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Research Article

Global Attractivity and Extinction of a Discrete Competitive System with Infinite Delays and Single Feedback Control

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A nonautonomous discrete two-species competition system with infinite delays and single feedback control is considered in this paper. Based on the discrete comparison theorem, a set of sufficient conditions which guarantee the permanence of the system is obtained. Then, by constructing some suitable discrete Lyapunov functionals, some sufficient conditions for the global attractivity and extinction of the system are obtained. It is shown that, by choosing some suitable feedback control variable, one of two species will be driven to extinction.

1. Introduction

Two or more species compete for the same limited food source or in some way inhibit each other's growth. For example, competition may be for territory which is directly related to food resources. The importance of species competition in nature is obvious. Tradition two-species Lotka-Volterra competition system is as follows:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[r_1 - a_{11}x_1(t) - a_{12}x_2(t)\right], \\ \dot{x}_2(t) &= x_2(t)\left[r_2 - a_{21}x_1(t) - a_{22}x_2(t)\right]. \end{aligned} \tag{1}$$

However, system (1) has a property which is considered as a disadvantage and that is the linearity of the above system. Ayala et al. [1] presented the following nonlinear competitive system with continuous time version:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[r_1 - x_1(t) - a_1x_2(t) - c_1x_2^2(t)\right], \\ \dot{x}_2(t) &= x_2(t)\left[r_2 - x_2(t) - a_2x_1(t) - c_2x_1^2(t)\right]. \end{aligned} \tag{2}$$

Assume that each species needs some time to mature and the competition occurs after some time lag required for maturity of the species; Gopalsamy [2] discussed the following system with discrete delays:

$$\begin{aligned} &\dot{x}_1(t) \\ &\quad = x_1(t)\left[r_1 - x_1(t) - a_1x_2(t-\tau_2) - c_1x_2^2(t-\tau_2)\right], \\ &\dot{x}_2(t) \\ &\quad = x_2(t)\left[r_2 - x_2(t) - a_2x_1(t-\tau_1) - c_2x_1^2(t-\tau_1)\right]. \end{aligned} \tag{3}$$

Such systems are not well studied in the sense that most results are continuous time cases related (see [3, 4]). As we know, a discrete time system governed by difference equations is more approximate than the continuous ones when the populations have nonoverlapping generations or a short-life expectancy. Discrete time system can also provide efficient computation for numerical simulations (see [5–10]). Considering the biological parameters naturally being subject to almost periodic fluctuation in time, Tan and

Liao [11] established the following nonautonomous discrete competition system:

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\left[r_1(k) - x_1(k)\right. \\ &\quad \left. - a_1(k)\,x_2(k-\tau_2) - c_1(k)\,x_2^2(k-\tau_2)\right], \\ x_2(k+1) &= x_2(k)\exp\left[r_2(k) - x_2(k)\right. \\ &\quad \left. - a_2(k)\,x_1(k-\tau_1) - c_2(k)\,x_1^2(k-\tau_1)\right]. \end{aligned} \tag{4}$$

In the real world, ecosystems are disturbed by unpredictable forces which can result in some changes of parameters. In order to accurately describe such a system, scholars introduced feedback control into ecosystems. Recently, the ecosystems with feedback controls have been extensively studied and obtained many interesting results (see [12–18]), noting that models in [10, 12–15] considered at least two feedback controls variables, which means that, for the different species, different control strategy is adopted, whereas, in the real world, the strategy adopted for one species may also affect the other species. For example, spraying pesticide not only can reduce the number of weeds but also have a negative impact on the growth of corps or beneficial animals [19]. Therefore, how to keep these negative effects caused by feedback controls to a minimum? One strategy is to reduce the number of feedback controls like [16–18]. Motivated by the above, in this paper, we study the following discrete competitive system with delays and single feedback control:

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\Bigg[r_1(n) - a_{11}(n)\,x_1(n) \\ &\quad - a_{12}(n)\sum_{s=0}^{+\infty}K_{12}(s)\,x_2(n-s) \\ &\quad - b_{12}(n)\sum_{s=0}^{+\infty}H_{12}(s)\,x_2^2(n-s) \\ &\quad - c_1(n)\sum_{s=0}^{+\infty}Q_1(s)\,u(n-s)\Bigg], \\ x_2(n+1) &= x_2(n)\exp\Bigg[r_2(n) - a_{22}(n)\,x_2(n) \\ &\quad - a_{21}(n)\sum_{s=0}^{+\infty}K_{21}(s)\,x_1(n-s) \\ &\quad - b_{21}(n)\sum_{s=0}^{+\infty}H_{21}(s)\,x_1^2(n-s) \\ &\quad - c_2(n)\sum_{s=0}^{+\infty}Q_2(s)\,u(n-s)\Bigg], \\ u(n+1) &= (1-e(n))\,u(n) + d_1(n)\sum_{s=0}^{+\infty}P_1(s)\,x_1(n-s) \\ &\quad + d_2(n)\sum_{s=0}^{+\infty}P_2(s)\,x_2(n-s). \end{aligned} \tag{5}$$

In system (5), $x_i(n)$ $(i = 1, 2)$ is the density of x_i species at the nth generation and $u(n)$ is the single feedback control variable.

Throughout this paper, we assume the following.

(H_1): for any nonnegative bounded sequence $\{f(n)\}$ defined on Z, we use the notations $f^l = \inf_{n\in Z} f(n)$ and $f^u = \sup_{n\in Z} f(n)$.

(H_2): $r_i(n), a_{ij}(n), b_{ij}(n), c_i(n), d_i(n)$ $(i = 1, 2)$, and $e(n)$ are bounded nonnegative sequences of real numbers defined on Z such that

$$\begin{aligned} &0 < r_i^l \le r_i(n) \le r_i^u, \\ &0 < a_{ij}^l \le a_{ij}(n) \le a_{ij}^u, \\ &0 < b_{ij}^l \le b_{ij}(n) \le b_{ij}^u, \\ &0 < c_i^l \le c_i(n) \le c_i^u, \\ &0 < d_i^l \le d_i(n) \le d_i^u, \\ &0 < e^l \le e(n) \le e^u < 1. \end{aligned} \tag{6}$$

(H_3): $K_{ij}(s)$, $H_{ij}(s)$, $Q_i(s)$, and $P_i(s)$ $(i = 1, 2)$ are nonnegative bounded sequences such that

$$\begin{aligned} &\sum_{s=0}^{+\infty}K_{ij}(s) = 1, \\ &\sum_{s=0}^{+\infty}H_{ij}(s) = 1, \\ &\sum_{s=0}^{+\infty}Q_i(s) = 1, \\ &\sum_{s=0}^{+\infty}P_i(s) = 1; \\ &\Theta_{ij} = \sum_{s=0}^{+\infty}K_{ij}(s)\,s < +\infty, \\ &\Lambda_{ij} = \sum_{s=0}^{+\infty}H_{ij}(s)\,s < +\infty, \\ &\Delta_i = \sum_{s=0}^{+\infty}Q_i(s)\,s < +\infty, \\ &\Omega_i = \sum_{s=0}^{+\infty}P_i(s)\,s < +\infty. \end{aligned} \tag{7}$$

We consider the solution of system (5) with the following initial conditions:

$$\begin{aligned} x_i(\theta) &= \varphi_i(\theta) \geq 0, \\ x_i(0) &= \varphi_i(0) > 0, \\ \sup_{n\in Z^-} \varphi_i(n) &< +\infty, \\ u(\theta) &= \psi(\theta) \geq 0, \\ u(0) &= \psi(0) > 0, \\ \sup_{n\in Z^-} \psi(n) &< +\infty, \end{aligned} \tag{8}$$

where $i = 1, 2$ and $s = \cdots, -n, -n+1, \cdots, -1, 0$. One can easily show that the solutions of (5) with initial condition (7) are defined and remain positive for all $n \in N^+$.

The remaining part of this paper is organized as follows. We introduce some useful lemmas in Section 2 and then state and prove the main results in Sections 3, 4, and 5, respectively. Two examples together with their numeric simulations are presented to show the feasibility of the main results in Section 6.

2. Preliminaries

This section is concerned with some lemmas which will be used for our main results. Consider the following difference equation:

$$y(n+1) = ay(n) + b, \tag{9}$$

where a, b are positive constants.

Lemma 1 (see [20]). *Assume that $|a| < 1$, and, for any initial value $y(0)$, there exists a unique solution $y(n)$ of (9), which can be expressed as follows:*

$$y(n) = a^n \left(y(0) - y^*\right) + y^*, \tag{10}$$

where $y^* = b/(1-a)$. Thus, for any solution $y(n)$ of (10), we have

$$\lim_{n\to+\infty} y(n) = y^*. \tag{11}$$

Lemma 2 (see [20]). *Let $n \in N^+_{n_0} = \{n_0, n_0+1, \cdots, n_0+l, \cdots\}$ and $r \geq 0$. For any fixed n, $g(n, r)$ is nondecreasing function with respect to r, and, for $n \geq n_0$, the following inequalities hold:*

$$\begin{aligned} y(n+1) &\leq g(n, y(n)), \\ u(n+1) &\geq g(n, u(n)). \end{aligned} \tag{12}$$

If $y(n_0) \leq u(n_0)$, then $y(n) \leq u(n)$ for all $n \geq n_0$.

