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Research Article

Pullback-Forward Dynamics for Damped Schrödinger Equations with Time-Dependent Forcing

Lianbing She[1,2] **Yangrong Li**[1] **and Renhai Wang**[1]

[1]School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
[2]School of Mathematics and Information Engineering, Liupanshui Normal College, Liupanshui, Guizhou 553004, China

Correspondence should be addressed to Yangrong Li; liyr@swu.edu.cn

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This paper deals with pullback dynamics for the weakly damped Schrödinger equation with time-dependent forcing. An increasing, bounded, and pullback absorbing set is obtained if the forcing and its time-derivative are backward uniformly integrable. Also, we obtain the forward absorption, which is only used to deduce the backward compact-decay decomposition according to high and low frequencies. Based on a new existence theorem of a backward compact pullback attractor, we show that the nonautonomous Schrödinger equation has a pullback attractor which is compact in the past. The method of energy, high-low frequency decomposition, Sobolev embedding, and interpolation are quite involved in calculating a priori pullback or forward bound.

1. Introduction

This paper is concerned with the backward compact dynamics of space-periodic solutions for the nonautonomous complex-valued Schrödinger equation in $\mathbb{R}$:

$$u_t + iu_{xx} + i|u|^2 u + \alpha u - iu = f(t,x). \qquad (1)$$

The above equation for $\alpha, f = 0$ was introduced in [1] as a model for the propagation of solitons and laser beams. In such a case (without damping and forcing), it is easy to prove the energy conservation law; that is, $\|u(t)\| = \|u(0)\|$ (see, e.g., [2]). So no attractors exist. To obtain an attractor, we have to assume that the equation has a positive damping parameter $\alpha > 0$.

The dynamical behavior of the damped Schrödinger equation was widely investigated by many physicists and mathematicians (see, e.g., [3–11]) but restricted in the autonomous case; that is, the force f is time-independent (only space dependent).

This paper deals with dynamics for the nonautonomous Schrödinger equation; that is, the force f is time-dependent. To the best of our knowledge, there is no literature treating nonautonomous dynamics (including random dynamics) for the Schrödinger equation, even in the simple case for the existence of a pullback attractor, although the theory and application of pullback attractors had been widely developed for many other PDEs (see [12–16]), and for pullback random attractors, see, for example, [17–20].

When one tries to look for a bounded pullback absorbing set for (1) (see Lemma 7), it seems to be assumed that f is backward bounded in L^2; that is,

$$\sup_{s\leq t}\|f(s)\| < \infty, \quad \forall t \in \mathbb{R}, \qquad (2)$$

rather than the ordinary tempered integrable condition. Another difficulty is that we must treat the time-derivative f_t, which is assumed to be backward tempered. All those assumptions are different from dealing with the pullback dynamics for other nonautonomous dissipative equations (see, e.g., [17, 21–28]).

On the other hand, the backward condition (2) permits us to consider further properties of the pullback attractor, for example, backward compactness, as considered in [29–31], where a pullback attractor is called backward compact if the union over the past is precompact.

So, in Section 2, we establish a new abstract theorem on a backward compact pullback attractor for a *decomposable* evolution process; that is, it has a backward compact-decay decomposition. For such a decomposable process, we show that the existence of a backward compact attractor is equivalent to the existence of an *increasing*, bounded, and pullback absorbing set (see Theorem 4).

We then apply the abstract result to the nonautonomous Schrödinger equation. In Section 3, the increasingly pullback absorption is verified if f is assumed to be backward bounded and the time-derivative f_t is backward tempered.

The difficulty arises from verifying the compact-decay decomposition according to the high and low frequency of the Fourier series. In this case, the pullback absorption may not be suitable for verifying such a decomposable property. So, in Section 4, we have to give an auxiliary result on the *forward absorption*. It may be possible to deduce a forward attractor (cf. [32, 33]), but we do not pursue this forward attractor in the present paper.

In Section 5, we present some techniques of splitting the solutions of (1) into high and low frequency parts and establish a new equation with initial value zero in the high-frequency part. Then the forward absorption obtained in Section 4 can be applied to prove that the new equation has a forward uniformly bounded solution in H^2, which further proves that the component system is backward asymptotically compact in H^1. Also, we prove that the difference of solutions from both equations in the high-frequency part is backward exponential decay and so obtain the compact-decay decomposition as required.

The final existence result of a backward compact attractor is given in Theorem 14. It is worth pointing out that the pullback-forward method (involving the high-low frequency decomposition) may be special for the nonautonomous Schrödinger equation, which is different from treating the pullback dynamics for other nonautonomous dissipative equations.

2. Backward Compact Dynamics for Decomposable Systems

Let $(X, \|\cdot\|_X)$ be a Banach space equipped with the class $\mathfrak{B}$ of all bounded subset in X. We consider a *nonautonomous process* S on X, which means $S(t,s) : X \to X$ is a continuous nonlinear mapping such that $S(s,s) = \mathrm{id}_X$ and $S(t,s) = S(t,r)S(r,s)$ for all $t \geqslant r \geqslant s$ with $s \in \mathbb{R}$.

We assume that the process is *decomposable* in the following sense.

Definition 1. A nonautonomous process S is said to have a *backward compact-decay decomposition* $S(t,s) = S_1(t,s) + S_2(t,s)$ $(t \geqslant s)$ if

(i) S_1 is *backward asymptotically compact*, that is, the sequence $\{S_1(s_n, s_n - \tau_n)x_n\}_{n=1}^{\infty}$ is precompact whenever $s_n \leqslant t$ with fixed $t \in \mathbb{R}$, $\tau_n \to +\infty$ and $\{x_n\}_{n=1}^{\infty}$ is bounded in X;

(ii) S_2 is *backward exponential decay*, that is, for each $t \in \mathbb{R}$ and $B \in \mathfrak{B}$, there exist two positive constants $c_1 = c_1(t,B)$ and $c_2 = c_2(t,B)$ such that, for some $\tau_0 \geqslant 0$,

$$\sup_{s \leqslant t} \sup_{x \in B} \left\| S_2(s, s-\tau)x \right\|_X \leqslant c_1 e^{-c_2(\tau - \tau_0)}, \quad \forall \tau \geqslant \tau_0. \tag{3}$$

A nonautonomous set $\mathscr{D} = \{\mathscr{D}(t)\}_{t\in\mathbb{R}}$ in X means a set-valued mapping $\mathscr{D}(\cdot) : \mathbb{R} \to 2^X \setminus \emptyset$, which is said to have some topological properties (such as compactness, boundedness, and closedness) if the component set $\mathscr{D}(t)$ has the corresponding properties.

Definition 2. A nonautonomous set $\mathscr{D} = \{\mathscr{D}(t)\}_{t\in\mathbb{R}}$ is called

(i) *backward* (resp., *forward or globally*) *compact* if $\mathscr{D}(t)$ and $\overline{\cup_{s\leqslant t}\mathscr{D}(s)}$ (resp., $\overline{\cup_{r\geqslant t}\mathscr{D}(r)}$ or $\overline{\cup_{r\in\mathbb{R}}\mathscr{D}(r)}$) are compact for each $t \in \mathbb{R}$;

(ii) *increasing* (resp., *decreasing*) if $\mathscr{D}(s) \subset \mathscr{D}(t)$ (resp., $\mathscr{D}(s) \supset \mathscr{D}(t)$) for all $s \leqslant t$.

Definition 3. A backward compact set $\mathscr{A} = \{\mathscr{A}(t)\}_{t\in\mathbb{R}}$ in X is called a *backward compact attractor* for a nonautonomous process S if $\mathscr{A}$ is *invariant*, that is, $S(t,s)\mathscr{A}(s) = \mathscr{A}(t)$ for $t \geqslant s$, and *pullback attracting*, that is, for any $t \in \mathbb{R}$ and $B \in \mathfrak{B}$,

$$\lim_{\tau\to+\infty} \mathrm{dist}_X (S(t, t-\tau)B, \mathscr{A}(t)) = 0, \tag{4}$$

where the Hausdorff semi-distance $\mathrm{dist}_X(A,B) = \sup_{a\in A}\inf_{b\in B}\|a-b\|_X$.

A backward compact attractor must be unique and minimal, where the minimality means $\mathscr{A} \subset \mathscr{K}$ for any closed attracting set $\mathscr{K}$ (see [30]).

For the purpose of applying to the Schrödinger equation, we need to establish a new existence theorem of a backward compact attractor for a backward compact-decay process, although other existence criteria were established in [29, 30].

Recall that a nonautonomous set $\mathscr{K}$ is called *pullback absorbing* for S if for each $t \in \mathbb{R}$ and $B \in \mathfrak{B}$ there is a $\tau_0 = \tau_0(t,B)$ such that

$$S(t, t-\tau)B \subset \mathscr{K}(t), \quad \forall \tau \geqslant \tau_0. \tag{5}$$

Theorem 4. *Suppose a nonautonomous process S has a backward compact-decay decomposition $S(t,s) = S_1(t,s) + S_2(t,s)$ for $t \geqslant s$. Then the following statements are equivalent.*

(i) There exists an increasing, bounded, and pullback absorbing set $\mathscr{K} = \{\mathscr{K}(t)\}_{t\in\mathbb{R}}$.

(ii) There is a backward compact attractor $\mathscr{A} = \{\mathscr{A}(t)\}_{t\in\mathbb{R}}$ given by

$$\mathscr{A}(t) = \omega(\mathscr{K}(t), t) := \bigcap_{\tau_0>0} \overline{\bigcup_{\tau\geqslant\tau_0} S(t, t-\tau)\mathscr{K}(t)}, \quad t \in \mathbb{R}. \tag{6}$$

Proof. The necessity is easily proved by setting $\mathscr{K}(t) = N_1(\cup_{s\leqslant t}\mathscr{A}(s))$, the 1-neighborhood of $\cup_{s\leqslant t}\mathscr{A}(s)$, for each $t \in \mathbb{R}$. Since $\cup_{s\leqslant t}\mathscr{A}(s)$ is obviously an increasing family in $t \in \mathbb{R}$, it follows that $\mathscr{K}$ is increasing. Since $\cup_{s\leqslant t}\mathscr{A}(s)$ is precompact, it follows that $\mathscr{K}(t)$ is bounded (not necessarily precompact). Finally, it is easy to deduce the pullback absorption of $\mathscr{K}$ from the pullback attraction of $\mathscr{A}$.

