(f_{n-2}b + f_{n-1}f)(f_{n-2}d + f_{n-1}b)}. \end{aligned} \quad (10)$$

Now, it follows from system (8) substitution of the above equations that

$$\begin{aligned} x_{2n-1} &= \frac{y_{2n-2} y_{2n-4}}{x_{2n-3} + y_{2n-4}} = \frac{(bdf / (f_{n-2}b + f_{n-1}f)(f_{n-2}d + f_{n-1}b))(bdf / (f_{n-3}b + f_{n-2}f)(f_{n-3}d + f_{n-2}b))}{bdf / (f_{n-3}d + f_{n-2}b)(f_{n-2}b + f_{n-1}f) + bdf / (f_{n-3}b + f_{n-2}f)(f_{n-3}d + f_{n-2}b)} \\ &= \frac{(bdf / (f_{n-2}b + f_{n-1}f)(f_{n-2}d + f_{n-1}b))(1 / (f_{n-3}b + f_{n-2}f))}{1 / (f_{n-2}b + f_{n-1}f) + 1 / (f_{n-3}b + f_{n-2}f)} \\ &= \frac{bdf / (f_{n-2}d + f_{n-1}b)}{(f_{n-2}b + f_{n-1}f)(f_{n-3}b + f_{n-2}f)[1 / (f_{n-2}b + f_{n-1}f) + 1 / (f_{n-3}b + f_{n-2}f)]} = \frac{bdf / (f_{n-2}d + f_{n-1}b)}{[f_{n-3}b + f_{n-2}f + f_{n-2}b + f_{n-1}f]} \\ &= \frac{bdf / (f_{n-2}d + f_{n-1}b)}{[f_{n-1}b + f_n f]} = \frac{bdf}{(f_{n-2}d + f_{n-1}b)(f_{n-1}b + f_n f)}, \\ y_{2n-1} &= \frac{x_{2n-2} x_{2n-4}}{y_{2n-3} + x_{2n-4}} = \frac{(ace / (f_{n-2}a + f_{n-1}e)(f_{n-2}e + f_{n-1}c))(ace / (f_{n-3}a + f_{n-2}e)(f_{n-3}e + f_{n-2}c))}{ace / (f_{n-3}a + f_{n-2}e)(f_{n-2}e + f_{n-1}c) + ace / (f_{n-3}a + f_{n-2}e)(f_{n-3}e + f_{n-2}c)} \end{aligned}$$

$$\begin{aligned}
&= \frac{(ace/(f_{n-2}a + f_{n-1}e)(f_{n-2}e + f_{n-1}c))(1/(f_{n-3}e + f_{n-2}c))}{1/(f_{n-2}e + f_{n-1}c) + 1/(f_{n-3}e + f_{n-2}c)} \\
&= \frac{ace/(f_{n-2}a + f_{n-1}e)}{(f_{n-2}e + f_{n-1}c)(f_{n-3}e + f_{n-2}c)[1/(f_{n-2}e + f_{n-1}c) + 1/(f_{n-3}e + f_{n-2}c)]} = \frac{ace/(f_{n-2}a + f_{n-1}e)}{[f_{n-3}e + f_{n-2}c + f_{n-2}e + f_{n-1}c]} \\
&= \frac{ace/(f_{n-2}a + f_{n-1}e)}{[f_{n-1}e + f_{n-1}c]} = \frac{ace}{(f_{n-2}a + f_{n-1}e)(f_{n-1}e + f_{n-1}c)}.
\end{aligned} \tag{11}$$

Also, we obtain

$$\begin{aligned}
x_{2n} &= \frac{y_{2n-1}y_{2n-3}}{x_{2n-2} + y_{2n-3}} = \frac{(ace/(f_{n-2}a + f_{n-1}e)(f_{n-1}e + f_{n-1}c))(ace/(f_{n-3}a + f_{n-2}e)(f_{n-2}e + f_{n-1}c))}{ace/(f_{n-2}a + f_{n-1}e)(f_{n-2}e + f_{n-1}c) + ace/(f_{n-3}a + f_{n-2}e)(f_{n-2}e + f_{n-1}c)} \\
&= \frac{(ace/(f_{n-2}a + f_{n-1}e)(f_{n-1}e + f_{n-1}c))(1/(f_{n-3}a + f_{n-2}e))}{1/(f_{n-2}a + f_{n-1}e) + 1/(f_{n-3}a + f_{n-2}e)} \\
&= \frac{ace/(f_{n-1}e + f_{n-1}c)}{(f_{n-2}a + f_{n-1}e)(f_{n-3}a + f_{n-2}e)[1/(f_{n-2}a + f_{n-1}e) + 1/(f_{n-3}a + f_{n-2}e)]} = \frac{ace/(f_{n-1}e + f_{n-1}c)}{[f_{n-3}a + f_{n-2}e + f_{n-2}a + f_{n-1}e]} \\
&= \frac{ace/(f_{n-1}e + f_{n-1}c)}{[f_{n-1}a + f_{n-1}e]} = \frac{ace}{(f_{n-1}a + f_{n-1}e)(f_{n-1}e + f_{n-1}c)}, \\
y_{2n} &= \frac{x_{2n-1}x_{2n-3}}{y_{2n-2} + x_{2n-3}} = \frac{(bdf/(f_{n-2}d + f_{n-1}b)(f_{n-1}b + f_{n-1}f))(bdf/(f_{n-3}d + f_{n-2}b)(f_{n-2}b + f_{n-1}f))}{bdf/(f_{n-2}d + f_{n-1}b)(f_{n-2}d + f_{n-1}b) + bdf/(f_{n-3}d + f_{n-2}b)(f_{n-2}b + f_{n-1}f)} \\
&= \frac{(bdf/(f_{n-2}d + f_{n-1}b)(f_{n-1}b + f_{n-1}f))(1/(f_{n-3}d + f_{n-2}b))}{1/(f_{n-2}d + f_{n-1}b) + 1/(f_{n-3}d + f_{n-2}b)} \\
&= \frac{bdf/(f_{n-1}b + f_{n-1}f)}{(f_{n-2}d + f_{n-1}b)(f_{n-3}d + f_{n-2}b)[1/(f_{n-2}d + f_{n-1}b) + 1/(f_{n-3}d + f_{n-2}b)]} = \frac{bdf/(f_{n-1}b + f_{n-1}f)}{[f_{n-3}d + f_{n-2}b + f_{n-2}d + f_{n-1}b]} \\
&= \frac{bdf/(f_{n-1}b + f_{n-1}f)}{[f_{n-1}d + f_{n-1}b]} = \frac{bdf}{(f_{n-1}b + f_{n-1}f)(f_{n-1}d + f_{n-1}b)}.
\end{aligned} \tag{12}$$

