Hindawi
Discrete Dynamics in Nature and Society
Volume 2018, Article ID 2325057, 11 pages
https://doi.org/10.1155/2018/2325057

*Research Article*

# Extinction in a Nonautonomous Discrete Lotka-Volterra Competitive System with Time Delay

**Meng Hu and Jingyuan Wei**

*School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan 455000, China*

Correspondence should be addressed to Meng Hu; humeng2001@126.com

Received 23 April 2018; Accepted 19 June 2018; Published 8 July 2018

Academic Editor: Antonia Vecchio



This paper is concerned with a nonautonomous discrete Lotka-Volterra competitive system with time delay. By using some analytical techniques, we prove that, under certain conditions, one of the species will be driven to extinction while the other one will be globally attractive with any positive solution of a discrete logistic equation.

## 1. Introduction

In this paper, we study the following Lotka-Volterra competitive system of difference equations

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left[r_1(n)\left(1-\frac{x_1(n)}{k_1(n)}-\mu_2(n)x_2(n-\tau)\right)\right], \\ x_2(n+1) &= x_2(n)\exp\left[r_2(n)\left(1-\mu_1(n)x_1(n)-\frac{x_2(n-\tau)}{k_2(n)}\right)\right], \end{aligned} \tag{1}$$

where $x_1(n), x_2(n)$ are population density of species $x_1$ and $x_2$ at time $n$, respectively, and $\{r_i(n)\}$, $\{k_i(n)\}$, and $\{\mu_i(n)\}$ for $i = 1, 2$ are bounded positive sequences such that

$$\begin{aligned} &0 < k_{i*} \le k_i(n) \le k_i^*, \\ &0 < r_{i*} \le r_i(n) \le r_i^*, \\ &0 < \mu_{i*} \le \mu_i(n) \le \mu_i^*, \\ &\qquad n \in N. \end{aligned} \tag{2}$$

Here, for any bounded sequence $\{a(n)\}$, $a^* = \sup_{n\in N} a(n)$ and $a_* = \inf_{n\in N} a(n)$.

We consider system (1) with the following initial conditions:

$$\begin{aligned} &x_1(0) > 0, \\ &x_2(\vartheta) = \varphi(\vartheta) > 0, \\ &\qquad \vartheta \in N[-\tau, 0] = \{-\tau, -\tau+1, \ldots, 0\}. \end{aligned} \tag{3}$$

It is not difficult to see that solutions of (1) are well defined for all $n \ge 0$ and satisfy $x_i(n) > 0, i = 1, 2$.

During the last decades, the study of extinction and permanence of the species has become one of the most important topics in population dynamics, and most of the studies are based on the traditional Lotka-Volterra competitive systems; see, for example, [1–15].

Consider the following nonautonomous Lotka-Volterra system of differential equations:

$$\begin{aligned} x_i'(t) &= x_i(t)\left[b_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\right], \\ &\qquad i = 1, 2, \cdots, n,\ n \ge 2, \end{aligned} \tag{4}$$

where $x_i(t)$ is population density of the $i$th species at time $t$ and $a_{ij}(t)$ and $b_i(t)$, $i, j = 1, 2, \cdots, n$, are continuous bounded functions defined on $R$.

Assume that

$$a_{ij}^{l} > 0,$$
$$a_{ij}^{u} < +\infty, \quad i, j = 1, 2, \cdots, n, \tag{5}$$

$$b_{i}^{l} > 0,$$
$$b_{i}^{u} < +\infty, \quad i = 1, 2, \cdots, n; \tag{6}$$

that is, the coefficients of system (4) are bounded above and below by strictly positive reals. Here, for any bounded function $f(t)$, $f^{u} = \sup_{t\in R} f(t)$ and $f^{l} = \inf_{t\in R} f(t)$.