Conversely, suppose (i) is true; we show that $\mathscr{A}(t) := \omega(\mathscr{K}(t), t)$ is a backward compact attractor in three steps.

Step 1. We show the invariance. As usual (cf. [12]), it is easy to prove the positive invariance $S(t,s)\mathscr{A}(s) \subset \mathscr{A}(t)$. On the other hand, let $y \in \mathscr{A}(t) = \omega(\mathscr{K}(t), t)$. We choose $y_n \in \mathscr{K}(t)$ and $\tau_n \to +\infty$ such that

$$\lim_{n\to\infty} S(t, t-\tau_n) y_n = y. \tag{7}$$

Let $s \leqslant t$ and $\tilde{\tau}_n = s - t + \tau_n \to +\infty$. Then $\tilde{\tau}_n > 0$ if n is large enough. By the compact-decay decomposition, we know that $\{S_1(s, s-\tilde{\tau}_n)y_n\}$ is precompact and $\{S_2(s, s-\tilde{\tau}_n)y_n\}$ is exponential decay. Then, passing to subsequences, there is an $z \in X$ such that

$$\begin{aligned} &S_1\left(s, s-\tilde{\tau}_{n_k}\right) y_{n_k} \longrightarrow z, \\ &S_2\left(s, s-\tilde{\tau}_{n_k}\right) y_{n_k} \longrightarrow 0, \end{aligned} \tag{8}$$

where $\tilde{\tau}_{n_k} > 0$ for all $k \in \mathbb{N}$. Hence, it further implies

$$\begin{aligned} S\left(s, s-\tilde{\tau}_{n_k}\right) y_{n_k} &= S_1\left(s, s-\tilde{\tau}_{n_k}\right) y_{n_k} \\ &\quad + S_2\left(s, s-\tilde{\tau}_{n_k}\right) y_{n_k} \longrightarrow z. \end{aligned} \tag{9}$$

Since $\mathscr{K}$ is pullback absorbing and increasing, for each $i \in N$, we can choose a k_i in $\{n_k\}$ such that $\tilde{\tau}_{k_i} - i$ is large enough; it follows that

$$\begin{aligned} z_i &:= S\left(s-i, s-i-\left(\tilde{\tau}_{k_i} - i\right)\right) y_{k_i} \in \mathscr{K}(s-i) \\ &\subset \mathscr{K}(s). \end{aligned} \tag{10}$$

By the process property, as $i \to \infty$,

$$S(s, s-i) z_i = S\left(s, s-\tilde{\tau}_{k_i}\right) y_{k_i} \longrightarrow z. \tag{11}$$

By (10)-(11), we have $z \in \omega(\mathscr{K}(s), s) = \mathscr{A}(s)$. By the continuity of S, it follows from (7) and (9) that

$$\begin{aligned} y &= \lim_{i\to\infty} S\left(t, t-\tau_{k_i}\right) y_{k_i} = \lim_{k\to\infty} S(t,s) S\left(s, s-\tilde{\tau}_{k_i}\right) y_{k_i} \\ &= S(t,s) z \in S(t,s)\mathscr{A}(s), \end{aligned} \tag{12}$$

which proves the negative invariance.

Step 2. We show the attraction of $\mathscr{A}$. If it is not true, then there are $\delta > 0$, $t \in \mathbb{R}$, $\tau_n \to +\infty$ as $n \to \infty$ and $x_n \in B$ with $B \in \mathfrak{B}$ such that

$$\operatorname{dist}_X\left(S(t, t-\tau_n) x_n, \mathscr{A}(t)\right) \geqslant \delta, \quad \text{for each } n \in \mathbb{N}. \tag{13}$$

By using the compact-decay decomposition, we can take subsequences such that

$$\begin{aligned} &\lim_{k\to\infty} S_1\left(t, t-\tau_{n_k}\right) x_{n_k} = x, \\ &\lim_{k\to\infty} S_2\left(t, t-\tau_{n_k}\right) x_{n_k} = 0, \end{aligned} \tag{14}$$

where $x \in X$. Therefore,

$$\begin{aligned} \lim_{k\to\infty} S\left(t, t-\tau_{n_k}\right) x_{n_k} &= \lim_{k\to\infty} S_1\left(t, t-\tau_{n_k}\right) x_{n_k} \\ &\quad + \lim_{k\to\infty} S_2\left(t, t-\tau_{n_k}\right) x_{n_k} = x. \end{aligned} \tag{15}$$

Like we did in Step 1, we choose a subsequence $\{k_i\}$ of $\{n_k\}$ such that

$$\begin{aligned} y_i &:= S\left(t-i, t-i-\left(\tau_{k_i} - i\right)\right) x_{k_i} \in \mathscr{K}(t-i) \\ &\subset \mathscr{K}(t), \end{aligned} \tag{16}$$

and by (15),

$$\lim_{k\to\infty} S(t, t-i) y_i = \lim_{k\to\infty} S\left(t, t-\tau_{k_i}\right) x_{k_i} = x. \tag{17}$$

The above two formulations imply $x \in \mathscr{A}(t)$, which contradicts with (13).

Step 3. It remains to show the precompact of $B_t := \cup_{s\leqslant t}\mathscr{A}(s)$ with fixed $t \in \mathbb{R}$. Let $s \leqslant t$. Since $\mathscr{K}$ is pullback absorbing and increasing, there is a $\tau_0(s)$ such that

$$\overline{\bigcup_{\tau\geqslant\tau_0} S(s, s-\tau)\mathscr{K}(s)} \subset \overline{\mathscr{K}(s)} \subset \overline{\mathscr{K}(t)}, \tag{18}$$

and so $\omega(\mathscr{K}(s), s) \subset \overline{\mathscr{K}(t)}$, which further implies

$$B_t = \bigcup_{s\leqslant t}\mathscr{A}(s) = \bigcup_{s\leqslant t}\omega(\mathscr{K}(s), s) \subset \overline{\mathscr{K}(t)}. \tag{19}$$

Hence B_t is at least bounded. To prove the precompactness of B_t, we take a sequence $z_n \in B_t = \cup_{s\leqslant t}\mathscr{A}(s)$ and then $z_n \in \mathscr{A}(s_n)$ with $s_n \leqslant t$. Let $0 \leqslant \tau_n \to \infty$. Then invariance of $\mathscr{A}$ implies

$$\begin{aligned} z_n \in \mathscr{A}(s_n) &= S(s_n, s_n-\tau_n)\mathscr{A}(s_n-\tau_n) \\ &\subset S(s_n, s_n-\tau_n) B_t. \end{aligned} \tag{20}$$

Since B_t is proved to be bounded, there is a bounded sequence $\{x_n\}$ such that $z_n = S(s_n, s_n-\tau_n)x_n$. By the backward compact-decay decomposition, we know S_1 is backward asymptotically compact on the bounded set B_t. Then, passing to a subsequence, there is a $x \in X$ such that

$$S_1\left(s_{n_k}, s_{n_k}-\tau_{n_k}\right) x_{n_k} \longrightarrow x. \tag{21}$$

By using the decay property of S_2, we know

$$\left\| S_2\left(s_{n_k}, s_{n_k}-\tau_{n_k}\right) x_{n_k} \right\| \leqslant c_1 e^{-c_2(\tau_{n_k}-\tau_0)} \longrightarrow 0. \tag{22}$$

The above limits imply

$$\begin{aligned} z_{n_k} &= S\left(s_{n_k}, s_{n_k}-\tau_{n_k}\right) x_{n_k} \\ &= S_1\left(s_{n_k}, s_{n_k}-\tau_{n_k}\right) x_{n_k} + S_2\left(s_{n_k}, s_{n_k}-\tau_{n_k}\right) x_{n_k} \\ &\longrightarrow x. \end{aligned} \tag{23}$$

Hence, B_t is precompact. In particular, $\mathscr{A}(t)$ is precompact. But $\mathscr{A}(t)$ is obviously closed and so it is compact. The proof is complete. □

3. Pullback Absorption in Schrödinger Equations

We come back to the Schrödinger equation with time-dependent force as follows:

$$\begin{aligned} &u_t + iu_{xx} + i|u|^2 u + \alpha u - iu = f(t,x), \\ &u(t,0) = u(t,1), \quad t \geqslant s, \\ &u(s,x) = u_0(x), \\ &s \in \mathbb{R}, \ x \in \Omega = (0,1), \end{aligned} \tag{24}$$

where $\alpha > 0, u(t,x)$ is an unknown complex-valued function.

3.1. Hypotheses and Existence of Solutions. Let $L^2(\Omega)$ be the space of complex-valued L^2-functions whose norm is denoted by $\|\cdot\|$. Let H^1 be the space of all one-periodic H^1-function with the norm $\|u\|_{H^1}^2 = \|u\|^2 + \|u_x\|^2$. The norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$. We then give some hypotheses on the time-space dependent force $f = f(t,x)$.

*Assumption **A1**.* f is continuous and *backward bounded*:

$$F_1(t) \coloneqq \sup_{s\leqslant t} \|f(s,\cdot)\|^2 < +\infty, \quad \forall t \in \mathbb{R}. \tag{25}$$

*Assumption **A2**.* The time-derivative f_t is *backward tempered*:

$$F_2(t) \coloneqq \sup_{s\leqslant t} \int_{-\infty}^{s} e^{\alpha(r-s)} \|f_r(r,\cdot)\|^2 \, dr < +\infty, \quad \forall t \in \mathbb{R}, \tag{26}$$

where $\alpha > 0$ as given in (24).

Assumption **A1** implies $f \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\Omega))$ and the finiteness of $F_3(t)$, where

$$F_3(t) \coloneqq \sup_{s\leqslant t} \int_{-\infty}^{s} e^{\alpha(r-s)} \|f(r,\cdot)\|^2 \, dr \leqslant \frac{F_1(t)}{1-e^{-\alpha}}. \tag{27}$$

The last inequality can be proved as follows.

$$\begin{aligned} F_3(t) &= \sup_{s\leqslant t} \sum_{m=0}^{\infty} \int_{s-(m+1)}^{s-m} e^{\alpha(r-s)} \|f(r,\cdot)\|^2 \, dr \\ &\leqslant \sup_{s\leqslant t} \sum_{m=0}^{\infty} e^{-m\alpha} \int_{s-(m+1)}^{s-m} \|f(r,\cdot)\|^2 \, dr \\ &\leqslant \sum_{m=0}^{\infty} e^{-m\alpha} \sup_{r\leqslant t} \|f(r,\cdot)\|^2 = \frac{F_1(t)}{1-e^{-\alpha}}. \end{aligned} \tag{28}$$

Note that all functions $F_i(\cdot)$ $(i = 1,2,3)$ are finite, nonnegative, and *increasing*.