Hence the proof is complete. $\square$

Lemma 2. Let $\{x_n, y_n\}$ be a positive solution of system (8); then every solution of system (8) is bounded and converges to zero.

Proof. It follows from system (8) that

$$\begin{aligned}
x_{n+1} &= \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}} \leq \frac{y_n y_{n-2}}{y_{n-2}} \leq y_n, \\
y_{n+1} &= \frac{x_n x_{n-2}}{y_{n-1} + x_{n-2}} \leq \frac{x_n x_{n-2}}{x_{n-2}} \leq x_n;
\end{aligned} \tag{13}$$

thus

$$\begin{aligned}
x_{n+1} &\leq y_n \leq x_{n-1}, \\
y_{n+1} &\leq x_n \leq y_{n-1}.
\end{aligned} \tag{14}$$

Then the subsequences $\{x_{2n-1}\}_{n=0}^{\infty}, \{x_{2n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by

$$M = \max\{x_{-2}, x_{-1}, x_0\}. \tag{15}$$

Similarly the subsequences $\{y_{2n-1}\}_{n=0}^{\infty}, \{y_{2n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by

$$N = \max\{y_{-2}, y_{-1}, y_0\}. \tag{16}$$

$\square$

2.2. The Second System: $x_{n+1} = y_n y_{n-2}/(x_{n-1} + y_{n-2}), y_{n+1} = x_n x_{n-2}/(y_{n-1} + x_{n-2})$. The structure of the solutions of the following system of the difference equations is examined in this subsection.

$$\begin{aligned}
x_{n+1} &= \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}}, \\
y_{n+1} &= \frac{x_n x_{n-2}}{y_{n-1} + x_{n-2}}, \\
n &= 0, 1, \dots,
\end{aligned} \tag{17}$$

with nonzero positive real numbers of initial conditions x_{-2}, x_{-1}, x_0 and y_{-2}, y_{-1}, y_0 with $x_{-2} \neq y_{-1}$.

Theorem 3. Suppose that $\{x_n, y_n\}$ are solutions of the system

$$\begin{aligned} x_{n+1} &= \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}}, \\ y_{n+1} &= \frac{x_n x_{n-2}}{y_{n-1} - x_{n-2}}. \end{aligned} \quad (18)$$

Then for $n = 0, 1, 2, \dots$, one has

$$\begin{aligned} x_{2n-1} &= \frac{(-1)^n bdf}{(f_{n-2}d - f_{n-1}b)(f_{n-1}b + f_n f)}, \\ x_{2n} &= \frac{(-1)^{n+1} ace}{(f_{n-1}a + f_n e)(f_{n-1}e - f_n c)}, \\ y_{2n-1} &= \frac{(-1)^{n+1} ace}{(f_{n-2}a + f_{n-1}e)(f_{n-1}e - f_n c)}, \\ y_{2n} &= \frac{(-1)^{n+1} bdf}{(f_{n-1}b + f_n f)(f_{n-1}d - f_n b)}, \end{aligned} \quad (19)$$

where $x_0 = a, x_{-1} = b, x_{-2} = c, y_0 = d, y_{-1} = e, y_{-2} = f$ and where f_n is the Fibonacci Sequence with $f_{-2} = 1, f_{-1} = 0$.

Proof. For $n = 0$ the result holds. By using mathematical induction now suppose that $n > 0$ and that our assumption holds for $n - 1, n - 2$. That is,

$$\begin{aligned} x_{2n-4} &= \frac{(-1)^{n-1} ace}{(f_{n-3}a + f_{n-2}e)(f_{n-3}e - f_{n-2}c)}, \\ x_{2n-3} &= \frac{(-1)^{n-1} bdf}{(f_{n-3}d - f_{n-2}b)(f_{n-2}b + f_{n-1}f)}, \\ x_{2n-2} &= \frac{(-1)^n ace}{(f_{n-2}a + f_{n-1}e)(f_{n-2}e - f_{n-1}c)}, \\ y_{2n-4} &= \frac{(-1)^{n-1} bdf}{(f_{n-3}b + f_{n-2}f)(f_{n-3}d - f_{n-2}b)}, \\ y_{2n-3} &= \frac{(-1)^n ace}{(f_{n-3}a + f_{n-2}e)(f_{n-2}e - f_{n-1}c)}, \\ y_{2n-2} &= \frac{(-1)^n bdf}{(f_{n-2}b + f_{n-1}f)(f_{n-2}d - f_{n-1}b)}. \end{aligned} \quad (20)$$