Lemma 3 (see [5]). *Assume that $r(n) > 0$, $x(n)$ satisfies $x(n) > 0$, and*

$$x(n+1) \leq x(n) \exp\{r(n)(1 - ax(n))\}, \tag{13}$$

for $n \in [n_1, +\infty)$, where a is a positive constant. Then

$$\limsup_{n\to+\infty} x(n) \leq \frac{1}{ar^u} \exp\left(r^u - 1\right). \tag{14}$$

Lemma 4 (see [5]). *Assume that $r(n) > 0$, $x(n)$ satisfies $x(n) > 0$, and*

$$x(n+1) \geq x(n) \exp\{r(n)(1 - ax(n))\}, \tag{15}$$

for $n \in [n_1, +\infty)$, $\limsup_{n\to+\infty} x(n) \leq x^*$, and $x(n_1) > 0$, where a and x^* are positive constants such that $ax^* > 1$. Then

$$\liminf_{n\to+\infty} x(n) \geq \frac{1}{a} \exp\left(r^u\left(1 - ax^*\right)\right). \tag{16}$$

Similarly, according to the proof of lemma 2.3 in [5], we have the following lemma.

Lemma 5. *Let $x : Z \to R$ be a nonnegative bounded sequences, and let $H : N \to R$ be a nonnegative sequence such that $\sum_{n=0}^{+\infty} H(n) = 1$. For any fixed $k \in N^+$, then*

$$\begin{aligned} \liminf_{n\to+\infty} x^k(n) &\leq \liminf_{n\to+\infty} \sum_{s=-\infty}^{n} H(n-s)\, x^k(s) \\ &\leq \limsup_{n\to+\infty} \sum_{s=-\infty}^{n} H(n-s)\, x^k(s) \\ &\leq \limsup_{n\to+\infty} x^k(n). \end{aligned} \tag{17}$$

If $m \leq \liminf_{n\to+\infty} x(n) \leq \limsup_{n\to+\infty} x(n) \leq M$ holds, then

$$m^k \leq \liminf_{n\to+\infty} x^k(n) \leq \limsup_{n\to+\infty} x^k(n) \leq M^k. \tag{18}$$

Proof. Let $x^* = \limsup_{n\to+\infty} x^k(n)$ and $\sup\{x^k(n) : n \in Z\} < M$. Given $\varepsilon > 0$, let N be an integer such that, for all $n \geq N$,

$$\begin{aligned} x^k(n) &< x^* + \frac{\varepsilon}{2}, \\ \sum_{s=n}^{+\infty} H(s) &< \frac{\varepsilon}{2M}. \end{aligned} \tag{19}$$

Therefore, for all $n \geq 2N$,

$$\begin{aligned} \sum_{s=-\infty}^{n} H(n-s)\, x^k(s) &\leq \sum_{s=-\infty}^{N} H(n-s)\, x^k(s) + \sum_{s=N}^{n} H(n-s)\, x^k(s) \\ &< \sum_{s=-\infty}^{N} H(n-s)\, M + \left(x^* + \frac{\varepsilon}{2}\right) \sum_{s=N}^{n} H(n-s) \\ &= \sum_{s=n-N}^{+\infty} H(s)\, M + \left(x^* + \frac{\varepsilon}{2}\right) \sum_{s=0}^{n-N} H(s) \\ &< M \frac{\varepsilon}{2M} + \left(x^* + \frac{\varepsilon}{2}\right) = x^* + \varepsilon. \end{aligned} \tag{20}$$

Then

$$\limsup_{n\to+\infty}\sum_{s=-\infty}^{n}H(n-s)\,x^{k}(s)\le x^{*}+\varepsilon. \tag{21}$$

Setting $\varepsilon\to 0$, we have

$$\limsup_{n\to+\infty}\sum_{s=-\infty}^{n}H(n-s)\,x^{k}(s)\le \limsup_{n\to+\infty}x^{k}(n). \tag{22}$$

Let $x_{*}=\liminf_{n\to+\infty}x^{k}(n)$. If $x_{*}=0$, the result is trivial. If $x_{*}>0$, then, given $\varepsilon\in(0,x_{*})$, there exists an integer N such that, for all $n\ge N$,

$$\begin{gathered}x^{k}(n)>x_{*}-\frac{\varepsilon}{2},\\ \sum_{s=0}^{N}H(s)>1-\varepsilon.\end{gathered} \tag{23}$$

Therefore, for all $n\ge 2N$,

$$\begin{aligned}\sum_{s=-\infty}^{n}H(n-s)\,x^{k}(s)&=\sum_{s=-\infty}^{N-1}H(n-s)\,x^{k}(s)\\&\quad+\sum_{s=N}^{n}H(n-s)\,x^{k}(s)\\&\ge\sum_{s=N}^{n}H(n-s)\,x^{k}(s)\\&=\left(x_{*}-\frac{\varepsilon}{2}\right)\sum_{s=0}^{n-N}H(s)\\&>\left(x_{*}-\frac{\varepsilon}{2}\right)(1-\varepsilon).\end{aligned} \tag{24}$$

Then

$$\liminf_{n\to+\infty}\sum_{s=-\infty}^{n}H(n-s)\,x^{k}(s)\ge\left(x_{*}-\frac{\varepsilon}{2}\right)(1-\varepsilon). \tag{25}$$

Setting $\varepsilon\to 0$, we have

$$\liminf_{n\to+\infty}\sum_{s=-\infty}^{n}H(n-s)\,x^{k}(s)\ge\liminf_{n\to+\infty}x^{k}(n). \tag{26}$$

If $\limsup_{n\to+\infty}x(n)\le M$, Given $\varepsilon>0$, there exists an integer N such that, for all $n>N$, $x(n)<M+\varepsilon$, therefore, $x^{k}(n)<(M+\varepsilon)^{k}$, and then $\limsup_{n\to+\infty}x^{k}(n)\le(M+\varepsilon)^{k}$. Setting $\varepsilon\to 0$, we have $\limsup_{n\to+\infty}x^{k}(n)\le M^{k}$.

If $\liminf_{n\to+\infty}x(n)\ge m$ and if $m=0$, the result is trivial. If $m>0$, given $\varepsilon\in(0,m)$, there exists an integer N such that, for all $n>N$, $x(n)>m-\varepsilon$, therefore, $x^{k}(n)>(m-\varepsilon)^{k}$, and then $\liminf_{n\to+\infty}x^{k}(n)\ge(m-\varepsilon)^{k}$. Setting $\varepsilon\to 0$, we have $\limsup_{n\to+\infty}x^{k}(n)\ge m^{k}$.

This ends the proof of Lemma 5. □

3. Permanence

Concerned with the persistent property of system (5), we have the following result.

Theorem 6. *Assume that*

$$r_i^{l}-a_{ij}^{u}M_j-b_{ij}^{u}M_j^{2}-c_i^{u}B>0,\quad i,j=1,2;\ i\ne j, \tag{27}$$

holds, and, then, for any positive solution $(x_1(n),x_2(n),u(n))$ of system (5), we have

$$\begin{gathered}m_i\le\liminf_{n\to+\infty}x_i(n)\le\limsup_{n\to+\infty}x_i(n)\le M_i,\\ A\le\liminf_{n\to+\infty}u(n)\le\limsup_{n\to+\infty}u(n)\le B,\quad i=1,2,\end{gathered} \tag{28}$$

where

$$\begin{aligned}D_i&=\frac{a_{ii}^{u}}{r_i^{l}-a_{ij}^{u}M_j-b_{ij}^{u}M_j^{2}-c_i^{u}B},\\ m_i&=\frac{1}{D_i}\cdot\exp\left(\frac{a_{ii}^{u}}{D_i}-a_{ii}^{u}M_i\right),\\ M_i&=\frac{1}{a_{ii}^{l}}\exp\left(r_i^{u}-1\right),\\ A&=\frac{\sum_{i=1}^{2}d_i^{l}m_i}{e^{u}},\\ B&=\frac{\sum_{i=1}^{2}d_i^{u}M_i}{e^{l}},\\ &\qquad i,j=1,2;\ i\ne j.\end{aligned} \tag{29}$$

Proof. From the first and second equations of system (5), we have

$$x_i(n+1)=x_i(n)\exp\left[r_i(n)\left(1-\frac{a_{ii}^{l}}{r_i^{u}}x_i(n)\right)\right],\quad i=1,2. \tag{30}$$