We will repeatedly use the following two energy inequalities.

Lemma 5. *Let $u(t) = u(t,s,u_0)$, $t \in [s,+\infty)$, be the approximation solution of (24). Then*

$$\frac{d}{dt}\|u\|^2 + \alpha\|u\|^2 \leqslant \frac{1}{\alpha}\|f(t,\cdot)\|^2, \tag{29}$$

$$\begin{aligned} \frac{d}{dt}\Phi_1(u) + \alpha\Phi_1(u) &\leqslant 9\alpha\|u\|^6 + 3\alpha\|u\|^4 \\ &\quad + \frac{1}{\alpha}\|f_t(t,\cdot)\|^2, \end{aligned} \tag{30}$$

where

$$\Phi_1(u) = \|u_x\|^2 + \|u\|^2 - \frac{1}{2}\|u\|_4^4 + 2\,\mathrm{Im}\int_\Omega f\bar{u}\,dx. \tag{31}$$

Proof. Equation (29) is easily obtained by multiplying (24) with $\bar{u}$ (the complex conjugate of u) and taking the real part of the final equation.

To prove (30), we multiply (24) by $-(\bar{u}_t + \alpha\bar{u})$ and take the imaginary part; after some complex-valued calculations, we obtain

$$\frac{d}{dt}\Phi_1(u) + 2\alpha\Phi_2(u) - 2\mathrm{Re}\int_\Omega f_t\bar{u}\,dx = 0, \tag{32}$$

where $\Phi_1(u)$ is given by (31), and

$$\Phi_2(u) = \|u_x\|^2 + \|u\|^2 - \|u\|_4^4 + \mathrm{Im}\int_\Omega f\bar{u}\,dx. \tag{33}$$

By the Agmon inequality $\|u\|_\infty^2 \leqslant 2\|u\|(\|u_x\| + \|u\|)$, we have $\|u\|_4^4 \leqslant 2\|u\|^3(\|u_x\| + \|u\|)$. Hence,

$$\begin{aligned} \Phi_1(u) - 2\Phi_2(u) &= -\|u_x\|^2 - \|u\|^2 + \frac{3}{2}\|u\|_4^4 \\ &\leqslant -\|u_x\|^2 - \|u\|^2 \\ &\quad + 3\|u\|^3(\|u_x\| + \|u\|) \\ &\leqslant 9\|u\|^6 + 3\|u\|^4 - \|u\|^2. \end{aligned} \tag{34}$$

We then rewrite the energy equation (32) as follows:

$$\begin{aligned} \frac{d}{dt}\Phi_1(u) + \alpha\Phi_1(u) &= \alpha(\Phi_1(u) - 2\Phi_2(u)) \\ &\quad + 2\,\mathrm{Im}\int_\Omega f_t\bar{u}\,dx \\ &\leqslant 9\alpha\|u\|^6 + 3\alpha\|u\|^4 + \frac{1}{\alpha\|f_t\|^2}, \end{aligned} \tag{35}$$

which is just the needed energy inequality. □

Based on the above energy inequalities, one can obtain a priori estimate (for absorption, see the next subsection). Then it is similar as the autonomous case (see [5, 8]) to prove that (24) is well posed in H^1. Namely, for each $u_0 \in H^1$ and $s \in \mathbb{R}$, (24) has a unique solution $u(\cdot,s,u_0) \in C([s,+\infty),H^1)$ and the solution $u(t,s,\cdot) : H^1 \to H^1$ is continuous. This well-posed property permits us to define an evolution process S on H^1 by

$$S(t,s)u_0 = u(t,s,u_0) \quad \text{for } t \geqslant s, \ u_0 \in H^1. \tag{36}$$

3.2. Increasing, Bounded, and Pullback Absorbing Sets. In order to use the results of Theorem 4, we need to look for an increasing, bounded, and pullback absorbing set.

Lemma 6. *Let **A1** be satisfied. Suppose B_R be a ball in H^1 (R is the radius) and $t \in \mathbb{R}$, then there are $\tau_0 = \tau_0(R)$ such that, for all $u_0 \in B_R$ and $\tau \geqslant \tau_0$,*

$$\sup_{s\leqslant t}\left\|u(s,s-\tau,u_0)\right\|^2 \leqslant 1+\frac{F_3(t)}{\alpha} \leqslant 1+\frac{F_1(t)}{\alpha(1-e^{-\alpha})}, \quad (37)$$

$$\sup_{s\leqslant t}\int_{s-\tau}^{s} e^{\alpha(r-s)}\left\|u(r,s-\tau,u_0)\right\|^4 dr \leqslant c_1\left(1+F_1^2(t)\right), \quad (38)$$

$$\sup_{s\leqslant t}\int_{s-\tau}^{s} e^{\alpha(r-s)}\left\|u(r,s-\tau,u_0)\right\|^6 dr \leqslant c_2\left(1+F_1^3(t)\right), \quad (39)$$

*where c_1, c_2 are positive constants and F_1, F_3 are functions given in **A1**.*

Proof. Let $t \in \mathbb{R}$ and $\tau \geqslant 0$. We apply the Gronwall inequality on (29) over the interval $[s-\tau, r]$ with $s-\tau \leqslant r \leqslant s \leqslant t$ and then find

$$\begin{aligned}\left\|u(r,s-\tau,u_0)\right\|^2 &\leqslant e^{-\alpha(r-s+\tau)}\left\|u_0\right\|^2 \\ &\quad+\frac{1}{\alpha}\int_{s-\tau}^{r} e^{\alpha(\hat{r}-r)}\left\|f(\hat{r},\cdot)\right\|^2 d\hat{r}.\end{aligned} \quad (40)$$

Letting $r = s$, we have

$$\left\|u(s,s-\tau,u_0)\right\|^2 \leqslant e^{-\alpha\tau}\left\|u_0\right\|_{H^1}^2+\frac{1}{\alpha}F_3(t), \quad (41)$$

which yields (37) if we take $\tau_0 = 2\ln(R+1)/\alpha$. On the other hand, by (40) again,

$$\left\|u(r,s-\tau,u_0)\right\|^2 \leqslant e^{-\alpha(r-s+\tau)}\left\|u_0\right\|^2+\frac{1}{\alpha^2}F_1(t), \quad (42)$$

where we use the fact that $\|f(\hat{r},\cdot)\|^2 \leqslant F_1(t)$ for $\hat{r} \leqslant r \leqslant t$. We then consider the sixth power to obtain

$$\left\|u(r,s-\tau,u_0)\right\|^6 \leqslant 8e^{-3\alpha(r-s+\tau)}\left\|u_0\right\|^6+\frac{8}{\alpha^6}F_1^3(t), \quad (43)$$

which yields

$$\begin{aligned}&\int_{s-\tau}^{s} e^{\alpha(r-s)}\left\|u(r,s-\tau,u_0)\right\|^6 dr \\ &\quad\leqslant 8\left\|u_0\right\|^6\int_{s-\tau}^{s} e^{\alpha(r-s)}e^{-3\alpha(r-s+\tau)}dr \\ &\qquad+cF_1^3(t)\int_{s-\tau}^{s} e^{\alpha(r-s)}dr \\ &\quad\leqslant 8e^{-3\alpha\tau}\left\|u_0\right\|^6\int_{-\tau}^{0} e^{-2\alpha r}dr+cF_1^3(t)\int_{-\infty}^{s} e^{\alpha(r-s)}dr \\ &\quad\leqslant \frac{4}{\alpha}e^{-\alpha\tau}\left\|u_0\right\|^6+cF_1^3(t).\end{aligned} \quad (44)$$

From this, it is easy to deduce (39) with $\tau_0(R) = (2\ln 2 + 6\ln R - \ln\alpha)/\alpha$ and similarly obtain (38). □

We then consider the backward bound of solutions in H^1 as follows.

Lemma 7. *Let **A1**, **A2** be satisfied. Then for each ball B_R in H^1 and $t \in \mathbb{R}$, there exists $\tau_1 = \tau_1(R,t)$, such that, for all $\tau \geqslant \tau_1$ and $u_0 \in B_R$,*

$$\begin{aligned}\sup_{s\leqslant t}\left\|u(s,s-\tau,u_0)\right\|_{H^1}^2 &\leqslant M_1(t) \\ &:= c_3\left(1+F_1^3(t)+F_2(t)\right),\end{aligned} \quad (45)$$

where $c_3 := c_3(\alpha)$ is a constant and thus $M_1(\cdot)$ is finite and increasing.