Montes de Oca and Zeeman [11] studied system (4), under which the functions $a_{ij}(t)$ and $b_i(t)$ were assumed to satisfy conditions (5) and (6). It was shown that if, for each $k > 1$, there exists $i_k < k$ such that for any $j \le k$ the inequality

$$\frac{b_k^u}{a_{kj}^l} < \frac{b_{i_k}^l}{a_{i_k j}^u} \tag{7}$$

holds, then every solution $(x_1(t), x_2(t), \cdots, x_n(t))$ of system (4) with $x_i(t_0) > 0$, $i = 1, 2, \cdots, n$, for some $t_0 \in R$ has the property

$$\lim_{t\longrightarrow+\infty} \left(x_1(t) - u_1(t)\right) = 0,$$
$$\lim_{t\longrightarrow+\infty} x_j(t) = 0, \quad j = 2, 3, \cdots, n, \tag{8}$$

where $u_1(t)$ is the unique solution of the logistic differential equation

$$u'(t) = u(t)\left[b_1(t) - a_{11}(t)\, u(t)\right] \tag{9}$$

which is bounded above and below by strictly positive reals for all $t \in R$.

Zeeman [12], Ahmad [13], Teng [14], and Zhao et al. [15] have also studied the extinction of species in system (4), especially in [15], and Zhao et al. obtained the same results as [11–13] did under the weaker assumption that, for each $k > 1$, there exists $i_k < k$ such that for any $j \le k$ the inequality

$$\sup_{t\in[t_0,+\infty)} \frac{b_k(t)}{b_{i_k}(t)} < \inf_{t\in[t_0,+\infty)} \frac{a_{kj}(t)}{a_{i_k j}(t)} \tag{10}$$

holds for some $t_0 \in R$.

In addition, the nonautonomous discrete population models also received much attention from many scholars in the last decades, since the discrete time models governed by difference equation are more appropriate than the continuous ones when the populations have nonoverlapping generations. One of the famous models is the discrete Lotka-Volterra competitive system. Owing to its theoretical and practical significance, various discrete Lotka-Volterra competitive systems have been studied; see, for example, [16–18]. In these literatures, systems which have been studied are all delayed systems. Research has shown that time delays have a great destabilizing influence on species populations [19]. However, there are seldom results on the extinction and stability of species in a discrete population dynamic system, especially for a population dynamic system with time delay.

Motivated by the above works, the main purpose of this paper is to study the extinction and stability of system (1) and derive some sufficient conditions which guarantee one of the species will be driven to extinction while the other one will be globally attractive with any positive solution of a discrete logistic equation.

The organization of this paper is as follows. In Section 2, preliminary results are presented. In Sections 3 and 4, the main results are stated and proved. In Section 5, two examples together with their numerical simulations are given to illustrate the feasibility of the obtained results. In the last section, a brief discussion is stated.

## 2. Preliminaries

In this section, we shall develop some preliminary results, which will be used to prove the main results.

**Lemma 1** (see [20]). *Assume that $\{x(n)\}$ satisfies*

$$x(n+1) \ge x(n) \exp\{r(n)(1 - ax(n))\}, \quad n \ge n_0, \tag{11}$$

*$\limsup_{n\longrightarrow+\infty} x(n) \le x^*$ and $x(n_0) > 0$, where $a$ is a positive constant such that $ax^* > 1$ and $n_0 \in N$. Then*

$$\liminf_{n\longrightarrow+\infty} x(n) \ge \frac{1}{a} \exp\{r^*(1 - ax^*)\}. \tag{12}$$

**Lemma 2** (see [20]). *Assume that $\{x(n)\}$ satisfies $x(n) > 0$ and*

$$x(n+1) \le x(n) \exp\{r(n)(1 - ax(n))\} \tag{13}$$

*for $n \in [n_1, +\infty)$, where $a$ is a positive constant. Then*

$$\limsup_{n\longrightarrow+\infty} x(n) \le \frac{1}{ar^*} \exp(r^* - 1). \tag{14}$$