Now, it follows from system (17) substitution of the above equations that

$$\begin{aligned} x_{2n-1} &= \frac{y_{2n-2} y_{2n-4}}{x_{2n-3} + y_{2n-4}} = \frac{((-1)^n bdf / (f_{n-2}b + f_{n-1}f)(f_{n-2}d - f_{n-1}b))((-1)^{n-1} bdf / (f_{n-3}b + f_{n-2}f)(f_{n-3}d - f_{n-2}b))}{(-1)^{n-1} bdf / (f_{n-3}d - f_{n-2}b)(f_{n-2}b + f_{n-1}f) + (-1)^{n-1} bdf / (f_{n-3}b + f_{n-2}f)(f_{n-3}d - f_{n-2}b)} \\ &= \frac{((-1)^n bdf / (f_{n-2}b + f_{n-1}f)(f_{n-2}d - f_{n-1}b))(1 / (f_{n-3}b + f_{n-2}f))}{1 / (f_{n-2}b + f_{n-1}f) + 1 / (f_{n-3}b + f_{n-2}f)} = \frac{(-1)^n bdf / (f_{n-2}d - f_{n-1}b)}{(f_{n-3}b + f_{n-2}f) + (f_{n-2}b + f_{n-1}f)} \\ &= \frac{(-1)^n bdf}{(f_{n-2}d - f_{n-1}b)(f_{n-1}b + f_n f)}, \\ y_{2n-1} &= \frac{x_{2n-2} x_{2n-4}}{y_{2n-3} - x_{2n-4}} = \frac{((-1)^n ace / (f_{n-2}a + f_{n-1}e)(f_{n-2}e - f_{n-1}c))((-1)^{n-1} ace / (f_{n-3}a + f_{n-2}e)(f_{n-3}e - f_{n-2}c))}{(-1)^n ace / (f_{n-3}a + f_{n-2}e)(f_{n-2}e - f_{n-1}c) - (-1)^{n-1} ace / (f_{n-3}a + f_{n-2}e)(f_{n-3}e - f_{n-2}c)} \\ &= \frac{((-1)^{n+1} ace / (f_{n-2}a + f_{n-1}e)(f_{n-2}e - f_{n-1}c))(1 / (f_{n-3}e - f_{n-2}c))}{1 / (f_{n-2}e - f_{n-1}c) + 1 / (f_{n-3}e - f_{n-2}c)} \\ &= \frac{(-1)^{n+1} ace / (f_{n-2}a + f_{n-1}e)}{(f_{n-2}e - f_{n-1}c)(f_{n-3}e - f_{n-2}c)[1 / (f_{n-2}e - f_{n-1}c) + 1 / (f_{n-3}e - f_{n-2}c)]} = \frac{(-1)^{n+1} ace / (f_{n-2}a + f_{n-1}e)}{[f_{n-3}e - f_{n-2}c + f_{n-2}e - f_{n-1}c]} \\ &= \frac{(-1)^{n+1} ace / (f_{n-2}a + f_{n-1}e)}{[f_{n-1}e - f_n c]} = \frac{(-1)^{n+1} ace}{(f_{n-2}a + f_{n-1}e)(f_{n-1}e - f_n c)}. \end{aligned} \quad (21)$$

Also, we obtain

$$\begin{aligned} x_{2n} &= \frac{y_{2n-1} y_{2n-3}}{x_{2n-2} + y_{2n-3}} = \frac{((-1)^{n+1} ace / (f_{n-2}a + f_{n-1}e)(f_{n-1}e - f_n c))((-1)^n ace / (f_{n-3}a + f_{n-2}e)(f_{n-2}e - f_{n-1}c))}{(-1)^n ace / (f_{n-2}a + f_{n-1}e)(f_{n-2}e - f_{n-1}c) + (-1)^n ace / (f_{n-3}a + f_{n-2}e)(f_{n-2}e - f_{n-1}c)} \\ &= \frac{((-1)^{n+1} ace / (f_{n-2}a + f_{n-1}e)(f_{n-1}e - f_n c))(1 / (f_{n-3}a + f_{n-2}e))}{1 / (f_{n-2}a + f_{n-1}e) + 1 / (f_{n-3}a + f_{n-2}e)} \end{aligned}$$

$$\begin{aligned}
&= \frac{(-1)^{n+1} ace / (f_{n-1}e - f_nc)}{(f_{n-2}a + f_{n-1}e)(f_{n-3}a + f_{n-2}e) [1/(f_{n-2}a + f_{n-1}e) + 1/(f_{n-3}a + f_{n-2}e)]} = \frac{(-1)^{n+1} ace / (f_{n-1}e - f_nc)}{[f_{n-3}a + f_{n-2}e + f_{n-2}a + f_{n-1}e]} \\
&= \frac{(-1)^{n+1} ace / (f_{n-1}e - f_nc)}{[f_{n-1}a + f_ne]} = \frac{(-1)^{n+1} ace}{(f_{n-1}a + f_ne)(f_{n-1}e - f_nc)}, \\
y_{2n} &= \frac{x_{2n-1}x_{2n-3}}{y_{2n-2} - x_{2n-3}} = \frac{((-1)^n bdf / (f_{n-2}d - f_{n-1}b)(f_{n-1}b + f_nf))((-1)^{n-1} bdf / (f_{n-3}d - f_{n-2}b)(f_{n-2}b + f_{n-1}f))}{(-1)^n bdf / (f_{n-2}b + f_{n-1}f)(f_{n-2}d - f_{n-1}b) - (-1)^{n-1} bdf / (f_{n-3}d - f_{n-2}b)(f_{n-2}b + f_{n-1}f)} \\
&= \frac{((-1)^{n+1} bdf / (f_{n-2}d - f_{n-1}b)(f_{n-1}b + f_nf))(1/(f_{n-3}d - f_{n-2}b))}{1/(f_{n-2}d - f_{n-1}b) + 1/(f_{n-3}d - f_{n-2}b)} \\
&= \frac{(-1)^{n+1} bdf / (f_{n-1}b + f_nf)}{(f_{n-2}d - f_{n-1}b)(f_{n-3}d - f_{n-2}b) [1/(f_{n-2}d - f_{n-1}b) + 1/(f_{n-3}d - f_{n-2}b)]} = \frac{(-1)^{n+1} bdf / (f_{n-1}b + f_nf)}{[f_{n-3}d - f_{n-2}b + f_{n-2}d - f_{n-1}b]} \\
&= \frac{(-1)^{n+1} bdf / (f_{n-1}b + f_nf)}{[f_{n-1}d - f_nb]} = \frac{(-1)^{n+1} bdf}{(f_{n-1}b + f_nf)(f_{n-1}d - f_nb)}.
\end{aligned} \tag{22}$$