Proof. Let $s \leqslant t$ and $\tau \geqslant 0$. Applying the Gronwall inequality on (30) over $[s-\tau, s]$, we get

$$\begin{aligned}&\Phi_1(u(s,s-\tau,u_0)) \\ &\quad\leqslant e^{-\alpha\tau}\Phi_1(u_0)+9\alpha\int_{s-\tau}^{s} e^{\alpha(r-s)}\left\|u(r,s-\tau)\right\|^6 dr \\ &\qquad+3\alpha\int_{s-\tau}^{s} e^{\alpha(r-s)}\left\|u(r,s-\tau)\right\|^4 dr \\ &\qquad+\frac{1}{\alpha}\int_{s-\tau}^{s} e^{\alpha(r-s)}\left\|f_r(r,\cdot)\right\|^2 dr.\end{aligned} \quad (46)$$

By Lemma 6 and **A2**, there is a $\tau_1 = \tau_1(R)$ such that, for all $\tau \geqslant \tau_1$ and $u_0 \in B_R$,

$$\begin{aligned}&\sup_{s\leqslant t}\Phi_1(u(s,s-\tau,u_0)) \\ &\quad\leqslant e^{-\alpha\tau}\Phi_1(u_0)+c\left(1+F_1^3(t)+F_1^2(t)+F_2(t)\right).\end{aligned} \quad (47)$$

On the other hand, by the definition of Φ_1 given in (31), we have

$$\begin{aligned}\Phi_1(u(s-\tau,s-\tau,u_0)) &\leqslant 2\left\|u_0\right\|_{H^1}^2+\left\|f(s-\tau,\cdot)\right\|^2 \\ &\leqslant 2\left\|u_0\right\|_{H^1}^2+F_1(t).\end{aligned} \quad (48)$$

In particular, at the initial value, we have, if τ is large enough, then

$$e^{-\alpha\tau}\Phi_1(u_0) \leqslant 2e^{-\alpha\tau}\left(\left\|u_0\right\|_{H^1}^2+F_1(t)\right) \leqslant 1. \quad (49)$$

Both (47) and (49) imply that, for all $u_0 \in B_R$ and $\tau \geqslant \tau_0$ (with a larger τ_0),

$$\sup_{s\leqslant t}\Phi_1(u(s,s-\tau,u_0)) \leqslant c\left(1+F_1^3(t)+F_2(t)\right). \quad (50)$$

By the Agmon inequality again, it follows from (31) that

$$\begin{aligned}\Phi_1(u) &\geqslant \left\|u_x\right\|^2+\left\|u\right\|^2-\left\|u\right\|^3\left(\left\|u_x\right\|+\left\|u\right\|\right) \\ &\quad+2\,\mathrm{Im}\int_{\Omega} f\bar{u}\,dx \\ &\geqslant \frac{1}{2}\left\|u\right\|_{H^1}^2-2\left\|u\right\|^6-\left\|u\right\|^4-2\left\|f(s,\cdot)\right\|^2,\end{aligned} \quad (51)$$

which together with (50) and Lemma 6 imply that, for all $u_0 \in B_R$ and $\tau \geqslant \tau_0$ (with a larger τ_0),

$$\begin{aligned}
&\sup_{s\leqslant t}\left\|u\left(s,s-\tau,u_0\right)\right\|_{H^1}^2 \\
&\quad\leqslant 2\sup_{s\leqslant t}\Phi_1\left(u\left(s,s-\tau,u_0\right)\right) \\
&\qquad + 4\sup_{s\leqslant t}\left\|u\left(s,s-\tau,u_0\right)\right\|^6 \\
&\qquad + 2\sup_{s\leqslant t}\left\|u\left(s,s-\tau,u_0\right)\right\|^4 + 4\sup_{s\leqslant t}\left\|f\left(s,\cdot\right)\right\|^2 \\
&\quad\leqslant c\left(1+F_1^3\left(t\right)+F_2\left(t\right)\right)+4\left(1+cF_1\left(t\right)\right)^3 \\
&\qquad + 2\left(1+cF_1\left(t\right)\right)^2+4F_1\left(t\right) \\
&\quad\leqslant c\left(1+F_1^3\left(t\right)+F_2\left(t\right)\right),
\end{aligned} \tag{52}$$

which shows the needed result. □

Under the light of Lemma 7, we have the following increasing absorption.

Theorem 8. *Let **A1**, **A2** be true. Then the nonautonomous Schrödinger equation possesses an increasing, bounded, and pullback absorbing set $\mathscr{K}=\{\mathscr{K}(t)\}_{t\in\mathbb{R}}$ in H^1 given by*

$$\begin{aligned}
\mathscr{K}\left(t\right) &= \left\{w\in H^1 : \|w\|_{H^1}^2 \leqslant M_1\left(t\right)\right. \\
&\left.\coloneqq c_3\left(1+F_1^3\left(t\right)+F_2\left(t\right)\right)\right\}, \quad t\in\mathbb{R}.
\end{aligned} \tag{53}$$

Proof. By **A1**, **A2**, we can see that $M_1(t)$ is an increasing and finite function with respect to t; this fact along with Lemma 7 shows that the nonautonomous set $\mathscr{K}$ is increasing, bounded, and pullback absorbing in H^1. In fact, by (45), the absorption is backward uniform. □

Remark 9. It seems not to prove that the nonautonomous Schrödinger equation has a bounded absorbing set in H^1 if the assumption **A1** is replaced by the weaker assumption that $F_3(t) < \infty$, although this weak assumption is enough for reaction-diffusion systems (see [29]), BBM equations (see [30]), and Navier-Stokes equations (see [31]).

4. Forward Absorption in Schrödinger Equations

This section establishes the *forward* absorption, which will be useful to deduce the compact-decay decomposition in the next section.

In this case, we need to strengthen Assumptions **A1** and **A2** as follows.

*Assumption **A1′**.* f is continuous such that $\rho_1 \coloneqq \lim_{t\to+\infty}F_1(t) < \infty$.

*Assumption **A2′**.* f_t exists such that $\rho_2 \coloneqq \lim_{t\to+\infty}F_2(t) < \infty$.

*Assumption **A3**.* f_{tx} exists such that $f_{tx} \in L^2_{\mathrm{loc}}(\mathbb{R}, H^{-1}(\Omega))$.

Lemma 10. *Let **A1′**, **A2′** be satisfied. Then for each ball B_R in H^1, there exists $t_1 = t_1(R) > 0$ such that*

$$\sup_{t\geqslant t_1}\sup_{s\leqslant 0}\sup_{u_0\in B_R}\left\|u\left(s+t,s,u_0\right)\right\|_{H^1} \leqslant \rho_3, \tag{54}$$

$$\sup_{t\geqslant t_1}\sup_{s\leqslant 0}\sup_{u_0\in B_R}\left\|u_t\left(s+t,s,u_0\right)\right\|_{H^{-1}} \leqslant \rho_4, \tag{55}$$

where ρ_3, ρ_4 are constants depending on α and f.

Proof. Let $s \leqslant 0$ and $t \geqslant 0$. We apply the Gronwall inequality on (29) over $[s,r]$ with $r \in [s, s+t]$ and then find

$$\left\|u\left(r,s,u_0\right)\right\|^2 \leqslant e^{-\alpha(r-s)}\left\|u_0\right\|^2 + \frac{1}{\alpha}F_3\left(r\right). \tag{56}$$

Taking $r = s+t$ in (56), we have

$$\begin{aligned}
\left\|u\left(s+t,s,u_0\right)\right\|^2 &\leqslant e^{-\alpha t}\left\|u_0\right\|_{H^1}^2 + \frac{1}{\alpha}F_3\left(s+t\right) \\
&\leqslant e^{-\alpha t}\left\|u_0\right\|_{H^1}^2 + cF_1\left(t\right).
\end{aligned} \tag{57}$$

Let $t_1 = 2\ln(R+1)/\alpha > 0$. By **A1′**, we know, for all $t \geqslant t_1$ and $u_0 \in B_R$,

$$\left\|u\left(s+t,s,u_0\right)\right\|^2 \leqslant 1 + c\rho_1. \tag{58}$$

We then consider the third power of (56) to obtain

$$\left\|u\left(r,s,u_0\right)\right\|^6 \leqslant 8e^{-3\alpha(r-s)}\left\|u_0\right\|^6 + c\rho_1^3, \quad s\leqslant r\leqslant s+t, \tag{59}$$

which yields

$$\begin{aligned}
&\int_s^{s+t} e^{\alpha(r-s-t)}\left\|u\left(r,s,u_0\right)\right\|^6 dr \\
&\quad\leqslant 8e^{-\alpha t}\left\|u_0\right\|^6\int_s^{s+t} e^{-2\alpha(r-s)}dr \\
&\qquad + c\rho_1^3\int_s^{s+t} e^{\alpha(r-s-t)}dr \\
&\quad\leqslant \frac{4}{\alpha}e^{-\alpha t}\left\|u_0\right\|^6\left(1-e^{-2\alpha t}\right) + c\rho_1^3 \\
&\quad\leqslant \frac{4}{\alpha}e^{-\alpha t}\left\|u_0\right\|^6 + c\rho_1^3.
\end{aligned} \tag{60}$$

Rewriting t_1 by $t_1(R) = (2\ln 2 + 6\ln(R+1) - \ln\alpha)/\alpha$, we deduce for all $t \geqslant t_1$,

$$\int_s^{s+t} e^{\alpha(r-s-t)}\left\|u\left(r,s,u_0\right)\right\|^6 dr \leqslant 1 + c\rho_1^3, \tag{61}$$

and similarly we have

$$\int_s^{s+t} e^{\alpha(r-s-t)}\left\|u\left(r,s,u_0\right)\right\|^4 dr \leqslant 1 + c\rho_1^2. \tag{62}$$

To prove (54), we apply the Gronwall inequality on (30) over $[s, s+t]$ to get

$$\begin{aligned}\Phi_1(u(s+t,s,u_0)) \leqslant{}& e^{-\alpha t}\Phi_1(u_0) \\ &+ 9\alpha\int_s^{s+t} e^{\alpha(r-s-t)}\|u(r,s)\|^6\,dr \\ &+ 3\alpha\int_s^{s+t} e^{\alpha(r-s-t)}\|u(r,s)\|^4 \\ &+ \frac{1}{\alpha}\int_s^{s+t} e^{\alpha(r-s-t)}\left\|f_r(r,\cdot)\right\|^2 dr.\end{aligned} \tag{63}$$

By (61)-(62) and Assumption **A2**, we easily deduce that, for $t \geqslant t_1$,

$$\begin{aligned}\Phi_1(u(s+t,s,u_0)) \leqslant{}& e^{-\alpha t}\Phi_1(u_0) + c\left(\rho_1^3+\rho_1^2\right) \\ &+ cF_2(s+t).\end{aligned} \tag{64}$$

By the definition of Φ_1 given in (31), we have

$$\begin{aligned}\Phi_1(u(r,s,u_0)) &\leqslant 2\|u\|_{H^1}^2 + \left\|f(r,\cdot)\right\|^2 \\ &\leqslant 2\|u\|_{H^1}^2 + F_1(r), \quad r \geqslant s.\end{aligned} \tag{65}$$

In particular, if t is large enough, then

$$\begin{aligned}e^{-\alpha t}\Phi_1(u(s,s,u_0)) &\leqslant e^{-\alpha t}\left(2\left\|u_0\right\|_{H^1}^2 + F_1(s)\right) \\ &\leqslant e^{-\alpha t}\left(2\left\|u_0\right\|_{H^1}^2 + F_1(0)\right) \leqslant 1.\end{aligned} \tag{66}$$

By (61)-(62) and Assumption **A2′**, for $t \geqslant t_1$ (with a larger t_1),

$$\Phi_1(u(s+t,s,u_0)) \leqslant 1 + c\left(\rho_1^3+\rho_1^2\right) + c\rho_2 \leqslant C. \tag{67}$$

By the Agmon inequality again, it is similar as (51) to prove that

$$\begin{aligned}\Phi_1(u(s+t,s,u_0)) \geqslant{}& \frac{1}{2}\|u\|_{H^1}^2 - 2\|u\|^6 - \|u\|^4 \\ &- 2\left\|f(s+t,\cdot)\right\|^2,\end{aligned} \tag{68}$$

which together with (67) and (58) imply that, for all $u_0 \in B_R$, $s \leqslant 0$, and $t \geqslant t_1$ (with a larger τ_0),

$$\begin{aligned}\left\|u(s+t,s,u_0)\right\|_{H^1}^2 \leqslant{}& 2C + 4\left\|u(s+t,s,u_0)\right\|^6 \\ &+ 2\left\|u(s+t,s,u_0)\right\|^4 \\ &+ 4\left\|f(s+t,\cdot)\right\|^2 \\ \leqslant{}& c\left(1+\rho_1^3+\rho_1^2+F_1(t)\right) \\ \leqslant{}& c\left(1+\rho_1^3+\rho_1^2+\rho_1\right),\end{aligned} \tag{69}$$

which shows the needed result (54).