**Lemma 3.** *For every solution $(x_1(n), x_2(n))^T$ of (1) we have*

$$\limsup_{n\longrightarrow+\infty} x_i(n) \le x_i^*, \quad i = 1, 2, \tag{15}$$

*where $x_1^* = (k_1^*/r_1^*)\exp(r_1^* - 1)$, $x_2^* = (k_2^*/r_2^*)\exp(r_2^*(\tau+1) - 1)$.*

*Proof.* Let $\tilde{x}(n) = (x_1(n), x_2(n))^T$ be any positive solution of system (1). From the first equation of (1),

$$\begin{aligned} x_1(n+1) &\le x_1(n) \exp\left[r_1(n)\left(1 - \frac{x_1(n)}{k_1(n)}\right)\right] \\ &\le x_1(n) \exp\left[r_1(n)\left(1 - \frac{x_1(n)}{k_1^*}\right)\right]. \end{aligned} \tag{16}$$

By Lemma 2, we have

$$\limsup_{n\longrightarrow+\infty} x_1(n) \le \frac{k_1^*}{r_1^*} \exp(r_1^* - 1) \triangleq x_1^*. \tag{17}$$

From the second equation of (1), we have

$$x_2(n+1) \le x_2(n)\exp(r_2(n)), \tag{18}$$

and, then,

$$x_2(n-\tau) \ge x_2(n)\exp(-\tau r_2^*). \tag{19}$$

Substituting (19) into the second equation of (1), then

$$\begin{aligned} x_2(n+1) &\le x_2(n) \\ &\cdot \exp\left[r_2(n)\left(1-\frac{x_2(n-\tau)}{k_2(n)}\right)\right] \le x_2(n) \\ &\cdot \exp\left[r_2(n)\left(1-\frac{\exp(-\tau r_2^*)}{k_2^*}x_2(n)\right)\right]. \end{aligned} \tag{20}$$

By Lemma 2, we have

$$\limsup_{n\longrightarrow+\infty} x_2(n) \le \frac{k_2^*}{r_2^*}\exp(r_2^*(\tau+1)-1) \triangleq x_2^*. \tag{21}$$

This completes the proof. □

## 3. Extinction of $x_2$ and Stability of $x_1$

In this section, we firstly present the extinction of the species $x_2$.

**Theorem 4.** *Assume that the inequality*

$$\begin{aligned} &\limsup_{n\longrightarrow+\infty}\frac{r_2(n)}{r_1(n)} \\ &\quad < \liminf_{n\longrightarrow+\infty}\left\{\frac{r_2(n)\mu_1(n)k_1(n)}{r_1(n)}, \frac{r_2(n)}{r_1(n)\mu_2(n)k_2(n)}\right\} \end{aligned} \tag{22}$$

*holds, and then the species $x_2$ will be driven to extinction; that is, for any positive solution $(x_1(n), x_2(n))^T$ of system (1), $x_2(n) \longrightarrow 0$ exponentially as $n \longrightarrow +\infty$.*

*Proof.* Let $\tilde{x}(n) = (x_1(n), x_2(n))^T$ be a solution of system (1) with initial conditions (3). First we show that $x_2(n) \longrightarrow 0$ exponentially as $n \longrightarrow +\infty$.

From (1), we have

$$\begin{aligned} &\ln x_1(n+1) - \ln x_1(n) \\ &\quad = r_1(n)\left(1-\frac{x_1(n)}{k_1(n)} - \mu_2(n)x_2(n-\tau)\right), \\ &\ln x_2(n+1) - \ln x_2(n) \\ &\quad = r_2(n)\left(1-\mu_1(n)x_1(n) - \frac{x_2(n-\tau)}{k_2(n)}\right). \end{aligned} \tag{23}$$