Hence the proof is complete. $\square$

2.3. The Third System: $x_{n+1} = y_n y_{n-2} / (x_{n-1} + y_{n-2})$, $y_{n+1} = x_n x_{n-2} / (-y_{n-1} + x_{n-2})$. In this subsection, the framework of the solution of the following system of difference equations is acquired.

$$\begin{aligned}
x_{n+1} &= \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}}, \\
y_{n+1} &= \frac{x_n x_{n-2}}{-y_{n-1} + x_{n-2}}, \\
n &= 0, 1, \dots,
\end{aligned} \tag{23}$$

where the initial conditions x_{-2} , x_{-1} , x_0 , y_{-2} , y_{-1} , y_0 are arbitrary positive real numbers with $x_{-2} \neq y_{-1}$.

Theorem 4. Suppose that $\{x_n, y_n\}$ are solutions of system (23). Then for $n = 0, 1, 2, \dots$, one sees that all solutions of system (23) are given by the following formulas:

$$\begin{aligned}
x_{6n-2} &= \frac{(-1)^n ac}{(f_{3n-2}a + f_{3n-1}e)}, \\
x_{6n-1} &= \frac{(-1)^n bf}{(f_{3n-1}b + f_{3n}f)}, \\
x_{6n} &= \frac{(-1)^n ae}{(f_{3n-1}a + f_{3n}e)}, \\
x_{6n+1} &= \frac{(-1)^n df}{(f_{3n}b + f_{3n+1}f)}, \\
x_{6n+2} &= \frac{(-1)^n ace}{(c-e)(f_{3n}a + f_{3n+1}e)},
\end{aligned}$$

$$\begin{aligned}
x_{6n+3} &= \frac{(-1)^n bdf}{(b-d)(f_{3n+1}b + f_{3n+2}f)}, \\
y_{6n-2} &= \frac{(-1)^n bf}{(f_{3n-2}b + f_{3n-1}f)}, \\
y_{6n-1} &= \frac{(-1)^n ae}{(f_{3n-2}a + f_{3n-1}e)}, \\
y_{6n} &= \frac{(-1)^n df}{(f_{3n-1}b + f_{3n}f)}, \\
y_{6n+1} &= \frac{(-1)^n ace}{(c-e)(f_{3n-1}a + f_{3n}e)}, \\
y_{6n+2} &= \frac{(-1)^n bdf}{(b-d)(f_{3n}b + f_{3n+1}f)}, \\
y_{6n+3} &= \frac{(-1)^{n+1} ac}{(f_{3n}a + f_{3n+1}e)},
\end{aligned} \tag{24}$$

where $x_0 = a$, $x_{-1} = b$, $x_{-2} = c$, $y_0 = d$, $y_{-1} = e$, $y_{-2} = f$ and where f_n is the Fibonacci Sequence with $f_{-2} = 1$, $f_{-1} = 0$.

Proof. For $n = 0$ the result holds. By using mathematical induction now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned}
x_{6n-8} &= \frac{(-1)^{n-1} ac}{(f_{3n-5}a + f_{3n-4}e)}, \\
x_{6n-7} &= \frac{(-1)^{n-1} bf}{(f_{3n-4}b + f_{3n-3}f)}, \\
x_{6n-6} &= \frac{(-1)^{n-1} ae}{(f_{3n-4}a + f_{3n-3}e)},
\end{aligned}$$

$$\begin{aligned}
x_{6n-5} &= \frac{(-1)^{n-1} df}{(f_{3n-3}b + f_{3n-2}f)}, & y_{6n-6} &= \frac{(-1)^{n-1} df}{(f_{3n-4}b + f_{3n-3}f)}, \\
x_{6n-4} &= \frac{(-1)^{n-1} ace}{(c-e)(f_{3n-3}a + f_{3n-2}e)}, & y_{6n-5} &= \frac{(-1)^{n-1} ace}{(c-e)(f_{3n-4}a + f_{3n-3}e)}, \\
x_{6n-3} &= \frac{(-1)^{n-1} bdf}{(b-d)(f_{3n-2}b + f_{3n-1}f)}, & y_{6n-4} &= \frac{(-1)^{n-1} bdf}{(b-d)(f_{3n-3}b + f_{3n-2}f)}, \\
y_{6n-8} &= \frac{(-1)^{n-1} bf}{(f_{3n-5}b + f_{3n-4}f)}, & y_{6n-3} &= \frac{(-1)^n ac}{(f_{3n-3}a + f_{3n-2}e)}. \\
y_{6n-7} &= \frac{(-1)^{n-1} ae}{(f_{3n-5}a + f_{3n-4}e)}, & &
\end{aligned} \tag{25}$$