And so, from Lemma 3, for any solution $(x_1(n),x_2(n),u(n))$ of system (5), we can obtain

$$\limsup_{n\to+\infty}x_i(n)\le\frac{1}{a_{ii}^{l}}\exp\left(r_i^{u}-1\right)\overset{\text{def}}{=}M_i,\quad i=1,2. \tag{31}$$

According to Lemma 5 and the above inequality, for $i, j = 1, 2\ (i \neq j)$, one has

$$\begin{aligned}
&\limsup_{n\to+\infty}\sum_{s=0}^{+\infty}K_{ji}(s)\,x_i(n-s) \\
&\quad = \limsup_{n\to+\infty}\sum_{s=-\infty}^{n}K_{ji}(n-s)\,x_i(s) \le \limsup_{n\to+\infty}x_i(n) \\
&\quad \le M_i, \\
&\limsup_{n\to+\infty}\sum_{s=0}^{+\infty}P_i(s)\,x_i(n-s) \\
&\quad = \limsup_{n\to+\infty}\sum_{s=-\infty}^{n}P_i(n-s)\,x_i(s) \le \limsup_{n\to+\infty}x_i(n) \\
&\quad \le M_i, \\
&\limsup_{n\to+\infty}\sum_{s=0}^{+\infty}H_{ji}(s)\,x_i^2(n-s) \\
&\quad = \limsup_{n\to+\infty}\sum_{s=-\infty}^{n}H_{ji}(n-s)\,x_i^2(s) \le \limsup_{n\to+\infty}x_i^2(n) \\
&\quad \le M_i^2.
\end{aligned} \tag{32}$$

For any $\varepsilon > 0$, there exists a positive integer N_1 such that, for all $n > N_1$,

$$\begin{aligned}
&x_i(n) \le M_i + \varepsilon, \\
&\sum_{s=0}^{+\infty}K_{ji}(s)\,x_i(n-s) \le M_i + \varepsilon, \\
&\sum_{s=0}^{+\infty}P_i(s)\,x_i(n-s) \le M_i + \varepsilon, \\
&\sum_{s=0}^{+\infty}H_{ji}(s)\,x_i^2(n-s) \le M_i^2 + \varepsilon, \quad i, j = 1, 2;\ i \neq j.
\end{aligned} \tag{33}$$

By the third equation of system (5) and (33), we have

$$u(n+1) \le \left(1 - e^l\right)u(n) + \sum_{i=1}^{2}d_i^u\left(M_i + \varepsilon\right). \tag{34}$$

Hence, by applying Lemma 1 and Lemma 2 to (34), we obtain

$$\limsup_{n\to+\infty}u(n) \le \frac{\sum_{i=1}^{2}d_i^u\left(M_i+\varepsilon\right)}{e^l}. \tag{35}$$

Setting $\varepsilon \to 0$, it follows that

$$\limsup_{n\to+\infty}u(n) \le \frac{\sum_{i=1}^{2}d_i^u M_i}{e^l} \overset{\text{def}}{=} B. \tag{36}$$

According to Lemma 5 and the above inequality,

$$\begin{aligned}
&\limsup_{n\to+\infty}\sum_{s=0}^{+\infty}Q_i(s)\,u(n-s) \\
&\quad = \limsup_{n\to+\infty}\sum_{s=-\infty}^{n}Q_i(n-s)\,u(s) \le \limsup_{n\to+\infty}u(n) \le B.
\end{aligned} \tag{37}$$

Condition (27) implies that, for enough small positive constant ε, the following inequalities hold:

$$\begin{aligned}
&r_i^l - a_{ij}^u\left(M_j + \varepsilon\right) - b_{ij}^u\left(M_j^2 + \varepsilon\right) - c_i^u\left(B + \varepsilon\right) > 0, \\
&\qquad\qquad i, j = 1, 2;\ i \neq j.
\end{aligned} \tag{38}$$

It follows from (37) that there exists a positive integer $N_2 > N_1$ such that, for all $n > N_2$,

$$\begin{aligned}
&u(n) \le B + \varepsilon, \\
&\sum_{s=0}^{+\infty}Q_i(s)\,u(n-s) \le B + \varepsilon.
\end{aligned} \tag{39}$$

Thus, for all $n > N_2$, from (33), (39), and the first two equations of system (5), we have

$$\begin{aligned}
&x_i(n+1) \ge x_i(n) \\
&\exp\left[\left(r_i^l - a_{ij}^u\left(M_j+\varepsilon\right) - b_{ij}^u\left(M_j^2+\varepsilon\right) - c_i^u\left(B+\varepsilon\right)\right)(1 \\
&\quad - D_i^\varepsilon x_i(n))\right],
\end{aligned} \tag{40}$$

where $D_i^\varepsilon = a_{ii}^u/(r_i^l - a_{ij}^u(M_j+\varepsilon) - b_{ij}^u(M_j^2+\varepsilon) - c_i^u(B+\varepsilon))$ for $i, j = 1, 2, i \neq j$. Noting the fact that $\exp(x-1) > x$, for $x > 0$, we have

$$\frac{\exp\left(r_i^u - 1\right)}{r_i^l - a_{ij}^u\left(M_j+\varepsilon\right) - b_{ij}^u\left(M_j^2+\varepsilon\right) - c_i^u\left(B+\varepsilon\right)} > 1, \tag{41}$$

and then

$$\begin{aligned}
&D_i^\varepsilon \cdot M_i \\
&\quad = \frac{\exp\left(r_i^u - 1\right)}{r_i^l - a_{ij}^u\left(M_j+\varepsilon\right) - b_{ij}^u\left(M_j^2+\varepsilon\right) - c_i^u\left(B+\varepsilon\right)} \\
&\quad \cdot \frac{a_{ii}^u}{a_{ii}^l} > 1.
\end{aligned} \tag{42}$$

Hence, according to Lemma 4,

$$\begin{aligned}
\liminf_{n\to+\infty}x_i(n) \ge{}& \frac{1}{D_i^\varepsilon}\cdot\exp\left(r_i^l - a_{ij}^u\left(M_j+\varepsilon\right)\right. \\
&\left. - b_{ij}^u\left(M_j^2+\varepsilon\right) - c_i^u\left(B+\varepsilon\right) - a_{ii}^u M_i\right).
\end{aligned} \tag{43}$$

Setting $\varepsilon \to 0$, it follows that

$$\begin{aligned}&\liminf_{n\to+\infty} x_i(n)\\&\quad\geq \frac{3}{4}\cdot\frac{1}{D_i}\cdot\exp\left(r_i^l - a_{ij}^u M_j - b_{ij}^u M_j^2 - c_i^u B - a_{ii}^u M_i\right)\\&\quad\stackrel{\text{def}}{=} m_i,\end{aligned}\tag{44}$$

where $D_i = a_{ii}^u/(r_i^l - a_{ij}^u M_j - b_{ij}^u M_j^2 - c_i^u B)$ for $i, j = 1, 2, i \neq j$.

According to Lemma 5, from (44) we have that, for any $\varepsilon_1 > 0$ small enough (without loss of generality, assume that $\varepsilon_1 \leq (1/2)\min\{m_1, m_2\}$), there exists an $N_3 > N_2$, such that

$$\sum_{s=0}^{+\infty} P_i(s)\, x_i(n-s) \geq m_i - \varepsilon_1, \quad \forall n > N_3,\ i = 1, 2. \tag{45}$$

For $n > N_3$, from (45) and the last equation of system (5), we have

$$u(n+1) \geq \left(1 - e^u\right) u(n) + \sum_{i=1}^{2} d_i^l \left(m_i - \varepsilon_1\right). \tag{46}$$

Hence, by applying Lemmas 1 and 2 to (44),

$$\liminf_{n\to+\infty} u(n) \geq \frac{\sum_{i=1}^{2} d_i^l \left(m_i - \varepsilon_1\right)}{e^u}. \tag{47}$$

Setting $\varepsilon_1 \to 0$, it follows that

$$\liminf_{n\to+\infty} u(n) \geq \frac{3}{4}\cdot\frac{\sum_{i=1}^{2} d_i^l m_i}{e^u} \stackrel{\text{def}}{=} A. \tag{48}$$

This ends the proof Theorem 6. □

4. Global Attractivity

Concerned with the stability property of system (5), we have the following result.