To prove (55), we multiply (24) by a test function φ with $\|\varphi\|_{H^1} = 1$, then by the Agmon inequality,

$$\begin{aligned}\langle u_t,\varphi\rangle ={}& -i\int u_{xx}\varphi - i\int |u|^2 u\varphi + (i-\alpha)\int u\varphi \\ &+ \int f\varphi \\ \leqslant{}& \left\|u_x\right\|^2 + \left\|\varphi_x\right\|^2 + \|u\|_4^4 + \|\varphi\|_4^4 \\ &+ c\left(\|u\|^2+\|\varphi\|^2\right) + \left\|f(s+t)\right\|^2 \\ \leqslant{}& c\left\|u_x\right\|^2 + C + F_1(t) \leqslant c\left\|u_x\right\|^2 + C.\end{aligned} \tag{70}$$

Then (55) follows from (54). □

5. Compact-Decay Decomposition

5.1. High-Low Frequency Decomposition. We expand $u(t)$ (the solution of (24)) into its Fourier series

$$u(t) = \sum_{k\in\mathbb{Z}} u_k(t)\,e^{2k\pi ix} \tag{71}$$

and split $u(t)$ into low frequency part and high-frequency part $u(t) = P_N u(t) + Q_N u(t)$ with

$$\begin{aligned}P_N u(t) &= \sum_{|k|\leqslant N} u_k(t)\,e^{2k\pi ix}, \\ Q_N u(t) &= \sum_{|k|>N} u_k(t)\,e^{2k\pi ix}.\end{aligned} \tag{72}$$

Let $s \leqslant 0$ be arbitrary but fixed and take the initial value u_0 in a ball B_R of H^1. Then we are concerned with two functions of $t \geqslant 0$:

$$\begin{aligned}y &= y(t) = y(t;s) = P_N u(s+t,s,u_0), \\ z &= z(t) = z(t;s) = Q_N u(s+t,s,u_0).\end{aligned} \tag{73}$$

By the forward absorption given in Lemma 10, we know there is a $t_1 > 0$ (depending on the radius of initial ball) such that

$$\sup_{t\geqslant t_1}\sup_{s\leqslant 0}\left(\left\|y(t;s)\right\|_{H^1}^2 + \|z(t;s)\|_{H^1}^2\right) \leqslant C. \tag{74}$$

The high-frequency part $z(t)$ satisfies the following equation.

$$\begin{aligned}&z_t + \alpha z + iz_{xx} - iz + iQ_N\left(\left|y+z\right|^2(y+z)\right) \\ &\quad = Q_N f(s+t), \quad t \geqslant t_1, \\ &z(t_1) = Q_N u(s+t_1,s,u_0).\end{aligned} \tag{75}$$

Then we consider the following equation on $Q_N H^1$ with the initial value zero.

$$\begin{aligned}&Z_t + \alpha Z + iZ_{xx} - iZ + iQ_N\left(\left|y+Z\right|^2(y+Z)\right) \\ &\quad = Q_N f(s+t), \quad t \geqslant t_1, \\ &Z(t_1) = Z(t_1;s) = 0,\end{aligned} \tag{76}$$

where $Z := Z(t)$ (if exists) is actually a function w.r.t. $t \geqslant t_1$, $s \leqslant 0$, $u_0 \in B_R$, and $N \in \mathbb{N}$. Sometimes, we write it as $Z = Z(t;s)$. We can decompose the evolution process S by

$$u(s+t, s, u_0) = v(t) + w(t), \quad \text{where } v := y + Z,\ w := z - Z \tag{77}$$

for all $t \geqslant t_1$, $s \leqslant 0$, and $u_0 \in H^1$. In the sequel, the main task is to prove the asymptotic compactness of v and the exponential decay of w in $Q_N H^1$ for large N.

5.2. The Uniformly Bounded Estimate for Z. To prove the existence of solutions for (76), we need to consider the approximation Z^m, which is the solution for the projection of (76) on the subspace

$$P_m Q_N H^1 = \left\{ Z^m(t) : Z^m(t) = \sum_{N<|k|\leqslant m} u_k(t) e^{2k\pi i x} \right\}, \quad m > N. \tag{78}$$

Then, by the standard Galerkin method and a priori estimate (see the next proposition), one can prove the existence of Z in $Q_N H^1$ if N is large enough.

In the following proposition, we actually prove the result for Z^m. However, for the sake of simplicity, we omit the subscript m and also omit the proof of convergence as $m \to \infty$.

Proposition 11. *Suppose* $\mathbf{A1}'$, $\mathbf{A2}'$ *are satisfied. Then there exists* $N_0 = N_0(\alpha, f)$ *such that, for* $N \geqslant N_0$, *(76) has a uniformly bounded solution* Z *in* $Q_N H^1$ *such that*

$$\sup_{t\geqslant t_1}\sup_{s\leqslant 0} \|Z(t;s)\|_{H^1}^2 \leqslant C, \tag{79}$$

where C is a constant and t_1 *is the forward absorbing entering time.*

Proof. Multiplying (76) by $-\overline{Z}_t - \alpha\overline{Z}$, taking the imaginary part, after some computations, we obtain an energy equation:

$$\frac{1}{2}\frac{d}{dt}\Psi_1(Z) + \alpha\Psi_1(Z) = \Psi_2(Z), \quad t \geqslant t_1 \tag{80}$$

with

$$\begin{aligned}\Psi_1(Z) &= \|Z\|_{H^1}^2 + 2\,\mathrm{Im}\int_\Omega f\overline{Z}dx - \frac{1}{2}\|Z\|_4^4 \\ &\quad - 2\,\mathrm{Re}\int_\Omega |y|^2 y\overline{Z}\,dx \\ &\quad - 2\,\mathrm{Re}\int_\Omega |Z|^2 Z\overline{y}\,dx - 2\int_\Omega |y|^2|Z|^2\,dx \\ &\quad - 2\int_\Omega (\mathrm{Re}(\overline{y}Z))^2\,dx,\end{aligned}$$

$$\begin{aligned}\Psi_2(Z) &= \alpha\,\mathrm{Im}\int_\Omega f\overline{Z}\,dx + \mathrm{Im}\int_\Omega f_t\overline{Z}\,dx + \frac{\alpha}{2}\|Z\|_4^4 \\ &\quad - \mathrm{Re}\int_\Omega \left(|y|^2 y\right)_t \overline{Z}\,dx \\ &\quad - \alpha\,\mathrm{Re}\int_\Omega |y|^2 y\overline{Z}\,dx \\ &\quad - \int_\Omega \mathrm{Re}(\overline{y}_t Z)|Z|^2\,dx \\ &\quad + \alpha\int_\Omega \mathrm{Re}(\overline{y}Z)|Z|^2\,dx \\ &\quad - \mathrm{Re}\int_\Omega \overline{y}y_t|Z|^2\,dx \\ &\quad - 2\int_\Omega \mathrm{Re}(\overline{y}_t Z)\,\mathrm{Re}(\overline{y}Z)\,dx.\end{aligned} \tag{81}$$

We now consider the upper bound of $\Psi_2(Z)$; by Assumption $\mathbf{A1}'$, we have

$$\begin{aligned} I_1 &:= \alpha\,\mathrm{Im}\int_\Omega f\overline{Z}\,dx + \mathrm{Im}\int_\Omega f_t\overline{Z}\,dx \\ &\leqslant \frac{\alpha}{16}\|Z\|^2 + c\|f(s+t,\cdot)\|^2 + c\|f_t(s+t,\cdot)\|^2 \\ &\leqslant \frac{\alpha}{16}\|Z\|_{H^1}^2 + c\rho_1 + c\|f_t(s+t,\cdot)\|^2. \end{aligned} \tag{82}$$

By the classical interpolation and the Poincaré inequality on $Q_N H^1$

$$\begin{aligned}\|Z\|_4 &\leqslant c\|Z\|^{3/4}\|Z\|_{H^1}^{1/4}, \\ \|Z\| &\leqslant \frac{c}{N}\|Z\|_{H^1},\end{aligned} \tag{83}$$

the second term of $\Psi_2(Z)$ can be bounded by

$$I_2 := \frac{\alpha}{2}\|Z\|_4^4 \leqslant c\|Z\|^3\|Z\|_{H^1} \leqslant \frac{c}{N^3}\|Z\|_{H^1}^4. \tag{84}$$

Note that H^1 forms a Banach multiplicative algebra in one-dimension; that is,

$$\|uvw\|_{H^1} \leqslant c\|u\|_{H^1}\|v\|_{H^1}\|w\|_{H^1}. \tag{85}$$

Then, by Lemma 10 and (74), the third term of $\Psi_2(Z)$ can be bounded by

$$\begin{aligned} I_3 &:= -\mathrm{Re}\int_\Omega \left(|y|^2 y\right)_t \overline{Z}\,dx \\ &\leqslant \|y_t\|_{H^{-1}}\left\||y|^2 Z\right\|_{H^1} + 2\|y_t\|_{H^{-1}}\|y\,\mathrm{Re}(\overline{y}Z)\|_{H^1} \\ &\leqslant 3\|y_t\|_{H^{-1}}\|y\|_{H^1}^2\|Z\|_{H^1} \leqslant C\|Z\|_{H^1} \\ &\leqslant \frac{\alpha}{32}\|Z\|_{H^1}^2 + C. \end{aligned} \tag{86}$$