By inequality (22), we can choose $\alpha, \beta, \varepsilon > 0$ such that

$$\begin{aligned} &\limsup_{n\longrightarrow+\infty}\frac{r_2(n)}{r_1(n)} < \frac{\alpha}{\beta} - \varepsilon < \frac{\alpha}{\beta} \\ &\quad < \liminf_{n\longrightarrow+\infty}\left\{\frac{r_2(n)\mu_1(n)k_1(n)}{r_1(n)}, \frac{r_2(n)}{r_1(n)\mu_2(n)k_2(n)}\right\}, \end{aligned} \tag{24}$$

and then there exists an $N_1 > 0$ such that, for all $n > N_1$,

$$r_2(n)\beta - r_1(n)\alpha < -\varepsilon\beta r_1(n) < -\varepsilon\beta r_{1*} < 0; \tag{25}$$

$$\alpha r_1(n) - \beta r_2(n)\mu_1(n)k_1(n) < 0; \tag{26}$$

$$\alpha r_1(n)\mu_2(n)k_2(n) - \beta r_2(n) < 0. \tag{27}$$

It follows from (23) and (25)-(27) that

$$\begin{aligned} &\beta(\ln x_2(n+1) - \ln x_2(n)) \\ &\quad - \alpha(\ln x_1(n+1) - \ln x_1(n)) \\ &\quad = (r_2(n)\beta - r_1(n)\alpha) \\ &\quad - \left(\beta r_2(n)\mu_1(n) - \frac{\alpha r_1(n)}{k_1(n)}\right)x_1(n) \\ &\quad - \left(\frac{\beta r_2(n)}{k_2(n)} - \alpha r_1(n)\mu_2(n)\right)x_2(n-\tau) \le r_2(n)\beta \\ &\quad - r_1(n)\alpha < -\varepsilon\beta r_{1*} < 0. \end{aligned} \tag{28}$$

Summating both sides of inequality (28) from 0 to $n-1$, then

$$\begin{aligned} &\beta(\ln x_2(n) - \ln x_2(0)) - \alpha(\ln x_1(n) - \ln x_1(0)) \\ &\quad < -\varepsilon\beta r_{1*}n. \end{aligned} \tag{29}$$

So, we can get

$$\begin{aligned} x_2(n) &< \left[\left(\frac{x_1(n)}{x_1(0)}\right)^\alpha (x_2(0))^\beta\right]^{1/\beta}\exp(-\varepsilon r_{1*}n) \\ &< \left[\left(\frac{x_1^*}{x_1(0)}\right)^\alpha (x_2(0))^\beta\right]^{1/\beta}\exp(-\varepsilon r_{1*}n). \end{aligned} \tag{30}$$

Therefore, we have $x_2(n) \longrightarrow 0$ exponentially as $n \longrightarrow +\infty$. This completes the proof. □

**Lemma 5.** *Under the assumption of Theorem 4. Let $\tilde{x}(n) = (x_1(n), x_2(n))^T$ be any positive solution of system (1), and then there exists a positive constant $x_{1^*}$ such that*

$$\liminf_{n\longrightarrow+\infty} x_1(n) \ge x_{1^*}, \tag{31}$$

*where $x_{1*} = k_{1*}\exp[r_1^*(1-x_1^*/k_{1*})]$ is a constant independent of any positive solution of system (1); i.e., the first species $x_1$ of system (1) is permanent.*

*Proof.* By Lemma 3 and Theorem 4,

$$\begin{aligned} &\limsup_{n\longrightarrow+\infty} x_1(n) \le x_1^*, \\ &\lim_{n\longrightarrow+\infty} x_2(n) = 0, \end{aligned} \tag{32}$$

and, for arbitrarily small positive constant $\varepsilon(0 < \varepsilon < 1/\mu_2^*)$, there exists an $N_2 > 0$ such that

$$\begin{aligned} &x_1(n) < x_1^* + \varepsilon, \\ &x_2(n) < \varepsilon \end{aligned} \tag{33}$$

for all $n > N_2$.