Theorem 7. *Assume that there exist positive constants α_1, α_2, and α_3, such that*

$$\begin{aligned}&\alpha_1 A_{11} - \alpha_2\left(a_{21}^u + 2b_{21}^u M_1\right) - \alpha_3 d_1^u > 0,\\&\alpha_2 A_{22} - \alpha_1\left(a_{12}^u + 2b_{12}^u M_2\right) - \alpha_3 d_2^u > 0,\\&\alpha_3 e^l - \alpha_1 c_1^u - \alpha_2 c_2^u > 0\end{aligned}\tag{49}$$

hold; then, for any two positive solutions $(x_1(n), x_2(n), u(n))$ and $(x_1^*(n), x_2^*(n), u^*(n))$ of system (5), we have

$$\begin{aligned}&\lim_{n\to+\infty}\left(x_i(n) - x_i^*(n)\right) = 0,\\&\lim_{n\to+\infty}\left(u(n) - u^*(n)\right) = 0,\\&\qquad i = 1, 2,\end{aligned}\tag{50}$$

where

$$A_{ii} = \min\left\{a_{ii}^l, \frac{2}{M_i} - a_{ii}^u\right\} \quad i = 1, 2. \tag{51}$$

Proof. By (49), we can choose enough small positive constants δ and ε such that

$$\begin{aligned}&\alpha_1 A_{11}^{\varepsilon} - \alpha_2\left(a_{21}^u + 2b_{21}^u M_1\right) - \alpha_3 d_1^u > \delta,\\&\alpha_2 A_{22}^{\varepsilon} - \alpha_1\left(a_{12}^u + 2b_{12}^u M_2\right) - \alpha_3 d_2^u > \delta,\\&\alpha_3 e^l - \alpha_1 c_1^u - \alpha_2 c_2^u > \delta,\end{aligned}\tag{52}$$

where

$$A_{ii}^{\varepsilon} = \min\left\{a_{ii}^l, \frac{2}{M_i + \varepsilon} - a_{ii}^u\right\} \quad i = 1, 2. \tag{53}$$

Let $(x_1(n), x_2(n), u(n))$ be any positive solution of system (5). For the above ε, from (31) and (36), there exists an enough large integer $n^* > N_1$, such that

$$\begin{aligned}&x_i(n) < M_i + \varepsilon,\\&u(n) < B + \varepsilon,\\&\qquad \forall n > n^*,\ i = 1, 2.\end{aligned}\tag{54}$$

Now, let us define a Lyapunov functional

$$V(n) = \sum_{i=1}^{3} \alpha_i V_i(n), \tag{55}$$

where α_i, $i = 1, 2, 3$ are positive constants and

$$\begin{aligned}V_1(n) ={}& \left|\ln x_1(n) - \ln x_1^*(n)\right|\\&+ \sum_{s=0}^{+\infty} K_{12}(s) \sum_{r=n-s}^{n-1} a_{12}(r+s)\left|x_2(r) - x_2^*(r)\right|\\&+ \sum_{s=0}^{+\infty} H_{12}(s) \sum_{r=n-s}^{n-1} b_{12}(r+s)\left|x_2^2(r) - x_2^{*2}(r)\right|\\&+ \sum_{s=0}^{+\infty} Q_1(s) \sum_{r=n-s}^{n-1} c_1(r+s)\left|u(r) - u^*(r)\right|,\\V_2(n) ={}& \left|\ln x_2(n) - \ln x_2^*(n)\right|\\&+ \sum_{s=0}^{+\infty} K_{21}(s) \sum_{r=n-s}^{n-1} a_{21}(r+s)\left|x_1(r) - x_1^*(r)\right|\\&+ \sum_{s=0}^{+\infty} H_{21}(s) \sum_{r=n-s}^{n-1} b_{21}(r+s)\left|x_1^2(r) - x_1^{*2}(r)\right|\\&+ \sum_{s=0}^{+\infty} Q_2(s) \sum_{r=n-s}^{n-1} c_2(r+s)\left|u(r) - u^*(r)\right|,\\V_3(n) ={}& \left|u(n) - u^*(n)\right|\\&+ \sum_{s=0}^{+\infty} P_1(s) \sum_{r=n-s}^{n-1} d_1(r+s)\left|x_1(r) - x_1^*(r)\right|\\&+ \sum_{s=0}^{+\infty} P_2(s) \sum_{r=n-s}^{n-1} d_2(r+s)\left|x_2(r) - x_2^*(r)\right|.\end{aligned}\tag{56}$$

Then, from the definition of $V_i(n)$, $i = 1, 2, 3$, one can easily see that $V_i(n) > 0$ for all $n \in Z^+$. Also, for any fixed $n^* \in Z^+$,

$$\begin{aligned} V(n^*) \le{} & \alpha_1 \Big\{ \left|\ln x_1(n^*) - \ln x_1^*(n^*)\right| \\ & + a_{12}^u \sup_{r\in Z^+, r\le n^*} \left|x_2(r) - x_2^*(r)\right| \sum_{s=0}^{+\infty} K_{12}(s)\, s \\ & + b_{12}^u \sup_{r\in Z^+, r\le n^*} \left|x_2^2(r) - x_2^{*2}(r)\right| \sum_{s=0}^{+\infty} H_{12}(s)\, s \\ & + c_1^u \sup_{r\in Z^+, r\le n^*} \left|u(r) - u^*(r)\right| \sum_{s=0}^{+\infty} Q_1(s)\, s \Big\} \\ & + \alpha_2 \Big\{ \left|\ln x_2(n^*) - \ln x_2^*(n^*)\right| \\ & + a_{21}^u \sup_{r\in Z^+, r\le n^*} \left|x_1(r) - x_1^*(r)\right| \sum_{s=0}^{+\infty} K_{21}(s)\, s \\ & + b_{21}^u \sup_{r\in Z^+, r\le n^*} \left|x_1^2(r) - x_1^{*2}(r)\right| \sum_{s=0}^{+\infty} H_{21}(s)\, s \\ & + c_2^u \sup_{r\in Z^+, r\le n^*} \left|u(r) - u^*(r)\right| \sum_{s=0}^{+\infty} Q_2(s)\, s \Big\} \\ & + \alpha_3 \Big\{ \left|u(n) - u^*(n)\right| \\ & + d_1^u \sup_{r\in Z^+, r\le n^*} \left|x_1(r) - x_1^*(r)\right| \sum_{s=0}^{+\infty} P_1(s)\, s \\ & + d_2^u \sup_{r\in Z^+, r\le n^*} \left|x_2(r) - x_2^*(r)\right| \sum_{s=0}^{+\infty} P_2(s)\, s \Big\} < +\infty. \end{aligned} \tag{57}$$

Also, from the first equation of system (5) and using the Mean Value Theorem, for all $n > n^*$,

$$\begin{aligned} & \Delta V_1(n) \\ & \le \left|\ln x_1(n) - \ln x_1^*(n) - a_{11}(n)\left(x_1(n) - x_1^*(n)\right)\right| \\ & \quad - \left|\ln x_1(n) - \ln x_1^*(n)\right| \\ & \quad + \sum_{s=0}^{+\infty} K_{12}(s)\, a_{12}(n+s) \left|x_2(n) - x_2^*(n)\right| \\ & \quad + \sum_{s=0}^{+\infty} H_{12}(s)\, b_{12}(n+s) \left|x_2^2(n) - x_2^{*2}(n)\right| \\ & \quad + \sum_{s=0}^{+\infty} Q_1(s)\, c_1(n+s) \left|u(n) - u*(n)\right| \\ & \le -\left(\frac{1}{\varphi_1(n)} - \left|\frac{1}{\varphi_1(n)} - a_{11}(n)\right|\right) \left|x_1(n) - x_1^*(n)\right| \\ & \quad + a_{12}^u \left|x_2(n) - x_2^*(n)\right| \\ & \quad + b_{12}^u \left|x_2(n) + x_2^*(n)\right| \left|x_2(n) - x_2^*(n)\right| \\ & \quad + c_1^u \left|u(n) - u*(n)\right| \\ & \le -\min\left\{a_{11}^l, \frac{2}{M_1+\varepsilon} - a_{11}^u\right\} \left|x_1(n) - x_1^*(n)\right| \\ & \quad + \left(a_{12}^u + 2b_{12}^u\left(M_2+\varepsilon\right)\right) \left|x_2(n) - x_2^*(n)\right| \\ & \quad + c_1^u \left|u(n) - u^*(n)\right|. \end{aligned} \tag{58}$$

Similarly to the analysis of (58), we can obtain

$$\begin{aligned} & \Delta V_2(n) \\ & \quad \le -\min\left\{a_{22}^l, \frac{2}{M_2+\varepsilon} - a_{22}^u\right\} \left|x_2(n) - x_2^*(n)\right| \\ & \qquad + \left(a_{21}^u + 2b_{21}^u\left(M_1+\varepsilon\right)\right) \left|x_1(n) - x_1^*(n)\right| \\ & \qquad + c_2^u \left|u(n) - u^*(n)\right|, \\ & \Delta V_3(n) \\ & \quad \le -e^l \left|u(n) - u^*(n)\right| + \sum_{i=1}^{2} d_i^u \left|x_i(n) - x_i^*(n)\right|, \end{aligned} \tag{59}$$

where $\varphi_i(n)$ lies between $x_i(n)$ and $x_i^*(n)$, $i = 1, 2$.