By (74) and the embedding $L^6 \hookrightarrow H^1$, the fourth term is bounded by

$$I_4 := -\alpha \operatorname{Re} \int_\Omega |y|^2 y \overline{Z} dx \leqslant \frac{\alpha}{32} \|Z\|^2 + c \|y\|_6^6 \leqslant \frac{\alpha}{32} \|Z\|^2 + C. \tag{87}$$

By the Agmon inequality and (83), we have

$$\|Z\|_\infty \leqslant c \|Z\|^{1/2} \|Z\|_{H^1}^{1/2} \leqslant \frac{c}{\sqrt{N}} \|Z\|_{H^1}. \tag{88}$$

Then the fifth term of $\Psi_2(Z)$ is bounded by

$$I_5 := -\int_\Omega \operatorname{Re}(\overline{y}_t Z) |Z|^2 dx \leqslant c \|y_t\|_{H^{-1}} \|Z\|_\infty^2 \|Z\|_{H^1} \leqslant \frac{c}{N} \|Z\|_{H^1}^3 \leqslant \frac{\alpha}{32} \|Z\|_{H^1}^2 + \frac{c}{N^2} \|Z\|_{H^1}^4. \tag{89}$$

Similarly, by (83) and the Hölder inequality, the sixth term of $\Psi_2(Z)$ is bounded by

$$\begin{aligned} I_6 &:= \alpha \int_\Omega \operatorname{Re}(\overline{y} Z) |Z|^2 dx \leqslant \alpha \|y\|_4 \left\|Z^3\right\|_{4/3} \\ &= \alpha \|y\|_4 \|Z\|_4^3 \leqslant c \|y\|_{H^1} \|Z\|^{9/4} \|Z\|_{H^1}^{3/4} \\ &\leqslant \frac{c}{N^{9/4}} \|Z\|_{H^1}^3 \leqslant \frac{\alpha}{32} \|Z\|_{H^1}^2 + \frac{c}{N^{9/2}} \|Z\|_{H^1}^4. \end{aligned} \tag{90}$$

By (88), the rest terms can be bounded by

$$\begin{aligned} I_7 &:= -\operatorname{Re} \int_\Omega \overline{y} y_t |Z|^2 dx - 2 \int_\Omega \operatorname{Re}(\overline{y}_t Z) \operatorname{Re}(\overline{y} Z) dx \\ &\leqslant 3 \|Z\|_\infty \int_\Omega |y_t y Z| dx \leqslant 3 \|Z\|_\infty \|y_t\|_{H^{-1}} \|yZ\|_{H^1} \\ &\leqslant c \|Z\|_\infty \|y_t\|_{H^{-1}} \|y\|_{H^1} \|Z\|_{H^1} \leqslant \frac{c_1}{\sqrt{N}} \|Z\|_{H^1}^2. \end{aligned} \tag{91}$$

By the above estimates, we know that $\Psi_2(Z)$ can be bounded by

$$\begin{aligned} \Psi_2(Z) &= \sum_{i=1}^7 I_i \\ &\leqslant \left(\frac{3\alpha}{16} + \frac{c_1}{\sqrt{N}}\right) \|Z\|_{H^1}^2 + \frac{c}{N^2} \|Z\|_{H^1}^4 + c\rho_1 \\ &\quad + c \left\|f_t(s+t,\cdot)\right\|^2. \end{aligned} \tag{92}$$

Letting $N_0' = [256 c_1/\alpha^2] + 1$, where $[\cdot]$ denotes the integer-valued function, we have, for $N \geqslant N_0'$,

$$\Psi_2(Z) \leqslant \frac{\alpha}{4} \|Z\|_{H^1}^2 + \frac{c}{N^2} \|Z\|_{H^1}^4 + c\rho_1 + c \left\|f_t(s+t,\cdot)\right\|^2. \tag{93}$$

On the other hand, we similarly obtain a lower bound of $\Psi_1(Z)$.

$$\begin{aligned} \Psi_1(Z) &\geqslant \|Z\|_{H^1}^2 - c \left\|f(s+t)\right\|^2 - \frac{1}{2} \|Z\|_{H^1}^2 - \frac{1}{2} \|Z\|_4^4 \\ &\quad - c \left(\|y\|_4^3 \|Z\|_4 + \|y\|_4 \|Z\|_4^3 + \|y\|_4^2 \|Z\|_4^2\right) \\ &\geqslant \frac{1}{2} \|Z\|_{H^1}^2 - C - \|Z\|_4^4 - g(\|y\|_4) \\ &\geqslant \frac{1}{2} \|Z\|_{H^1}^2 - \frac{c}{N^3} \|Z\|_{H^1}^4 - C \\ &\geqslant \frac{1}{2} \|Z\|_{H^1}^2 - \frac{c}{N^2} \|Z\|_{H^1}^4 - C, \end{aligned} \tag{94}$$

where $g(\|y\|_4)$ is uniformly bounded in view of (74). By (93)-(94), we have

$$2\Psi_2(Z) - \alpha \Psi_1(Z) \leqslant \frac{c}{N^2} \|Z\|_{H^1}^4 + C + c \left\|f_t(s+t,\cdot)\right\|^2. \tag{95}$$

Then it follows from (80) that

$$\frac{d}{dt} \Psi_1(Z) + \alpha \Psi_1(Z) \leqslant \frac{c}{N^2} \|Z\|_{H^1}^4 + C + c \left\|f_t(s+t,\cdot)\right\|^2. \tag{96}$$

Since $Z(t_1) = 0$ and $\Psi_1(Z(t_1)) = \Psi_1(0) = 0$, it follows from the Gronwall inequality on $[t_1, t]$ that

$$\begin{aligned} \Psi_1(Z(t)) &\leqslant \frac{c}{N^2} \int_{t_1}^t \|Z(r)\|_{H^1}^4 e^{\alpha(r-t)} dr \\ &\quad + C \int_{t_1}^t e^{\alpha(r-t)} dr \\ &\quad + c \int_{t_1}^t e^{\alpha(r-t)} \left\|f_t(s+r,\cdot)\right\|^2 dr \\ &\leqslant \frac{c}{N^2} \int_{t_1}^t \|Z(r)\|_{H^1}^4 e^{\alpha(r-t)} dr + \frac{C}{\alpha} \\ &\quad + c F_2(s+t), \end{aligned} \tag{97}$$

which along with (94) and Assumption **A2′** imply that, for all $t \geqslant t_1$,

$$\|Z(t)\|_{H^1}^2 \leqslant \frac{c}{N^2} \|Z(t)\|_{H^1}^4 + \frac{c}{N^2} \int_{t_1}^t \|Z(r)\|_{H^1}^4 e^{\alpha(r-t)} dr + C. \tag{98}$$

Hence, $\xi(t) := \sup_{r\in[t_1,t]}\|Z(r)\|_{H^1}^2$ satisfies the following twice inequality:

$$\xi(t) \leqslant \frac{c_2}{N^2}\xi^2(t) + C_2, \qquad \text{i.e. } c_2\xi^2(t) - N^2\xi(t) + C_2N^2 \geqslant 0, \tag{99}$$

where c_2, C_2 are positive constants. Let $N_0'' = [2\sqrt{c_2C_2}]+1$ and $N_0 = \max(N_0', N_0'')$. Then for all $N \geqslant N_0$ the discriminant of twice inequality is positive:

$$\Delta = N^4 - 4c_2C_2N^2 = N^2\left(N^2 - 4c_2C_2\right) > 0. \tag{100}$$

In this case, the equation $c_2\xi^2 - N^2\xi + C_2N^2 = 0$ has two positive roots.

$$\alpha_1 = \frac{N^2 - N\sqrt{N^2 - 4c_2C_2}}{2c_2}, \qquad \alpha_2 = \frac{N^2 + N\sqrt{N^2 - 4c_2C_2}}{2c_2}. \tag{101}$$

We show that $\alpha_1 \leqslant 2C_2$. Indeed,

$$\alpha_1 \leqslant 2C_2 \Longleftrightarrow N^2 - 4c_2C_2 \leqslant N\sqrt{N^2 - 4c_2C_2} \Longleftrightarrow \sqrt{N^2 - 4c_2C_2} \leqslant N, \tag{102}$$

where the last inequality is obviously true. Now the twice inequality (99) has the solution

$$\xi(t) \leqslant \alpha_1 \quad \text{or } \xi(t) \geqslant \alpha_2, \quad t \geqslant t_1. \tag{103}$$

Since $\xi(t_1) = 0$ and the mapping $t \to \xi(t)$ is continuous on $[t_1, +\infty)$, it follows that

$$\xi(t) \leqslant \alpha_1 \leqslant 2C_2, \quad t \geqslant t_1, \tag{104}$$

which proves the required bound. □

5.3. Further Regularity Estimates. We further show a regularity result for Z in H^2.

Proposition 12. *Suppose* $\mathbf{A1}'$, $\mathbf{A2}'$, *and* $\mathbf{A3}$ *are satisfied. Then there exists* $N_1 = N_1(\alpha, f) \geqslant N_0$ *such that, for* $N \geqslant N_1$, *the solution* Z *of (76) is uniformly bounded in* Q_NH^2:

$$\sup_{t\geqslant t_1}\sup_{s\leqslant 0}\|Z(t;s)\|_{H^2}^2 \leqslant C(N+1), \tag{105}$$

where C *is a constant and* t_1 *is the forward absorbing entering time.*

Proof. Just like we did in the above proposition, we need to firstly show the bound of Z_x^m in H^1 and then let $m \to +\infty$ to obtain the bound of Z_x in H^1. Thus, for the sake of convenience, we omit the detail of letting $m \to +\infty$ and also drop the superscript m of Z^m and write $Z = Z^m$, $v = v^m = y + Z^m$, $Q_N = P_mQ_N$ in this proof.

The first thing we shall do is to obtain the following equation by differentiating (76):

$$Z_{xt} + \alpha Z_x + Q_N\left(A - F'(v)\right)Z_x = G, \quad t \geqslant t_1, \tag{106}$$

with initial value $Z_x(t_1) = 0$, where A, F', and G are given by

$$\begin{aligned} AZ_x &= iZ_{xxx} - iZ_x, \\ F'(v)Z_x &= -i|v|^2 Z_x - 2i\operatorname{Re}(\bar{v}Z_x)v, \\ G &= Q_N\left(f_x(s+t,\cdot) + F'(v)y_x\right). \end{aligned} \tag{107}$$

In order to show the needed result (105), we divide it into three steps.