From the first equation of (1), for $n > N_2 + \tau$,

$$\begin{aligned} x_1(n+1) &= x_1(n) \\ &\cdot \exp\left[r_1(n)\left(1 - \frac{x_1(n)}{k_1(n)} - \mu_2(n)x_2(n-\tau)\right)\right] \\ &> x_1(n)\exp\left[r_1(n)\left(1 - \frac{x_1(n)}{k_{1*}} - \mu_2^*\varepsilon\right)\right] = x_1(n) \\ &\cdot \exp\left[r_1(n)(1-\mu_2^*\varepsilon)\left(1 - \frac{x_1(n)}{(1-\mu_2^*\varepsilon)k_{1*}}\right)\right]. \end{aligned} \tag{34}$$

Let $\varepsilon \longrightarrow 0$, and then

$$x_1(n+1) \geq x_1(n)\exp\left[r_1(n)\left(1 - \frac{x_1(n)}{k_{1*}}\right)\right]. \tag{35}$$

It is easy to check that the inequality $x_1^*/k_{1*} > 1$ holds. By Lemma 1, we have

$$\liminf_{n\longrightarrow+\infty} x_1(n) \geq k_{1*}\exp\left[r_1^*\left(1 - \frac{x_1^*}{k_{1*}}\right)\right] \triangleq x_{1^*}. \tag{36}$$

This completes the proof. □

Consider the following discrete logistic equation:

$$x(n+1) = x(n)\exp\left[r_1(n)\left(1 - \frac{x(n)}{k_1(n)}\right)\right]. \tag{37}$$

**Lemma 6** (see [21]). *Assume that $\{r_1(n)\}$ and $\{k_1(n)\}$ satisfy (2), and then any positive solution $\{x(n)\}$ of (37) satisfies*

$$x_{1*} < \liminf_{n\longrightarrow+\infty} x(n) \leq \limsup_{n\longrightarrow+\infty} x(n) \leq x_1^*. \tag{38}$$

**Theorem 7.** *Under the assumptions of Theorem 4 and Lemmas 5 and 6, furthermore, suppose that*

$$\frac{k_1^*}{k_{1*}}\exp(r_1^* - 1) \leq 2. \tag{39}$$

*Let $\tilde{x}(n) = (x_1(n), x_2(n))^T$ be any positive solution of system (1), and then the species $x_2$ will be driven to extinction; that is, $x_2(n) \longrightarrow 0$ as $n \longrightarrow +\infty$, and $x_1(n) \longrightarrow x(n)$ as $n \longrightarrow +\infty$, where $x(n)$ is any positive solution of (37).*

*Proof.* Let $\tilde{x}(n) = (x_1(n), x_2(n))^T$ be a solution of system (1) with initial conditions (3). From Lemmas 3 and 5, $x_1(n)$ is bounded above and below by positive constants on $[0, +\infty)$. To finish the proof of Theorem 7, it is enough to show that $x_1(n) \longrightarrow x(n)$ as $n \longrightarrow +\infty$, where $x(n)$ is any positive solution of (37).

Let

$$y(n) = \ln x_1(n) - \ln x(n), \tag{40}$$

and then

$$x_1(n) = x(n)\exp\{y(n)\}. \tag{41}$$

From the first equation of (1) and (37), we have

$$\begin{aligned} &\ln x_1(n+1) - \ln x_1(n) \\ &\quad = r_1(n)\left(1 - \frac{x_1(n)}{k_1(n)} - \mu_2(n)x_2(n-\tau)\right), \\ &\ln x(n+1) - \ln x(n) = r_1(n)\left(1 - \frac{x(n)}{k_1(n)}\right), \end{aligned} \tag{42}$$

and then

$$\begin{aligned} y(n+1) &= \ln x_1(n+1) - \ln x(n+1) \\ &= \ln x_1(n) \\ &\quad + r_1(n)\left(1 - \frac{x_1(n)}{k_1(n)} - \mu_2(n)x_2(n-\tau)\right) \\ &\quad - \ln x(n) - r_1(n)\left(1 - \frac{x(n)}{k_1(n)}\right) \\ &= y(n) - \frac{r_1(n)}{k_1(n)}(x_1(n) - x(n)) \\ &\quad - r_1(n)\mu_2(n)x_2(n-\tau) \\ &= y(n) - \frac{r_1(n)}{k_1(n)}x(n)(\exp\{y(n)\} - 1) \\ &\quad - r_1(n)\mu_2(n)x_2(n-\tau). \end{aligned} \tag{43}$$