From (58) and (59), we have

$$\begin{aligned} & \Delta V(n) \\ & \quad \le -\left\{\alpha_1 A_{11}^{\varepsilon} - \alpha_2 a_{21}^u - 2\alpha_2 b_{21}^u\left(M_1+\varepsilon\right) - \alpha_3 d_1^u\right\} \\ & \quad \cdot \left|x_1(n) - x_1^*(n)\right| \\ & \quad - \left\{\alpha_2 A_{22}^{\varepsilon} - \alpha_1 a_{12}^u - 2\alpha_1 b_{12}^u\left(M_2+\varepsilon\right) - \alpha_3 d_2^u\right\} \\ & \quad \cdot \left|x_2(n) - x_2^*(n)\right| - \left\{\alpha_3 e^l - \alpha_1 c_1^u - \alpha_2 c_2^u\right\} \\ & \quad \cdot \left|u(n) - u^*(n)\right| \\ & \quad \le -\delta\left(\sum_{i=1}^{2}\left|x_i(n) - x_i^*(n)\right| + \left|u(n) - u^*(n)\right|\right). \end{aligned} \tag{60}$$

Summating both sides of the above inequalities from n^* to n,

$$\begin{aligned} & \sum_{p=n^*}^{n} \left(V(p+1) - V(p)\right) \\ & \quad \le -\delta \sum_{p=n^*}^{n} \left(\sum_{i=1}^{2}\left|x_i(p) - x_i^*(p)\right| + \left|u(p) - u^*(p)\right|\right). \end{aligned} \tag{61}$$

Hence,

$$\begin{aligned} & V(n+1) \\ & \quad + \delta \sum_{p=n^*}^{n} \left(\sum_{i=1}^{2} \left| x_i(p) - x_i^*(p) \right| + \left| u(p) - u^*(p) \right| \right) \\ & \quad \le V(n^*) < +\infty. \end{aligned} \tag{62}$$

Then, we have

$$\begin{aligned} & \sum_{p=n^*}^{n} \left(\sum_{i=1}^{2} \left| x_i(p) - x_i^*(p) \right| + \left| u(p) - u^*(p) \right| \right) \\ & \quad \le \frac{V(n^*)}{\delta}. \end{aligned} \tag{63}$$

Therefore

$$\begin{aligned} & \sum_{p=n^*}^{+\infty} \left(\sum_{i=1}^{2} \left| x_i(p) - x_i^*(p) \right| + \left| u(p) - u^*(p) \right| \right) \\ & \quad \le +\infty, \end{aligned} \tag{64}$$

which means that

$$\lim_{n\to+\infty} \left(\sum_{i=1}^{2} \left| x_i(n) - x_i^*(n) \right| + \left| u(n) - u^*(n) \right| \right) = 0. \tag{65}$$

Consequently

$$\begin{aligned} & \lim_{n\to+\infty} \left(u(n) - u^*(n) \right) = 0, \\ & \lim_{n\to+\infty} \left(x_i(n) - x_i^*(n) \right) = 0, \quad i = 1, 2. \end{aligned} \tag{66}$$

This completes the proof of Theorem 7. □

5. Extinction

Concerned with the extinction property of system (5), we have the following results.

Theorem 8. *Assume that*

$$\frac{r_1^u}{r_2^l} < \min \left\{ \frac{a_{12}^l e^l + c_1^l d_2^u}{a_{22}^u e^l + c_2^u d_2^u}, \frac{a_{11}^l e^l + c_1^l d_1^u}{\left(a_{21}^u + b_{21}^u M_1\right) e^l + c_2^u d_1^u} \right\}, \tag{67}$$

hold, let $x_1(n), x_2(n), u(n)$ be any positive solution of system (5), and then

$$\lim_{n\to+\infty} x_1(n) = 0, \tag{68}$$

where M_1 is defined in Theorem 6.

Theorem 9. *Assume that*

$$\frac{r_2^u}{r_1^l} < \min \left\{ \frac{a_{21}^l e^l + c_2^l d_1^u}{a_{11}^u e^l + c_1^u d_1^u}, \frac{a_{22}^l e^l + c_2^l d_2^u}{\left(a_{12}^u + b_{12}^u M_2\right) e^l + c_1^u d_2^u} \right\}, \tag{69}$$

hold, let $x_1(n), x_2(n), u(n)$ be any positive solution of system (5), and then

$$\lim_{n\to+\infty} x_2(n) = 0, \tag{70}$$

where M_2 is defined in Theorem 6.

Proof of Theorem 8. By condition (67), we can choose positive constants β_1 and β_2 such that

$$\begin{aligned} \frac{r_1^u}{r_2^l} & < \frac{\beta_2}{\beta_1} \\ & < \min \left\{ \frac{a_{12}^l e^l + c_1^l d_2^u}{a_{22}^u e^l + c_2^u d_2^u}, \frac{a_{11}^l e^l + c_1^l d_1^u}{\left(a_{21}^u + b_{21}^u M_1\right) e^l + c_2^u d_1^u} \right\}. \end{aligned} \tag{71}$$

Thus, there exists a positive constant η such that

$$\begin{aligned} & \beta_1 r_1^u - \beta_2 r_2^l < -\eta < 0, \\ & -\beta_1 a_{11}^l + \beta_2 \left(a_{21}^u + b_{21}^u M_1\right) + \frac{\left(-\beta_1 c_1^l + \beta_2 c_2^u\right) d_1^u}{e^l} < 0, \\ & \beta_2 a_{22}^u - \beta_1 a_{12}^l + \frac{\left(-\beta_1 c_1^l + \beta_2 c_2^u\right) d_2^u}{e^l} < 0. \end{aligned} \tag{72}$$

There exists a constant β_3 such that $(-\beta_1 c_1^l + \beta_2 c_2^u)/e^l < \beta_3$. Thus, for enough small positive constant ε, we have

$$\begin{aligned} -\beta_1 a_{11}^l + \beta_2 \left(a_{21}^u + b_{21}^u \left(M_1 + \varepsilon\right)\right) + \beta_3 d_1^u & < 0, \\ \beta_2 a_{22}^u - \beta_1 a_{12}^l + \beta_3 d_2^u & < 0, \\ -\beta_3 e^l - \beta_1 c_1^l + \beta_2 c_2^u & < 0. \end{aligned} \tag{73}$$

Consider the following Lyapunov functional:

$$\begin{aligned} V_4(n) = {} & x_1^{\beta_1}(n)\, x_2^{-\beta_2}(n) \\ & \cdot \exp \Bigg\{ -\beta_1 \Bigg[\sum_{s=0}^{+\infty} K_{12}(s) \sum_{r=n-s}^{n-1} a_{12}(r+s)\, x_2(r) \\ & + \sum_{s=0}^{+\infty} H_{12}(s) \sum_{r=n-s}^{n-1} b_{12}(r+s)\, x_2^2(r) \\ & + \sum_{s=0}^{+\infty} Q_1(s) \sum_{r=n-s}^{n-1} c_1(r+s)\, u(r) \Bigg] \\ & + \beta_2 \Bigg[\sum_{s=0}^{+\infty} K_{21}(s) \sum_{r=n-s}^{n-1} a_{21}(r+s)\, x_1(r) \\ & + \sum_{s=0}^{+\infty} H_{21}(s) \sum_{r=n-s}^{n-1} b_{21}(r+s)\, x_1^2(r) \\ & + \sum_{s=0}^{+\infty} Q_2(s) \sum_{r=n-s}^{n-1} c_2(r+s)\, u(r) \Bigg] + \beta_3 \Bigg[u(n) \\ & + \sum_{i=1}^{2} \sum_{s=0}^{+\infty} P_i(s) \sum_{r=n-s}^{n-1} d_i(r+s)\, x_i(r) \Bigg] \Bigg\}. \end{aligned} \tag{74}$$

From (74), we obtain

$$\begin{aligned}\frac{V_4(n+1)}{V_4(n)} &= \exp\Bigg\{(\beta_1 r_1(n) - \beta_2 r_2(n)) \\ &+ \beta_2 \sum_{s=0}^{+\infty} H_{21}(s)\, b_{21}(n+s)\, x_1^2(n) \\ &- \beta_1 \sum_{s=0}^{+\infty} H_{12}(s)\, b_{12}(n+s)\, x_2^2(n) + \Bigg(-\beta_1 a_{11}(n) \\ &+ \beta_2 \sum_{s=0}^{+\infty} K_{21}(s)\, a_{21}(n+s) + \beta_3 \sum_{s=0}^{+\infty} P_1(s)\, d_1(n+s)\Bigg) \\ &\cdot x_1(n) + \Bigg(\beta_2 a_{22}(n) - \beta_1 \sum_{s=0}^{+\infty} K_{12}(s)\, a_{12}(n+s) \\ &+ \beta_3 \sum_{s=0}^{+\infty} P_2(s)\, d_2(n+s)\Bigg) x_2(n) + \Bigg(-\beta_3 e(n) \\ &- \beta_1 \sum_{s=0}^{+\infty} Q_1(s)\, c_1(n+s) + \beta_2 \sum_{s=0}^{+\infty} Q_2(s)\, c_2(n+s)\Bigg) \\ &\cdot u(n)\Bigg\} \le \exp\Big\{\left(\beta_1 r_1^u - \beta_2 r_2^l\right) + \Big(-\beta_1 a_{11}^l \\ &+ \beta_2\left(a_{21}^u + b_{21}^u (M_1+\varepsilon)\right) + \beta_3 d_1^u\Big) x_1(n) + \Big(\beta_2 a_{22}^u \\ &- \beta_1 a_{12}^l + \beta_3 d_2^u\Big) x_2(n) + \left(-\beta_3 e^l - \beta_1 c_1^l + \beta_2 c_2^u\right) \\ &\cdot u(n)\Big\}.\end{aligned} \tag{75}$$