Now, it follows from system (23) substitution of the above equations that

$$\begin{aligned}
x_{6n-2} &= \frac{y_{6n-3}y_{6n-5}}{x_{6n-4} + y_{6n-5}} = \frac{((-1)^n ac / (f_{3n-3}a + f_{3n-2}e))((-1)^{n-1} ace / (c-e)(f_{3n-4}a + f_{3n-3}e))}{(-1)^{n-1} ace / (c-e)(f_{3n-3}a + f_{3n-2}e) + (-1)^{n-1} ace / (c-e)(f_{3n-4}a + f_{3n-3}e)} \\
&= \frac{((-1)^n ac / (f_{3n-3}a + f_{3n-2}e))(1 / (f_{3n-4}a + f_{3n-3}e))}{1 / (f_{3n-3}a + f_{3n-2}e) + 1 / (f_{3n-4}a + f_{3n-3}e)} = \frac{(-1)^n ac}{(f_{3n-3}a + f_{3n-2}e)(f_{3n-4}a + f_{3n-3}e)[1 / (f_{3n-3}a + f_{3n-2}e) + 1 / (f_{3n-4}a + f_{3n-3}e)]} \\
&= \frac{(-1)^n ac}{[(f_{3n-4}a + f_{3n-3}e) + (f_{3n-3}a + f_{3n-2}e)]} = \frac{(-1)^n ac}{(f_{3n-2}a + f_{3n-1}e)}, \\
y_{6n-2} &= \frac{x_{6n-3}x_{6n-5}}{-y_{6n-4} + x_{6n-5}} = \frac{((-1)^{n-1} bdf / (b-d)(f_{3n-2}b + f_{3n-1}f))((-1)^{n-1} df / (f_{3n-3}b + f_{3n-2}f))}{-(-1)^{n-1} bdf / (b-d)(f_{3n-3}b + f_{3n-2}f) + (-1)^{n-1} df / (f_{3n-3}b + f_{3n-2}f)} \\
&= \frac{((-1)^{n-1} bdf / (b-d)(f_{3n-2}b + f_{3n-1}f))(1 / (f_{3n-3}b + f_{3n-2}f))}{-b / (b-d)(f_{3n-3}b + f_{3n-2}f) + 1 / (f_{3n-3}b + f_{3n-2}f)} = \frac{(-1)^{n-1} bdf / (b-d)(f_{3n-2}b + f_{3n-1}f)}{-b / (b-d) + 1} \\
&= \frac{(-1)^{n-1} bdf}{(b-d)(f_{3n-2}b + f_{3n-1}f)[-b / (b-d) + 1]} = \frac{(-1)^{n-1} bdf}{(f_{3n-2}b + f_{3n-1}f)[-b + b - d]} = \frac{(-1)^{n-1} bdf}{(f_{3n-2}b + f_{3n-1}f)[-b + b - d]} = \frac{(-1)^n bf}{(f_{3n-2}b + f_{3n-1}f)}, \\
x_{6n} &= \frac{y_{6n-1}y_{6n-3}}{x_{6n-2} + y_{6n-3}} = \frac{((-1)^n ae / (f_{3n-2}a + f_{3n-1}e))((-1)^n ac / (f_{3n-3}a + f_{3n-2}e))}{(-1)^n ac / (f_{3n-2}a + f_{3n-1}e) + (-1)^n ac / (f_{3n-3}a + f_{3n-2}e)} = \frac{((-1)^n ae / (f_{3n-2}a + f_{3n-1}e))(1 / (f_{3n-3}a + f_{3n-2}e))}{1 / (f_{3n-2}a + f_{3n-1}e) + 1 / (f_{3n-3}a + f_{3n-2}e)} \\
&= \frac{(-1)^n ae}{(f_{3n-2}a + f_{3n-1}e)(f_{3n-3}a + f_{3n-2}e)[1 / (f_{3n-2}a + f_{3n-1}e) + 1 / (f_{3n-3}a + f_{3n-2}e)]} = \frac{(-1)^n ae}{[f_{3n-3}a + f_{3n-2}e + f_{3n-2}a + f_{3n-1}e]} \\
&= \frac{(-1)^n ae}{(f_{3n-1}a + f_{3n}e)}, \\
y_{6n} &= \frac{x_{6n-1}x_{6n-3}}{-y_{6n-2} + x_{6n-3}} = \frac{((-1)^n bf / (f_{3n-1}b + f_{3n}f))((-1)^{n-1} bdf / (b-d)(f_{3n-2}b + f_{3n-1}f))}{-(-1)^n bf / (f_{3n-2}b + f_{3n-1}f) + (-1)^{n-1} bdf / (b-d)(f_{3n-2}b + f_{3n-1}f)} \\
&= \frac{((-1)^n bf / (f_{3n-1}b + f_{3n}f))(d / (b-d)(f_{3n-2}b + f_{3n-1}f))}{1 / (f_{3n-2}b + f_{3n-1}f) + d / (b-d)(f_{3n-2}b + f_{3n-1}f)} \\
&= \frac{(-1)^n bdf}{(b-d)(f_{3n-2}b + f_{3n-1}f)(f_{3n-1}b + f_{3n}f)[1 / (f_{3n-2}b + f_{3n-1}f) + d / (b-d)(f_{3n-2}b + f_{3n-1}f)]} \\
&= \frac{(-1)^n bdf}{[(b-d)(f_{3n-2}b + f_{3n-1}f)(f_{3n-1}b + f_{3n}f) / (f_{3n-2}b + f_{3n-1}f) + d(b-d)(f_{3n-2}b + f_{3n-1}f)(f_{3n-1}b + f_{3n}f) / (b-d)(f_{3n-2}b + f_{3n-1}f)]} \\
&= \frac{(-1)^n bdf}{[(b-d)(f_{3n-1}b + f_{3n}f) + d(f_{3n-1}b + f_{3n}f)]} = \frac{(-1)^n bdf}{[b(f_{3n-1}b + f_{3n}f) - d(f_{3n-1}b + f_{3n}f) + d(f_{3n-1}b + f_{3n}f)]} = \frac{(-1)^n bdf}{b(f_{3n-1}b + f_{3n}f)} \\
&= \frac{(-1)^n df}{(f_{3n-1}b + f_{3n}f)}.
\end{aligned} \tag{26}$$