By the mean value theorem, $\exp\{y(n)\} - 1 = \exp\{\theta y(n)\}y(n)$, where $\theta \in (0,1)$. It follows from (43) that

$$\begin{aligned} y(n+1) &= y(n)\left[1 - \frac{r_1(n)}{k_1(n)}x(n)\exp\{\theta y(n)\}\right] \\ &\quad - r_1(n)\mu_2(n)x_2(n-\tau). \end{aligned} \tag{44}$$

Notice that $\theta \in (0,1)$ and (40) implies that $x(n)\exp\{\theta y(n)\}$ lies between $x(n)$ and $x_1(n)$. From Lemmas 3, 5, and 6 and Theorem 4, for arbitrarily small $\varepsilon > 0$, there exists an $N_3 > 0$ such that

$$\begin{aligned} &x_{1*} - \varepsilon < x_1(n) < x_1^* + \varepsilon, \\ &x_2(n) < \varepsilon, \\ &x_{1*} - \varepsilon < x(n) < x_1^* + \varepsilon \end{aligned} \tag{45}$$

for all $n > N_3$. Therefore,

$$|y(n+1)| = \lambda|y(n)| + r_1^*\mu_2^*\varepsilon, \tag{46}$$

where

$$\lambda = \max\left\{\left|1 - \frac{r_1^*}{k_{1*}}(x_1^* + \varepsilon)\right|, \left|1 - \frac{r_{1*}}{k_1^*}(x_{1*} - \varepsilon)\right|\right\}. \tag{47}$$

Repeated iteration of (46) is

$$|y(n+1)| < \lambda^{n-N_3}|y(N_3)| + \frac{1-\lambda^{n-N_3}}{1-\lambda}r_1^*\mu_2^*\varepsilon. \tag{48}$$

If $(k_1^*/k_{1*})\exp(r_1^*-1)<2$, then $-1<1-(r_1^*/k_{1*})x_1^*$. For the above small enough $\varepsilon$, there is $-1<1-(r_1^*/k_{1*})(x_1^*+\varepsilon)$. On the other hand, $1-(r_1^*/k_{1*})(x_1^*+\varepsilon)<1-(r_{1*}/k_1^*)(x_{1*}-\varepsilon)<1$; that is, $0<\lambda<1$. From (48), we can obtain

$$\lim_{n\longrightarrow+\infty} y(n)=0; \tag{49}$$

that is,

$$\lim_{n\longrightarrow+\infty}\left(x_1(n)-x(n)\right)=0. \tag{50}$$

This completes the proof. □

## 4. Extinction of $x_1$ and Stability of $x_2$

In this section, we firstly present the extinction of the species $x_1$.

**Theorem 8.** *Assume that the inequality*

$$\begin{aligned}&\liminf_{n\longrightarrow+\infty}\frac{r_2(n)}{r_1(n)}\\&\quad>\limsup_{n\longrightarrow+\infty}\left\{\frac{r_2(n)\mu_1(n)k_1(n)}{r_1(n)},\frac{r_2(n)}{r_1(n)\mu_2(n)k_2(n)}\right\}\end{aligned} \tag{51}$$

*holds, and then the species $x_1$ will be driven to extinction; that is, for any positive solution $(x_1(n),x_2(n))^T$ of system (1), $x_1(n)\longrightarrow 0$ exponentially as $n\longrightarrow+\infty$.*

*Proof.* Let $\tilde{x}(n)=(x_1(n),x_2(n))^T$ be a solution of system (1) with initial conditions (3). First we show that $x_1(n)\longrightarrow 0$ exponentially as $n\longrightarrow+\infty$.