From (73) and (75), we can obtain

$$V_4(n+1) \le V_4(n) \exp(-\eta). \tag{76}$$

Therefore,

$$V_4(n) \le V_4(0) \exp(-n\eta). \tag{77}$$

From (31) and (36) we know that there exists an $M > 0$ such that

$$\max_{i=1,2}\left\{x_i(n), x_i^2(n)\, u(n)\right\} < M, \quad \forall n \in Z, \tag{78}$$

and so

$$\begin{aligned}V_4(0) &< x_1^{\beta_1}(0)\, x_2^{-\beta_2}(0) \exp\Bigg\{\beta_1 M \Bigg[a_{12}^u \sum_{s=0}^{+\infty} K_{12}(s)\, s \\ &+ b_{12}^u \sum_{s=0}^{+\infty} H_{12}(s)\, s + c_1^u \sum_{s=0}^{+\infty} Q_1(s)\, s\Bigg] \\ &+ \beta_2 M \Bigg[a_{21}^u \sum_{s=0}^{+\infty} K_{21}(s)\, s + b_{21}^u \sum_{s=0}^{+\infty} H_{21}(s)\, s \\ &+ c_2^u \sum_{s=0}^{+\infty} Q_2(s)\, s\Bigg] + \left|\beta_3\right| M \Bigg[1 + d_1^u \sum_{s=0}^{+\infty} P_1(s)\, s \\ &+ d_2^u \sum_{s=0}^{+\infty} P_2(s)\, s\Bigg]\Bigg\} = x_1^{\beta_1}(0)\, x_2^{-\beta_2}(0) \\ &\cdot \exp\Bigg\{\beta_1 M \left[a_{12}^u \Theta_{12} + b_{12}^u \Lambda_{12} + c_1^u \Delta_1\right] \\ &+ \beta_2 M \left[a_{21}^u \Theta_{21} + b_{21}^u \Lambda_{21} + c_2^u \Delta_2\right] \\ &+ \left|\beta_3\right| M \left[1 + \sum_{i=1}^{2} d_i^u \Omega_i\right]\Bigg\} < +\infty.\end{aligned} \tag{79}$$

On the other hand, we also have

$$\begin{aligned}V_4(n) &\ge x_1^{\beta_1}(n)\, x_2^{-\beta_2}(n) \\ &\cdot \exp\Bigg\{-\beta_1 \Bigg[\sum_{s=0}^{+\infty} K_{12}(s) \sum_{r=n-s}^{n-1} a_{12}(r+s)\, x_2(r) \\ &+ \sum_{s=0}^{+\infty} H_{12}(s) \sum_{r=n-s}^{n-1} b_{12}(r+s)\, x_2^2(r) \\ &+ \sum_{s=0}^{+\infty} Q_1(s) \sum_{r=n-s}^{n-1} c_1(r+s)\, u(r)\Bigg]\Bigg\} > x_1^{\beta_1}(n)\, M^{-\beta_2} \\ &\cdot \exp\left\{-\beta_1 M \left[a_{12}^u \Theta_{12} + b_{12}^u \Lambda_{12} + c_1^u \Delta_1\right]\right\}.\end{aligned} \tag{80}$$

Combining inequalities (77), (79), and (80),

$$x_1(n) \le \lambda \exp\left\{-\frac{\eta}{\beta_1} n\right\}, \tag{81}$$

where

$$\begin{aligned}\lambda &= \left(V_4(0)\right)^{1/\beta_1} M^{\beta_2/\beta_1} \\ &\cdot \exp\left\{\beta_1 M \left[a_{12}^u \Theta_{12} + b_{12}^u \Lambda_{12} + c_1^u \Delta_1\right]\right\} < +\infty.\end{aligned} \tag{82}$$

Hence we obtain that

$$\lim_{n\to+\infty} x_1(n) = 0. \tag{83}$$

This ends the proof of Theorem 8. □

Proof of Theorem 9. Define the following Lyapunov functional:

$$\begin{aligned} V_5(n) &= x_1^{-\lambda_1}(n)\, x_2^{\lambda_2}(n) \\ &\cdot \exp\left\{\lambda_1\left[\sum_{s=0}^{+\infty} K_{12}(s) \sum_{r=n-s}^{n-1} a_{12}(r+s)\, x_2(r)\right.\right. \\ &+ \sum_{s=0}^{+\infty} H_{12}(s) \sum_{r=n-s}^{n-1} b_{12}(r+s)\, x_2^2(r) \\ &\left.+ \sum_{s=0}^{+\infty} Q_1(s) \sum_{r=n-s}^{n-1} c_1(r+s)\, u(r)\right] \\ &- \lambda_2\left[\sum_{s=0}^{+\infty} K_{21}(s) \sum_{r=n-s}^{n-1} a_{21}(r+s)\, x_1(r)\right. \\ &+ \sum_{s=0}^{+\infty} H_{21}(s) \sum_{r=n-s}^{n-1} b_{21}(r+s)\, x_1^2(r) \\ &\left.+ \sum_{s=0}^{+\infty} Q_2(s) \sum_{r=n-s}^{n-1} c_2(r+s)\, u(r)\right] + \lambda_3\left[u(n)\right. \\ &\left.\left.+ \sum_{i=1}^{2}\sum_{s=0}^{+\infty} P_i(s) \sum_{r=n-s}^{n-1} d_i(r+s)\, x_i(r)\right]\right\}. \end{aligned} \tag{84}$$

Similarly to the analysis of the proof of Theorem 8, we have $\lim_{n\to+\infty} x_2(n) = 0$.

From Theorems 8 and 9 we know that, under some suitable assumption, one of the species in the system may be driven to extinction; in this case, one interesting problem is to investigate the stability property of the rest of the species.

Consider the following discrete equations:

$$\begin{aligned} x_2(n+1) &= x_2(n) \exp\left[r_2(n) - a_{22}(n)\, x_2(n)\right. \\ &\left.- c_2(n) \sum_{s=0}^{+\infty} Q_2(s)\, u(n-s)\right], \\ u(n+1) &= (1 - e(n))\, u(n) + d_2(n) \sum_{s=0}^{+\infty} P_2(s)\, x_2(n \\ &- s). \end{aligned} \tag{85}$$

$$\begin{aligned} x_1(n+1) &= x_1(n) \exp\left[r_1(n) - a_{11}(n)\, x_1(n)\right. \\ &\left.- c_1(n) \sum_{s=0}^{+\infty} Q_1(s)\, u(n-s)\right], \\ u(n+1) &= (1 - e(n))\, u(n) + d_1(n) \sum_{s=0}^{+\infty} P_1(s)\, x_1(n \\ &- s). \end{aligned} \tag{86}$$

□

Theorem 10. *Assume that (67) holds and also*

$$d_2^u < \frac{A_{22} e^l}{c_2^u} \tag{87}$$

holds; then, for any positive solution $(x_1(n), x_2(n), u(n))$ of system (5) and any positive solution $(x_2^*(n), u^*(n))$ of system (85), we have

$$\begin{aligned} \lim_{n\to+\infty} x_1(n) &= 0, \\ \lim_{n\to+\infty} \left(x_2(n) - x_2^*(n)\right) &= 0, \\ \lim_{n\to+\infty} \left(u(n) - u^*(n)\right) &= 0, \end{aligned} \tag{88}$$

where A_{22} is defined in Theorem 7.

Theorem 11. *Assume that (69) holds and also*

$$d_1^u < \frac{A_{11} e^l}{c_1^u} \tag{89}$$

holds; then, for any positive solution $(x_1(n), x_2(n), u(n))$ of system (5) and any positive solution $(x_1^*(n), u^*(n))$ of system (86), we have

$$\begin{aligned} \lim_{n\to+\infty} x_2(n) &= 0, \\ \lim_{n\to+\infty} \left(x_1(n) - x_1^*(n)\right) &= 0, \\ \lim_{n\to+\infty} \left(u(n) - u^*(n)\right) &= 0, \end{aligned} \tag{90}$$

where A_{11} is defined in Theorem 7.