Also, we obtain

$$\begin{aligned}
 x_{6n+2} &= \frac{y_{6n+1}y_{6n-1}}{x_{6n} + y_{6n-1}} = \frac{((-1)^n ace / (c - e) (f_{3n-1}a + f_{3n}e)) ((-1)^n ae / (f_{3n-2}a + f_{3n-1}e))}{(-1)^n ae / (f_{3n-1}a + f_{3n}e) + (-1)^n ae / (f_{3n-2}a + f_{3n-1}e)} \\
 &= \frac{((-1)^n ace / (c - e) (f_{3n-1}a + f_{3n}e)) (1 / (f_{3n-2}a + f_{3n-1}e))}{1 / (f_{3n-1}a + f_{3n}e) + 1 / (f_{3n-2}a + f_{3n-1}e)} = \frac{(-1)^n ace / (c - e)}{[f_{3n-2}a + f_{3n-1}e + f_{3n-1}a + f_{3n}e]} \\
 &= \frac{(-1)^n ace}{(c - e) (f_{3n}a + f_{3n+1}e)}, \\
 y_{6n+2} &= \frac{x_{6n+1}x_{6n-1}}{-y_{6n} + x_{6n-1}} = \frac{((-1)^n df / (f_{3n}b + f_{3n+1}f)) ((-1)^n bf / (f_{3n-1}b + f_{3n}f))}{-(-1)^n df / (f_{3n-1}b + f_{3n}f) + (-1)^n bf / (f_{3n-1}b + f_{3n}f)} \\
 &= \frac{((-1)^n df / (f_{3n}b + f_{3n+1}f)) (b / (f_{3n-1}b + f_{3n}f))}{-d / (f_{3n-1}b + f_{3n}f) + b / (f_{3n-1}b + f_{3n}f)} \\
 &= \frac{(-1)^n bdf}{(f_{3n}b + f_{3n+1}f) (f_{3n-1}b + f_{3n}f) [-d / (f_{3n-1}b + f_{3n}f) + b / (f_{3n-1}b + f_{3n}f)]} = \frac{(-1)^n bdf}{(f_{3n}b + f_{3n+1}f) [-d + b]} \\
 &= \frac{(-1)^n bdf}{(b - d) (f_{3n}b + f_{3n+1}f)}.
 \end{aligned} \tag{27}$$

Thus, the other relations can be proven in a similar way. Hence the proof is complete. $\square$

2.4. The Fourth System: $x_{n+1} = y_n y_{n-2} / (x_{n-1} + y_{n-2})$, $y_{n+1} = x_n x_{n-2} / (-y_{n-1} - x_{n-2})$. In this subsection, we analyze the solutions of the system of the following two difference equations.

$$\begin{aligned}
 x_{n+1} &= \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}}, \\
 y_{n+1} &= \frac{x_n x_{n-2}}{-y_{n-1} - x_{n-2}}, \\
 n &= 0, 1, \dots,
 \end{aligned} \tag{28}$$

with the initial conditions x_{-2}, x_{-1}, x_0 and y_{-2}, y_{-1}, y_0 being nonzero positive real numbers.

Theorem 5. Let $\{x_n, y_n\}_{n=-1}^{+\infty}$ be solutions of system (28). Then $\{x_n\}_{n=-1}^{+\infty}$ and $\{y_n\}_{n=-1}^{+\infty}$ are given by the following formulas for $n = 0, 1, 2, \dots$:

$$\begin{aligned}
 x_{6n-2} &= \frac{ac}{(f_{3n-2}a + f_{3n-1}e)}, \\
 x_{6n-1} &= \frac{bf}{(f_{3n-1}b + f_{3n}f)}, \\
 x_{6n} &= \frac{ae}{(f_{3n-1}a + f_{3n}e)}, \\
 x_{6n+1} &= \frac{df}{(f_{3n}b + f_{3n+1}f)}, \\
 x_{6n+2} &= \frac{-ace}{(c + e) (f_{3n}a + f_{3n+1}e)},
 \end{aligned}$$

$$\begin{aligned}
 x_{6n+3} &= \frac{-bdf}{(b + d) (f_{3n+1}b + f_{3n+2}f)}, \\
 y_{6n-2} &= \frac{bf}{(f_{3n-2}b + f_{3n-1}f)}, \\
 y_{6n-1} &= \frac{ae}{(f_{3n-2}a + f_{3n-1}e)}, \\
 y_{6n} &= \frac{df}{(f_{3n-1}b + f_{3n}f)}, \\
 y_{6n+1} &= \frac{-ace}{(c + e) (f_{3n-1}a + f_{3n}e)}, \\
 y_{6n+2} &= \frac{-bdf}{(b + d) (f_{3n}b + f_{3n+1}f)}, \\
 y_{6n+3} &= \frac{ac}{(f_{3n}a + f_{3n+1}e)},
 \end{aligned} \tag{29}$$

where $x_0 = a, x_{-1} = b, x_{-2} = c, y_0 = d, y_{-1} = e, y_{-2} = f$ and where f_n is the Fibonacci Sequence with $f_{-2} = 1, f_{-1} = 0$.