Step 1. We show $G \in C_b([t_1, +\infty); H^{-1})$. Let φ be a test function taken from H^1 such that $\|\varphi\|_{H^1} = 1$. Then by Assumption $\mathbf{A1}'$,

$$\begin{aligned} \langle\varphi, f_x(s+t)\rangle &= \int \varphi f_x(s+t) \\ &\leqslant c\left(\|\varphi_x\|^2 + \|f(s+t)\|^2\right) \\ &\leqslant c\left(\|\varphi\|_{H^1}^2 + F_1(s+t)\right) \leqslant c(1+\rho_1), \end{aligned} \tag{108}$$

which implies that $\|f_x(s+t)\|_{H^{-1}} \leqslant C$. Since Q_N is a bounded operator from H^{-1} to H^{-1}, it follows that $\|Q_Nf_x(s+t)\|_{H^{-1}} \leqslant C$ for all $t \geqslant t_1$ and $s \leqslant 0$.

The rest term of G can be rewritten as

$$Q_NF'(v)y_x = -iQ_N|v|^2 y_x - 2iQ^N\operatorname{Re}(\bar{v}y_x)v. \tag{109}$$

Note that we have the inverse inequality $\|y_x\|_{H^1} \leqslant 2\pi N\|y\|_{H^1}$ on P_NH^1. By Lemma 10 and Proposition 11, we know $y, v \in C_b([t_1, +\infty); H^1)$, where $v = y+Z$. Then, by the multiplicative algebra (see (85)), we have, for all $t \geqslant t_1$,

$$\begin{aligned} &\left\|-iQ_N|v(t)|^2 y_x(t)\right\|_{H^1} \leqslant c\|v\bar{v}y_x\|_{H^1} \\ &\quad\leqslant c\|v\|_{H^1}^2\|y_x\|_{H^1} \leqslant cN\|y\|_{H^1} \leqslant CN, \\ &\left\|-2iQ_N\operatorname{Re}(\bar{v}(t)y_x(t))v(t)\right\|_{H^1} \leqslant cN\|v\|_{H^1}^2\|y\|_{H^1} \\ &\quad\leqslant CN. \end{aligned} \tag{110}$$

Therefore, $Q_NF'(v)y_x \in C_b([t_1, +\infty); H^1)$ and so $G \in C_b([t_1, +\infty); H^{-1})$ with $\|G(t)\|_{H^{-1}} \leqslant C(N+1)$.

Step 2. We give the estimates of $\|G_t\|_{H^{-1}}$, where it is easy to obtain

$$\begin{aligned} G_t = {} & Q_Nf_{xt}(s+t,\cdot) - 2iQ_N\operatorname{Re}(\bar{v}v_t)y_x \\ & - iQ_N|v|^2 y_{xt} - 2iQ_N\left(\operatorname{Re}(\bar{v}y_x)v\right)_t. \end{aligned} \tag{111}$$

By Assumption **A3**, for the test function $\|\varphi\|_{H^1} = 1$, we have

$$\begin{aligned}\langle \varphi, f_{xt}(s+t)\rangle &= \int \varphi f_{xt}(s+t) \\ &\leqslant c\left(\|\varphi_x\|^2 + \|f_t(s+t)\|^2\right).\end{aligned} \tag{112}$$

Hence, $\|Q_N f_{xt}(s+t)\|_{H^{-1}} \leqslant c(1 + \|f_t(s+t,\cdot)\|^2)$ for all $t \geqslant t_1$ and $s \leqslant 0$.

To estimate the rest terms, we note that $\sup_{t\geqslant t_1}\|Z_t(t)\|_{H^{-1}} \leqslant C$, which can be obtained from (76) and the uniform bound of Z. By Lemma 10, $\|y_t\|_{H^{-1}} \leqslant C$ and so $\|v_t\|_{H^{-1}} \leqslant C$. For the second term of G_t, by the multiplicative algebra (85) again, we have

$$\begin{aligned}\left|\langle -2i\,\mathrm{Re}(\bar{v}v_t)\, y_x, \varphi\rangle\right| &\leqslant \int \left|v_t v y_x \varphi\right| dx \\ &\leqslant \|v_t\|_{H^{-1}} \|v y_x \varphi\|_{H^1} \\ &\leqslant C\|v\|_{H^1}\|\varphi\|_{H^1}\|y_x\|_{H^1} \\ &\leqslant C\|y_x\|_{H^1} \leqslant CN\|y\|_{H^1} \\ &\leqslant CN,\end{aligned} \tag{113}$$

where $\|\varphi\|_{H^1} = 1$. Hence $\|\mathrm{Re}(\bar{v}v_t)y_x\|_{H^{-1}} \leqslant CN$ and so $\|Q_N\,\mathrm{Re}(\bar{v}v_t)y_x\|_{H^{-1}} \leqslant CN$ in view of the boundedness of the operator Q_N from H^{-1} to H^{-1}. It is similar to obtain the estimates for the rest terms in G_t and so

$$\|G_t(t;s)\|_{H^{-1}} \leqslant c\left(1 + N + \|f_t(s+t,\cdot)\|^2\right), \qquad t \geqslant t_1,\ s \leqslant 0. \tag{114}$$

Step 3. Finally, we bound $\|Z_x\|_{H^1}$. Multiplying (106) by $-\overline{Z}_{xt} - \alpha\overline{Z}_x$, taking the imaginary part of the resulting equation and then integrating over Ω, we obtain an energy equation:

$$\frac{d}{dt}\Lambda_1(Z_x) + 2\alpha\Lambda_1(Z_x) = 2\Lambda_2(Z_x) \tag{115}$$

where

$$\begin{aligned}\Lambda_1(Z_x) &= \|Z_x\|_{H^1}^2 - \int_\Omega |v|^2 |Z_x|^2\, dx \\ &\quad - 2\int_\Omega \mathrm{Re}(\bar{v}Z_x)^2\, dx \\ &\quad + 2\int_\Omega \mathrm{Im}\left(G\overline{Z}_x\right) dx,\end{aligned} \tag{116}$$

$$\begin{aligned}\Lambda_2(Z_x) &= -\int_\Omega \mathrm{Re}(\bar{v}v_t)\,|Z_x|^2\, dx \\ &\quad - 2\int_\Omega \mathrm{Re}(\bar{v}_t Z_x)\cdot\mathrm{Re}(\bar{v}Z_x)\, dx \\ &\quad + \int_\Omega \mathrm{Im}\left(G_t\overline{Z}_x\right) dx \\ &\quad + \alpha\int_\Omega \mathrm{Im}\left(G\overline{Z}_x\right) dx.\end{aligned} \tag{117}$$

By the bound of v in H^1 and v_t in H^{-1}, and by the inequality $\|v\|_\infty \leqslant (c/\sqrt{N})\|v\|_{H^1}$ on the space $Q_N H^1$ (see (88)), we have, for $N \geqslant N_0$,

$$\begin{aligned}\Lambda_2(Z_x) &\leqslant 3\|v_t\|_{H^{-1}}\|v\|_\infty\|Z_x\|_{H^1}^2 + \|G_t\|_{H^{-1}}\|Z_x\|_{H^1} \\ &\quad + \|G\|_{H^{-1}}\|Z_x\|_{H^1} \\ &\leqslant \frac{c}{\sqrt{N}}\|v_t\|_{H^{-1}}\|v\|_{H^1}\|Z_x\|_{H^1}^2 \\ &\quad + \left(\|G_t\|_{H^{-1}} + \|G\|_{H^{-1}}\right)\|Z_x\|_{H^1} \\ &\leqslant \left(\frac{c_0}{\sqrt{N}} + \frac{\alpha}{16}\right)\|Z_x\|_{H^1}^2 \\ &\quad + c\left(\|G_t\|_{H^{-1}}^2 + \|G\|_{H^{-1}}^2\right).\end{aligned} \tag{118}$$

Letting $N_1' = \max([256c_0/9\alpha^2] + 1, N_0)$, then we have

$$\Lambda_2(Z_x) \leqslant \frac{\alpha}{4}\|Z_x\|_{H^1}^2 + c\left(\|G_t\|_{H^{-1}}^2 + \|G\|_{H^{-1}}^2\right), \qquad \text{for } N \geqslant N_1'. \tag{119}$$

On the other hand, we have the lower bound of $\Lambda_1(Z_x)$:

$$\begin{aligned}\Lambda_1(Z_x) &\geqslant \|Z_x\|_{H^1}^2 - 3\|v\|_\infty^2\|Z_x\|_{L^2}^2 \\ &\quad - 2\|G\|_{H^{-1}}\|Z_x\|_{H^1} \\ &\geqslant \|Z_x\|_{H^1}^2 - \frac{c}{N^2}\|Z_x\|_{H^1}^2 - 2\|G\|_{H^{-1}}\|Z_x\|_{H^1} \\ &\geqslant \left(\frac{3}{4} - \frac{c_1}{N^2}\right)\|Z_x\|_{H^1}^2 - c\|G\|_{H^{-1}}^2.\end{aligned} \tag{120}$$

Letting $N_1 = \max([\sqrt{4c_1}] + 1, N_1')$, we see, that for $N \geqslant N_1$,

$$\Lambda_1(Z_x) \geqslant \frac{1}{2}\|Z_x\|_{H^1}^2 - c\|G\|_{H^{-1}}^2 \quad \text{for } t \geqslant t_1. \tag{121}$$

Therefore, we substitute (119)-(121) into (115) and use (114) to find

$$\begin{aligned}\frac{d}{dt}\Lambda_1(Z_x) + \alpha\Lambda_1(Z_x) &\leqslant c\left(\|G_t\|_{H^{-1}}^2 + \|G\|_{H^{-1}}^2\right) \\ &\leqslant c\left(1 + N + \|f_t(s+t,\cdot)\|^2\right).\end{aligned} \tag{122}$$

Applying the Gronwall inequality on $[t_1, t]$ and observing $\Lambda_1(Z_x(t_1)) = \Lambda_1(0) = 0$, we get

$$\begin{aligned}\Lambda_1(Z_x(t)) &\leqslant \int_{t_1}^t ce^{\alpha(r-t)}\left(1 + N + \|f_t(s+r,\cdot)\|^2\right) dr \\ &\leqslant c\left(1 + N + F_2(t)\right),\end{aligned} \tag{123}$$

which along with **A2′** and (121) imply the needed result (105). □

5.4. Exponential Decay for Large Times. Let $z = z(t) = z(t; s, u_0) = Q_N u(s+t, s, u_0)$, which is a solution of (75). Let $Z = Z(t) = Z(t; s, u_0)$ be the corresponding solution of (76). We need to prove the exponential decay of $z - Z$.