Proof of Theorem 10. By condition (87), we can choose positive constants ω_1 and ω_2 such that

$$\frac{d_2^u}{A_{22}} < \frac{\omega_1}{\omega_2} < \frac{e^l}{c_2^u}. \tag{91}$$

Thus, there exist enough small positive constants ε and γ such that

$$\begin{aligned} \omega_1 A_{22}^{\varepsilon} - \omega_2 d_2^u &> \gamma, \\ \omega_2 e^l - \omega_1 c_2^u &> \gamma, \end{aligned} \tag{92}$$

where A_{22}^{ε} is defined in (53).

From (81), we have

$$x_1^2(n) \le \lambda^2 \exp\left\{-\frac{2\eta}{\beta_1} n\right\}. \tag{93}$$

By applying the Direct Comparison Test to (81) and (93), we obtain $\sum_{n=1}^{+\infty} x_1(n)$ and $\sum_{n=1}^{+\infty} x_1^2(n)$ are absolute convergence.

Now, we define a Lyapunov functional:

$$\begin{aligned} V_6(n) = {} & \omega_1 \Bigg[\left|\ln x_2(n) - \ln x_2^*(n)\right| \\ & + \sum_{s=0}^{+\infty} K_{21}(s) \sum_{r=n-s}^{n-1} a_{21}(r+s)\, x_1(r) \\ & + \sum_{s=0}^{+\infty} H_{21}(s) \sum_{r=n-s}^{n-1} b_{21}(r+s)\, x_1^2(r) \\ & + \sum_{s=0}^{+\infty} Q_2(s) \sum_{r=n-s}^{n-1} c_2(r+s) \left|u(r) - u^*(r)\right| \Bigg] \\ & + \omega_2 \Bigg[\left|u(n) - u^*(n)\right| \\ & + \sum_{s=0}^{+\infty} P_1(s) \sum_{r=n-s}^{n-1} d_1(r+s)\, x_1(r) \\ & + \sum_{s=0}^{+\infty} P_2(s) \sum_{r=n-s}^{n-1} d_2(r+s) \left|x_2(r) - x_2^*(r)\right| \Bigg] \end{aligned} \tag{94}$$

and one could easily see that $V_6(n) \ge 0$ for all $n \in Z^+$. Also, for any fixed $n^* \in Z^+$, from (94) one could see that

$$\begin{aligned} V_6(n^*) < {} & \omega_1 \Bigg[\left|\ln x_2(n^*) - \ln x_2^*(n^*)\right| \\ & + a_{21}^u M \sum_{s=0}^{+\infty} K_{21}(s)\, s + b_{21}^u M \sum_{s=0}^{+\infty} H_{21}(s)\, s \\ & + c_2^u \sup_{q\in Z^+, q\le n^*} \left|u(q) - u^*(q)\right| \sum_{s=0}^{+\infty} Q_2(s)\, s \Bigg] \\ & + \omega_2 \Bigg[\left|u(n^*) - u^*(n^*)\right| + d_1^u M \sum_{s=0}^{+\infty} P_1(s)\, s \\ & + d_2^u \sup_{q\in Z^+, q\le n^*} \left|x_2(q) - x_2^*(q)\right| \sum_{s=0}^{+\infty} P_2(s)\, s \Bigg] < +\infty. \end{aligned} \tag{95}$$

It follows from system (5) and (85) and the Mean Value Theorem that

$$\begin{aligned} & \Delta V_6(n) \\ & \le - \left[\omega_1 \left(\frac{1}{\varphi_2(n)} - \left| \frac{1}{\varphi_2(n)} - a_{22}(n) \right| \right) - \omega_2 d_2^u \right] \\ & \cdot \left|x_2(n) - x_2^*(n)\right| - \left(\omega_2 e^l - \omega_1 c_2^u\right) \left|u(n) - u^*(n)\right| \\ & + \left(\omega_1 a_{21}^u + \omega_2 d_1^u\right) x_1(n) + \omega_1 b_{21}^u x_1^2(n) \\ & \le - \left(\omega_1 A_{22}^{\varepsilon} - \omega_2 d_2^u\right) \left|x_2(n) - x_2^*(n)\right| \\ & - \left(\omega_2 e^l - \omega_1 c_2^u\right) \left|u(n) - u^*(n)\right| + \left(\omega_1 a_{21}^u + \omega_2 d_1^u\right) \\ & \cdot x_1(n) + \omega_1 b_{21}^u x_1^2(n) \\ & \le -\gamma \left(\left|x_2(n) - x_2^*(n)\right| + \left|u(n) - u^*(n)\right| \right) \\ & + \left(\omega_1 a_{21}^u + \omega_2 d_1^u\right) x_1(n) + \omega_1 b_{21}^u x_1^2(n). \end{aligned} \tag{96}$$

Summating both sides of the above inequality from n^* to n, we have

$$\begin{aligned} & \sum_{p=n^*}^{n} \left(V_6(p+1) - V_6(p)\right) \\ & \quad < -\gamma \sum_{p=n^*}^{n} \left(\left|x_2(p) - x_2^*(p)\right| + \left|u(p) - u^*(p)\right| \right) \\ & \quad + \sum_{p=n^*}^{n} \left(\left(\omega_1 a_{21}^u + \omega_2 d_1^u\right) x_1(p) + \omega_1 b_{21}^u x_1^2(p) \right). \end{aligned} \tag{97}$$

Hence

$$\begin{aligned} & V_6(n+1) \\ & \quad + \gamma \sum_{p=n^*}^{n} \left(\left|x_2(p) - x_2^*(p)\right| + \left|u(p) - u^*(p)\right| \right) \\ & \quad < V_6(n^*) \\ & \quad + \sum_{p=n^*}^{n} \left(\left(\omega_1 a_{21}^u + \omega_2 d_1^u\right) x_1(p) + \omega_1 b_{21}^u x_1^2(p) \right). \end{aligned} \tag{98}$$

Then, from (95) we have

$$\begin{aligned} & \sum_{p=n^*}^{n} \left(\left|x_2(p) - x_2^*(p)\right| + \left|u(p) - u^*(p)\right| \right) \\ & < \frac{V_6(n^*) + \sum_{p=n^*}^{n} \left(\left(\omega_1 a_{21}^u + \omega_2 d_1^u\right) x_1(p) + \omega_1 b_{21}^u x_1^2(p) \right)}{\gamma} \\ & < +\infty. \end{aligned} \tag{99}$$

Therefore

$$\sum_{p=n^*}^{+\infty} \left(\left|x_2(p) - x_2^*(p)\right| + \left|u(p) - u^*(p)\right| \right) < +\infty, \tag{100}$$

which means that

$$\lim_{n\to+\infty} \left(\left|x_2(n) - x_2^*(n)\right| + \left|u(n) - u^*(n)\right| \right) = 0. \tag{101}$$

Consequently

$$\begin{aligned} \lim_{n\to+\infty} \left(x_2(n) - x_2^*(n)\right) &= 0, \\ \lim_{n\to+\infty} \left(u(n) - u^*(n)\right) &= 0. \end{aligned} \tag{102}$$

This completes the proof of Theorem 10. □

Proof of Theorem 11. The proof of Theorem 11 is similar to that of Theorem 10, and we omit the details here. □

6. Numerical Simulations

In this section, we give an example to check the feasibility of our result.

Example 12. Consider the following system:

$$\begin{aligned}
x_1(n+1) = x_1(n)\exp\Bigg[&0.87 + 0.02\sin\left(\sqrt{2}n\right) \\
&- x_1(n) - \left(0.025 - 0.001\sin\left(\sqrt{3}n\right)\right) \\
&\cdot \sum_{s=0}^{+\infty}\frac{e-1}{e}e^{-s}x_2(n-s) \\
&- \left(0.015 + 0.002\sin\left(\sqrt{2}n\right)\right) \\
&\cdot \sum_{s=0}^{+\infty}\frac{e^2-1}{e^2}e^{-2s}x_2^2(n-s) \\
&- \left(0.25 + 0.05\cos\left(\sqrt{3}n\right)\right) \\
&\cdot \sum_{s=0}^{+\infty}\frac{e^3-1}{e^3}e^{-3s}u(n-s)\Bigg], \\
x_2(n+1) = x_2(n)\exp\Bigg[&0.95 + 0.03\cos\left(\sqrt{3}n\right) \\
&- x_2(n) - \left(0.02 + 0.001\cos\left(\sqrt{3}n\right)\right) \\
&\cdot \sum_{s=0}^{+\infty}\frac{e^4-1}{e^4}e^{-4s}x_1(n-s) \\
&- \left(0.027 + 0.002\cos\left(\sqrt{2}n\right)\right) \\
&\cdot \sum_{s=0}^{+\infty}\frac{e^5-1}{e^5}e^{-5s}x_1^2(n-s) \\
&- \left(0.35 + 0.05\sin\left(\sqrt{3}n\right)\right)\sum_{s=0}^{+\infty}\frac{e^6-1}{e^6}e^{-6s}u(n-s)\Bigg] \\
u(n+1) = \Big(1 - &\left(0.93 + 0.03\cos\left(\sqrt{3}n\right)\right)\Big)u(n) \\
&+ \left(0.17 + 0.05\cos\left(\sqrt{2}n\right)\right)\sum_{s=0}^{+\infty}\frac{e^7-1}{e^7}e^{-7s}x_1(n-s) \\
&+ \left(0.16 + 0.03\sin\left(\sqrt{3}n\right)\right)\sum_{s=0}^{+\infty}\frac{e^8-1}{e^8}e^{-8s}x_2(n-s),
\end{aligned} \tag{103}$$