Proof. For $n = 0$ the result holds. By using mathematical induction now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned}
 x_{6n-8} &= \frac{ac}{(f_{3n-5}a + f_{3n-4}e)}, \\
 x_{6n-7} &= \frac{bf}{(f_{3n-4}b + f_{3n-3}f)}, \\
 x_{6n-6} &= \frac{ae}{(f_{3n-4}a + f_{3n-3}e)},
 \end{aligned}$$

$$\begin{aligned}
x_{6n-5} &= \frac{df}{(f_{3n-3}b + f_{3n-2}f)}, & y_{6n-6} &= \frac{df}{(f_{3n-4}b + f_{3n-3}f)}, \\
x_{6n-4} &= \frac{-ace}{(c+e)(f_{3n-3}a + f_{3n-2}e)}, & y_{6n-5} &= \frac{-ace}{(c+e)(f_{3n-4}a + f_{3n-3}e)}, \\
x_{6n-3} &= \frac{-bdf}{(b+d)(f_{3n-2}b + f_{3n-1}f)}, & y_{6n-4} &= \frac{-bdf}{(b+d)(f_{3n-3}b + f_{3n-2}f)}, \\
y_{6n-8} &= \frac{bf}{(f_{3n-5}b + f_{3n-4}f)}, & y_{6n-3} &= \frac{ac}{(f_{3n-3}a + f_{3n-2}e)}. \\
y_{6n-7} &= \frac{ae}{(f_{3n-5}a + f_{3n-4}e)}, & &
\end{aligned} \tag{30}$$

Now, it follows from system (28) substitution of the above equations that

$$\begin{aligned}
x_{6n-2} &= \frac{y_{6n-3}y_{6n-5}}{x_{6n-4} + y_{6n-5}} = \frac{(ac/(f_{3n-3}a + f_{3n-2}e))(-ace/(c+e)(f_{3n-4}a + f_{3n-3}e))}{-ace/(c+e)(f_{3n-3}a + f_{3n-2}e) + -ace/(c+e)(f_{3n-4}a + f_{3n-3}e)} = \frac{(ac/(f_{3n-3}a + f_{3n-2}e))(1/(f_{3n-4}a + f_{3n-3}e))}{1/(f_{3n-3}a + f_{3n-2}e) + 1/(f_{3n-4}a + f_{3n-3}e)} \\
&= \frac{ac}{(f_{3n-3}a + f_{3n-2}e)(f_{3n-4}a + f_{3n-3}e)[1/(f_{3n-3}a + f_{3n-2}e) + 1/(f_{3n-4}a + f_{3n-3}e)]} = \frac{ac}{[(f_{3n-4}a + f_{3n-3}e) + (f_{3n-3}a + f_{3n-2}e)]} \\
&= \frac{ac}{(f_{3n-2}a + f_{3n-1}e)}, \\
y_{6n-2} &= \frac{x_{6n-3}x_{6n-5}}{-y_{6n-4} - x_{6n-5}} = \frac{(-bdf/(b+d)(f_{3n-2}b + f_{3n-1}f))(df/(f_{3n-3}b + f_{3n-2}f))}{-(-bdf/(b+d)(f_{3n-3}b + f_{3n-2}f)) - df/(f_{3n-3}b + f_{3n-2}f)} \\
&= \frac{(-bdf/(b+d)(f_{3n-2}b + f_{3n-1}f))(1/(f_{3n-3}b + f_{3n-2}f))}{-(-b/(b+d)(f_{3n-3}b + f_{3n-2}f)) - 1/(f_{3n-3}b + f_{3n-2}f)} = \frac{-bdf/(b+d)(f_{3n-2}b + f_{3n-1}f)}{b/(b+d) - 1} = \frac{-bdf}{(b+d)(f_{3n-2}b + f_{3n-1}f)[b/(b+d) - 1]} \\
&= \frac{-bdf}{(f_{3n-2}b + f_{3n-1}f)[b - b - d]} = \frac{bf}{(f_{3n-2}b + f_{3n-1}f)}, \\
x_{6n} &= \frac{y_{6n-1}y_{6n-3}}{x_{6n-2} + y_{6n-3}} = \frac{(ae/(f_{3n-2}a + f_{3n-1}e))(ac/(f_{3n-3}a + f_{3n-2}e))}{ac/(f_{3n-2}a + f_{3n-1}e) + ac/(f_{3n-3}a + f_{3n-2}e)} = \frac{(ae/(f_{3n-2}a + f_{3n-1}e))(1/(f_{3n-3}a + f_{3n-2}e))}{1/(f_{3n-2}a + f_{3n-1}e) + 1/(f_{3n-3}a + f_{3n-2}e)} \\
&= \frac{ae}{(f_{3n-2}a + f_{3n-1}e)(f_{3n-3}a + f_{3n-2}e)[1/(f_{3n-2}a + f_{3n-1}e) + 1/(f_{3n-3}a + f_{3n-2}e)]} = \frac{ae}{[f_{3n-3}a + f_{3n-2}e + f_{3n-2}a + f_{3n-1}e]} \\
&= \frac{ae}{(f_{3n-1}a + f_{3n}e)}, \\
y_{6n} &= \frac{x_{6n-1}x_{6n-3}}{-y_{6n-2} - x_{6n-3}} = \frac{(bf/(f_{3n-1}b + f_{3n}f))(-bdf/(b+d)(f_{3n-2}b + f_{3n-1}f))}{-bf/(f_{3n-2}b + f_{3n-1}f) - (-bdf/(b+d)(f_{3n-2}b + f_{3n-1}f))} \\
&= \frac{(bf/(f_{3n-1}b + f_{3n}f))(-d/(b+d)(f_{3n-2}b + f_{3n-1}f))}{-1/(f_{3n-2}b + f_{3n-1}f) - (-d/(b+d)(f_{3n-2}b + f_{3n-1}f))} \\
&= \frac{-bdf}{(b+d)(f_{3n-2}b + f_{3n-1}f)(f_{3n-1}b + f_{3n}f)[-1/(f_{3n-2}b + f_{3n-1}f) - (-d/(b+d)(f_{3n-2}b + f_{3n-1}f))]} \\
&= \frac{-bdf}{[-(b+d)(f_{3n-2}b + f_{3n-1}f)(f_{3n-1}b + f_{3n}f)/(f_{3n-2}b + f_{3n-1}f) + d(b+d)(f_{3n-2}b + f_{3n-1}f)(f_{3n-1}b + f_{3n}f)/(b+d)(f_{3n-2}b + f_{3n-1}f)]} \\
&= \frac{-bdf}{[-(b+d)(f_{3n-1}b + f_{3n}f) + d(f_{3n-1}b + f_{3n}f)]} = \frac{-bdf}{[-b(f_{3n-1}b + f_{3n}f) - d(f_{3n-1}b + f_{3n}f) + d(f_{3n-1}b + f_{3n}f)]} = \frac{-bdf}{-b(f_{3n-1}b + f_{3n}f)} \\
&= \frac{df}{(f_{3n-1}b + f_{3n}f)}.
\end{aligned} \tag{31}$$