Proposition 13. *Suppose* $\mathbf{A1}'$, $\mathbf{A2}'$ *are satisfied. Then there exists* $N_2 \geqslant N_1$ *such that, for* $N \geqslant N_2$,

$$\sup_{s\leqslant 0} \|z(t; s) - Z(t; s)\|_{H^1} \leqslant ce^{-\alpha(t-t_1)}, \quad \text{for } t \geqslant t_1. \tag{124}$$

Proof. Recall that $v = y + Z$ and $w = z - Z = u - v$. Then it follows from (75) and (76) that w is the solution of

$$\begin{aligned} & w_t + \alpha w + i w_{xx} - i w \\ & \quad = i Q_N\left(\left(|u|^2 + |v|^2\right) w + uv\overline{w}\right), \quad t \geqslant t_1. \end{aligned} \tag{125}$$

Multiplying (125) by $-\overline{w}_t - \alpha\overline{w}$, taking the imaginary part of the resulting equation and then integrating over Ω, we obtain an energy equation:

$$\frac{1}{2}\frac{d}{dt}\Upsilon_1(w) + \alpha\Upsilon_1(w) = \Upsilon_2(w) \tag{126}$$

with

$$\begin{aligned} \Upsilon_1(w) &= \|w\|_{H^1}^2 - \int_\Omega \left(|u|^2 + |v|^2\right)|w|^2\,dx \\ &\quad - \operatorname{Re}\int_\Omega uv\overline{w}^2\,dx, \\ \Upsilon_2(w) &= -\frac{1}{2}\int_\Omega \left(|u|^2 + |v|^2\right)_t |w|^2\,dx \\ &\quad - \frac{1}{2}\operatorname{Re}\int_\Omega (uv)_t\,\overline{w}^2\,dx. \end{aligned} \tag{127}$$

By the boundedness of z, Z in $Q_N H^1$ and u_t, v_t in H^{-1}, we have the upper bound of $\Upsilon_2(w)$:

$$\begin{aligned} \Upsilon_2(w) &\leqslant c\left(\|u_t + v_t\|_{H^{-1}}\right)\left(\|u + v\|_{H^1}\right)\|w\|_{H^1}\|w\|_{L^\infty} \\ &\leqslant \frac{c}{\sqrt{N}}\|w\|_{H^1}^2, \end{aligned} \tag{128}$$

and also the lower bound of $\Upsilon_1(w)$:

$$\Upsilon_1(w) \geqslant \left(1 - \frac{c}{N^2}\right)\|w\|_{H^1}^2 \geqslant \frac{1}{2}\|w\|_{H^1}^2 \tag{129}$$

if N is large enough. By (126)–(129), we have

$$\frac{d}{dt}\Upsilon_1(w) + \alpha\Upsilon_1(w) \leqslant \left(\frac{c_2}{\sqrt{N}} - \alpha\right)\|w\|_{H^1}^2 \leqslant 0 \tag{130}$$

if $N \geqslant N_2 := \max([4c_2^2/\alpha^2] + 1, N_1)$ and $t \geqslant t_1$. By the Gronwall inequality on $[t_1, t]$, we obtain

$$\Upsilon_1(w(t)) \leqslant e^{-\alpha(t-t_1)}\Upsilon_1(w(t_1)). \tag{131}$$

By Lemma 10, $\|z(t_1)\|_{H^1} = \|Q_N u(s + t_1, s, u_0)\|_{H^1} \leqslant C$. Note that $Z(t_1) = 0$ and so $\|w(t_1)\|_{H^1} \leqslant C$, which easily implies that $\Upsilon_1(w(t_1))$ is bounded. Therefore, by (129), we have

$$\|w(t)\|_{H^1}^2 \leqslant Ce^{-\alpha(t-t_1)}, \quad \forall t \geqslant t_1, \tag{132}$$

which completes the proof. □

6. Backward Compact Attractors

Finally, we state and prove the main result. We need to spilt the evolution process S. Let $N \geqslant N_2$ be fixed, where N_2 is the level given in Proposition 13. Then for all $t \geqslant 0$, $s \in \mathbb{R}$, and $u_0 \in H^1$, we write $S = S_1 + S_2$ with

$$\begin{aligned} & S_1(s+t, s)u_0 \\ & \quad = \begin{cases} y(t; s) = P_N u(s+t, s, u_0) & \text{if } 0 \leqslant t \leqslant t_1 \text{ or } s > 0, \\ v(t; s) = y(t; s) + Z(t; s) & \text{for } t \geqslant t_1,\ s \leqslant 0, \end{cases} \end{aligned} \tag{133}$$

$$\begin{aligned} & S_2(s+t, s)u_0 \\ & \quad = \begin{cases} z(t; s) = Q_N u(s+t, s, u_0) & \text{for } 0 \leqslant t \leqslant t_1 \text{ or } s > 0, \\ w(t; s) = z(t; s) - Z(t; s) & \text{for } t \geqslant t_1,\ s \leqslant 0, \end{cases} \end{aligned} \tag{134}$$

where $t_1 = t_1(\|u_0\|_{H^1})$ is the forward entering time.

Theorem 14. *(a) Suppose* **A1**, **A2**. *Then the nonautonomous Schrödinger equation has an increasing, bounded, and pullback absorbing set* $\mathscr{K} = \{\mathscr{K}(s)\}_{s\in\mathbb{R}}$.

(b) Suppose $\mathbf{A1}'$, $\mathbf{A2}'$, **A3**. *Then the nonautonomous Schrödinger equation has a backward compact pullback attractor* $\mathscr{A} = \{\mathscr{A}(s)\}_{s\in\mathbb{R}}$ *given by*

$$\begin{aligned} \mathscr{A}(r) &= \omega(\mathscr{K}(r), r) = \bigcap_{t_0>0}\overline{\bigcup_{t\geqslant t_0} S(r, r-t)\mathscr{K}(r)} \\ &= \omega_1(\mathscr{K}(r), r) = \bigcap_{t_0>0}\overline{\bigcup_{t\geqslant t_0} S_1(r, r-t)\mathscr{K}(r)}, \\ &\qquad r \in \mathbb{R}. \end{aligned} \tag{135}$$

Proof. The assertion (a) follows from Theorem 8. To prove (b), by the abstract result (i.e., Theorem 4), it suffices to prove that the evolution process S has a backward compact-decay decomposition.

We need to prove that S_1 is backward asymptotically compact. Indeed, let $r \in \mathbb{R}$ and take some sequences $r_n \leqslant r$, $\tau_n \to +\infty$, and $u_{0,n} \in B_R \subset H^1$. Without loss of generality, we assume $\tau_n \geqslant \max\{t_1, r\}$, where t_1 is the forward entering time. Then by (133) we have

$$\begin{aligned} S_1(r_n, r_n - \tau_n)u_{0,n} &= S_1((r_n - \tau_n) + \tau_n, r_n - \tau_n)u_{0,n} \\ &= y(\tau_n; r_n - \tau_n, u_{0,n}) \\ &\quad + Z(\tau_n; r_n - \tau_n, u_{0,n}). \end{aligned} \tag{136}$$

By Lemma 10, we know that

$$\begin{aligned} & \left\|y(\tau_n; r_n - \tau_n, u_{0,n})\right\|_{H^1} \\ & \quad \leqslant \sup_{t\geqslant t_1}\sup_{s\leqslant 0}\sup_{u_0\in B_R}\left\|y(t; s, u_0)\right\|_{H^1} \leqslant C, \end{aligned} \tag{137}$$

which means that $\{y(\tau_n; r_n - \tau_n, u_{0,n})\}$ is a bounded sequence in the finite-dimensional space $P_N H^1$ and so, passing to a subsequence, it is convergent. By Proposition 12, we know

$$\begin{aligned}&\left\|Z\left(\tau_n; r_n - \tau_n, u_{0,n}\right)\right\|_{H^2} \\ &\quad \leqslant \sup_{t\geqslant t_1}\sup_{s\leqslant 0}\sup_{u_0\in B_R}\left\|Z\left(t; s, u_0\right)\right\|_{H^2} \leqslant C\left(N+1\right),\end{aligned} \tag{138}$$

which means that $\{Z(\tau_n; r_n - \tau_n, u_{0,n})\}$ is bounded in H^2. By the Sobolev compact embedding, it is precompact in H^1 and so is $\{S_1(r_n, r_n - \tau_n)u_{0,n}\}$.

On the other hand, by Proposition 13, we know, for $r \in \mathbb{R}$, $\tau \geqslant \max\{t_1, r\}$ and $u_0 \in B_R$,

$$\begin{aligned}&\sup_{\hat{r}\leqslant r}\left\|S_2\left(\hat{r}, \hat{r}-\tau\right)u_0\right\|_{H^1} \\ &\quad \leqslant \sup_{s\leqslant 0}\left\|z\left(\tau; s, u_0\right) - Z\left(\tau; s, u_0\right)\right\|_{H^1} \leqslant Ce^{-\alpha(\tau - t_1)},\end{aligned} \tag{139}$$

which implies S_2 is backward decay. □

Remark 15. (I) In fact, under Assumptions **A1**$'$-**A2**$'$, it is possible to prove the existence of a forward attractor (see [32, 33]), which may be different from the pullback attractor. In the present paper, we only impose the forward absorption to deduce a backward compact-decay composition, which seems not to be deduced from the pullback absorption for the nonautonomous Schrödinger equation.

(II) Suppose $f, f_t \in C(\mathbb{R}, L^2)$ and they are time-periodic, then f satisfies both conditions **A1**$'$ and **A2**$'$. In this case, by using the theoretical result as given in [17], one can obtain a periodic pullback attractor and maybe a periodic forward attractor.

Conflicts of Interest

There are no conflicts of interest to declare.

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