One could easily see that conditions (H_1), (H_2), and (H_3) are satisfied. Also, by calculating, one has

$$\begin{aligned}
A_{11} &= \min\left\{a_{11}^l, \frac{2}{M_1} - a_{11}^u\right\} = 1, \\
A_{22} &= \min\left\{a_{22}^l, \frac{2}{M_2} - a_{22}^u\right\} = 1.
\end{aligned} \tag{104}$$

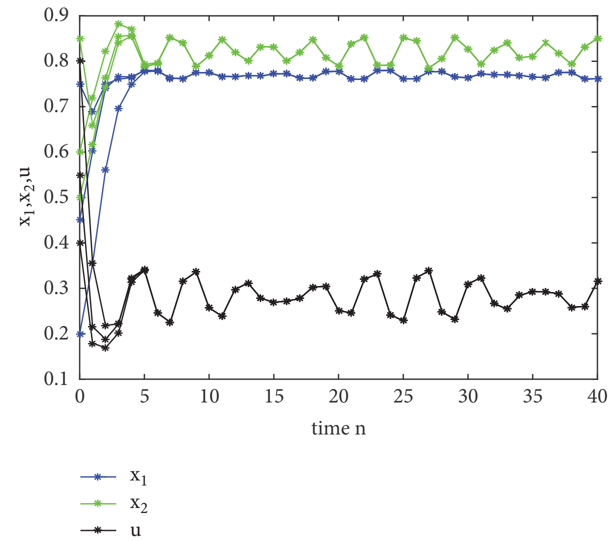

FIGURE 1: Dynamic behaviors of the solutions $(x_1(n), x_2(n), u(n))$ of system (103) with the initial conditions $(x_1(\theta), x_2(\theta), u(\theta)) = (0.2, 0.4, 0.6), (0.45, 0.5, 0.55), (0.75, 0.85, 0.8)$ for $\theta = \cdots, -n, -n+1, \cdots, -1, 0$, respectively.

Now, let us take $\alpha_1 = 0.0144$, $\alpha_2 = 0.0141$, and $\alpha_3 = 0.0222$, and then

$$\begin{aligned}
\alpha_1 A_{11} - \alpha_2\left(a_{21}^u + 2b_{21}^u M_1\right) - \alpha_3 d_1^u &= 0.01 > 0, \\
\alpha_2 A_{22} - \alpha_1\left(a_{12}^u + 2b_{12}^u M_2\right) - \alpha_3 d_2^u &= 0.01 > 0, \\
\alpha_3 e^l - \alpha_1 c_1^u - \alpha_2 c_2^u &= 0.01 > 0.
\end{aligned} \tag{105}$$

Clearly, condition (49) is satisfied, and so from Theorem 7 we have $\lim_{n\to+\infty}(x_i(n) - x_i^*(n)) = 0$ and $\lim_{n\to+\infty}(u(n) - u^*(n)) = 0$, where $(x_1(n), x_2(n), u(n))$ and $(x_1^*(n), x_2^*(n), u^*(n))$ are any two positive solutions of system (103).

Figure 1 shows the dynamic behaviors of system (103), which strongly supports the above assertions.

Example 13. Consider the following system:

$$\begin{aligned}
x_1(n+1) = x_1(n)\exp\Bigg[&0.4 + 0.2\sin\left(\sqrt{2}n\right) \\
&- 3x_1(n) - \left(1.4 + 0.1\sin\left(\sqrt{3}n\right)\right) \\
&\cdot \sum_{s=0}^{+\infty}\frac{e-1}{e}e^{-s}x_2(n-s) - \left(0.35 + 0.05\sin\left(\sqrt{3}n\right)\right) \\
&\cdot \sum_{s=0}^{+\infty}\frac{e^2-1}{e^2}e^{-2s}x_2^2(n-s) \\
&- \left(0.15 + 0.05\cos\left(\sqrt{3}n\right)\right) \\
&\cdot \sum_{s=0}^{+\infty}\frac{e^3-1}{e^3}e^{-3s}u(n-s)\Bigg],
\end{aligned}$$

$$
\begin{aligned}
x_2(n+1) &= x_2(n)\exp\Bigg[1.2+0.1\cos\left(\sqrt{3}n\right)-x_2(n) \\
&\quad -\left(2.8+0.2\sin\left(\sqrt{3}n\right)\right)\sum_{s=0}^{+\infty}\frac{e^4-1}{e^4}e^{-4s}x_1(n-s) \\
&\quad -\left(0.17+0.03\cos\left(\sqrt{2}n\right)\right)\sum_{s=0}^{+\infty}\frac{e^5-1}{e^5}e^{-5s}x_1^2(n-s) \\
&\quad -\left(0.15+0.05\sin\left(\sqrt{3}n\right)\right) \\
&\quad \cdot\sum_{s=0}^{+\infty}\frac{e^6-1}{e^6}e^{-6s}u(n-s)\Bigg], \\
u(n+1) &= \left(1-\left(0.7+0.1\cos\left(\sqrt{2}n\right)\right)\right)u(n)+\Big(1.1 \\
&\quad +0.1\sin\left(\sqrt{2}n\right)\Big)\sum_{s=0}^{+\infty}\frac{e^7-1}{e^7}e^{-7s}x_1(n-s)+\Big(0.9 \\
&\quad +0.3\sin\left(\sqrt{3}n\right)\Big)\sum_{s=0}^{+\infty}\frac{e^8-1}{e^8}e^{-8s}x_2(n-s).
\end{aligned}
\tag{106}
$$

One could easily see that conditions (H_1), (H_2), and (H_3) are satisfied. Also, by calculating, one has

$$
\begin{aligned}
&\min\left\{\frac{a_{12}^l e^l+c_1^l d_2^u}{a_{22}^u e^l+c_2^u d_2^u},\frac{a_{11}^l e^l+c_1^l d_1^u}{\left(a_{21}^u+b_{21}^u M_1\right)e^l+c_2^u d_1^u}\right\} \\
&\quad = 0.929, \\
&\frac{r_1^u}{r_2^l} = 0.5455, \\
&1.2 = d_2^u < \frac{A_{22}e^l}{c_2^u} = 1.449.
\end{aligned}
\tag{107}
$$

Clearly, conditions (67) and (87) are satisfied, and so from Theorems 8 and 10 we know that x_1 will be driven to extinction, while species x_2 is globally attractive.

Figure 2 shows the dynamic behaviors of system (106), which strongly supports our results.

7. Discussion

During the past decade, many scholars investigated the dynamic behaviors of the feedback control ecosystem. However, by using a feedback control variable to control all the species, it is much difficult. In this paper, we focused our attention on the nonlinear competition system with single feedback control, and the dynamic behavior of the system is investigated. The study shows that the feedback control variable plays a crucial role in both of global attractivity and partial extinction. i.e., under some suitable conditions the two species can survive well, but under other conditions one of two species will be driven to extinction.

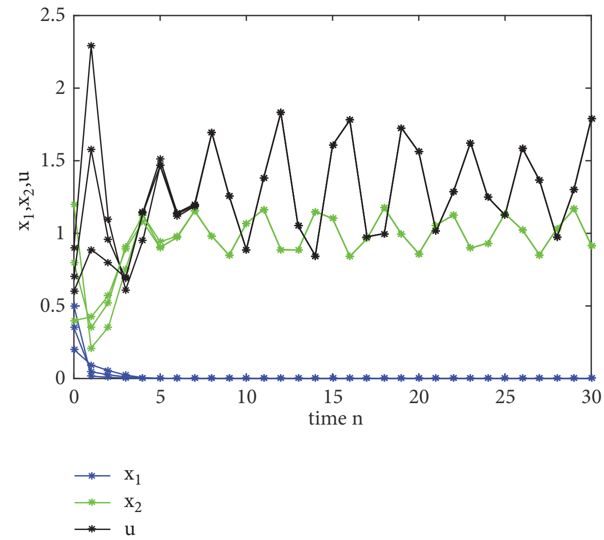

Figure 2: Dynamic behaviors of the solutions $(x_1(n), x_2(n), u(n))$ of system (106) with the initial conditions $(x_1(\theta), x_2(\theta), u(\theta)) = (0.2, 0.4, 0.6), (0.35, 0.8, 0.7), (0.5, 0.1.2, 0.9)$ for $\theta = \cdots, -n, -n+1, \cdots, -1, 0$, respectively.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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