Also, we obtain

$$\begin{aligned}
x_{6n+2} &= \frac{y_{6n+1}y_{6n-1}}{x_{6n} + y_{6n-1}} = \frac{(-ace/(c+e)(f_{3n-1}a + f_{3n}e))(ae/(f_{3n-2}a + f_{3n-1}e))}{ae/(f_{3n-1}a + f_{3n}e) + ae/(f_{3n-2}a + f_{3n-1}e)} \\
&= \frac{(-ace/(c+e)(f_{3n-1}a + f_{3n}e))(1/(f_{3n-2}a + f_{3n-1}e))}{1/(f_{3n-1}a + f_{3n}e) + 1/(f_{3n-2}a + f_{3n-1}e)}
\end{aligned}$$

FIGURE 1: This figure shows the solution of (8).

$$\begin{aligned}
&= \frac{-ace / (c + e)}{(f_{3n-1}a + f_{3n}e)(f_{3n-2}a + f_{3n-1}e) [1 / (f_{3n-1}a + f_{3n}e) + 1 / (f_{3n-2}a + f_{3n-1}e)]} \\
&= \frac{-ace / (c + e)}{[f_{3n-2}a + f_{3n-1}e + f_{3n-1}a + f_{3n}e]} = \frac{-ace}{(c + e)(f_{3n}a + f_{3n+1}e)}, \\
y_{6n+2} &= \frac{x_{6n+1}x_{6n-1}}{-y_{6n} - x_{6n-1}} = \frac{(df / (f_{3n}b + f_{3n+1}f))(bf / (f_{3n-1}b + f_{3n}f))}{-df / (f_{3n-1}b + f_{3n}f) - bf / (f_{3n-1}b + f_{3n}f)} = \frac{(df / (f_{3n}b + f_{3n+1}f))(b / (f_{3n-1}b + f_{3n}f))}{-d / (f_{3n-1}b + f_{3n}f) - b / (f_{3n-1}b + f_{3n}f)} \\
&= \frac{bdf}{(f_{3n}b + f_{3n+1}f)(f_{3n-1}b + f_{3n}f) [-d / (f_{3n-1}b + f_{3n}f) - b / (f_{3n-1}b + f_{3n}f)]} = \frac{bdf}{(f_{3n}b + f_{3n+1}f) [-d - b]} \\
&= \frac{-bdf}{(b + d)(f_{3n}b + f_{3n+1}f)}.
\end{aligned} \tag{32}$$

Thus, the other relations can be proven in a similar way. Hence the proof is complete. $\square$

3. Numerical Examples

To illustrate the results of foregoing sections and to support our theoretical discussions, we take into account several interesting numerical examples in this section.

Example 1 (first system: $x_{n+1} = y_n y_{n-2} / (x_{n-1} + y_{n-2})$, $y_{n+1} = x_n x_{n-2} / (y_{n-1} + x_{n-2})$). Figure 1 considers the solution of the difference equations system (8) with the initial conditions $x_{-2} = 1$, $x_{-1} = 2$, $x_0 = 3$ and $y_{-2} = 4$, $y_{-1} = 5$, $y_0 = 6$.

Example 2 (second system: $x_{n+1} = y_n y_{n-2} / (x_{n-1} + y_{n-2})$, $y_{n+1} = x_n x_{n-2} / (y_{n-1} - x_{n-2})$). Figure 2 indicates the solution of the difference equation system (17) with the initial conditions $x_{-2} = 4$, $x_{-1} = 7$, $x_0 = 2$ and $y_{-2} = 2$, $y_{-1} = 5$, $y_0 = 1$.

Example 3 (third system: $x_{n+1} = y_n y_{n-2} / (x_{n-1} + y_{n-2})$, $y_{n+1} = x_n x_{n-2} / (-y_{n-1} + x_{n-2})$). Figure 3 considers the solution of the difference equation system (23) with the initial conditions $x_{-2} = 4$, $x_{-1} = 7$, $x_0 = 2$ and $y_{-2} = 2$, $y_{-1} = 5$, $y_0 = 1$.

Example 4 (fourth system: $x_{n+1} = y_n y_{n-2} / (x_{n-1} + y_{n-2})$, $y_{n+1} = x_n x_{n-2} / (-y_{n-1} - x_{n-2})$). Figure 4 shows the solution of the difference equation system (28) with the initial conditions $x_{-2} = 4$, $x_{-1} = 7$, $x_0 = 2$ and $y_{-2} = 2$, $y_{-1} = 5$, $y_0 = 1$.

FIGURE 2: This figure shows the solution of (17).

FIGURE 4: This figure shows the solution of (28).

FIGURE 3: This figure shows the solution of (23).

4. Conclusion

In this analysis, we oversee the structure of the solutions of four cases of the difference equations system $x_{n+1} = y_n y_{n-2}/(x_{n-1} + y_{n-2})$, $y_{n+1} = x_n x_{n-2}/(\pm y_{n-1} \pm x_{n-2})$. Additionally, some behavior of the solutions such as boundedness is investigated. Subsequently, some numerical examples are displayed by presenting some numerical values for the initial values per case and figures provided to justify the behavior of the obtained solutions in the case of numerical examples.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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