From (1), we have

$$\begin{aligned}&\ln x_1(n+1)-\ln x_1(n)\\&\quad=r_1(n)\left(1-\frac{x_1(n)}{k_1(n)}-\mu_2(n)x_2(n-\tau)\right),\\&\ln x_2(n+1)-\ln x_2(n)\\&\quad=r_2(n)\left(1-\mu_1(n)x_1(n)-\frac{x_2(n-\tau)}{k_2(n)}\right).\end{aligned} \tag{52}$$

By inequality (51), we can choose $\xi,\eta,\varepsilon>0$ such that

$$\begin{aligned}&\liminf_{n\longrightarrow+\infty}\frac{r_2(n)}{r_1(n)}>\frac{\xi}{\eta}+\varepsilon>\frac{\xi}{\eta}\\&\quad>\limsup_{n\longrightarrow+\infty}\left\{\frac{r_2(n)\mu_1(n)k_1(n)}{r_1(n)},\frac{r_2(n)}{r_1(n)\mu_2(n)k_2(n)}\right\},\end{aligned} \tag{53}$$

and then there exists an $N_4>0$ such that, for all $n>N_4$,

$$r_2(n)\eta-r_1(n)\xi>\varepsilon\eta r_1(n)>\varepsilon\eta r_{1*}>0; \tag{54}$$

$$\xi r_1(n)-\eta r_2(n)\mu_1(n)k_1(n)>0; \tag{55}$$

$$\xi r_1(n)\mu_2(n)k_2(n)-\eta r_2(n)>0. \tag{56}$$

It follows from (52) and (54)-(56) that

$$\begin{aligned}&\xi\left(\ln x_1(n+1)-\ln x_1(n)\right)\\&\quad-\eta\left(\ln x_2(n+1)-\ln x_2(n)\right)\\&\quad=\left(r_1(n)\xi-r_2(n)\eta\right)\\&\quad-\left(\frac{\xi r_1(n)}{k_1(n)}-\eta r_2(n)\mu_1(n)\right)x_1(n)\\&\quad-\left(\xi r_1(n)\mu_2(n)-\frac{\eta r_2(n)}{k_2(n)}\right)x_2(n-\tau)\le r_1(n)\xi\\&\quad-r_2(n)\eta<-\varepsilon\eta r_{1*}<0.\end{aligned} \tag{57}$$

Summating both sides of inequality (57) from 0 to $n-1$, then

$$\begin{aligned}&\xi\left(\ln x_1(n)-\ln x_1(0)\right)-\eta\left(\ln x_2(n)-\ln x_2(0)\right)\\&\quad<-\varepsilon\eta r_{1*}n.\end{aligned} \tag{58}$$

So, we can get

$$\begin{aligned}x_1(n)&<\left[\left(\frac{x_2(n)}{x_2(0)}\right)^{\eta}(x_1(0))^{\xi}\exp\left(-\varepsilon\eta r_{1*}n\right)\right]^{1/\xi}\\&<x_1(0)\left[\left(\frac{x_2^*}{x_2(0)}\right)^{\eta}\exp\left(-\varepsilon\eta r_{1*}n\right)\right]^{1/\xi}.\end{aligned} \tag{59}$$

Therefore, we have $x_1(n)\longrightarrow 0$ exponentially as $n\longrightarrow+\infty$. This completes the proof. □

**Lemma 9.** *Under the assumption of Theorem 8, let $\tilde{x}(n)=(x_1(n),x_2(n))^T$ be any positive solution of system (1), and then there exists a positive constant $x_{2^*}$ such that*

$$\liminf_{n\longrightarrow+\infty}x_2(n)\ge x_{2^*}, \tag{60}$$

*where*

$$\begin{aligned}x_{2*}&=k_{2*}\exp\left[\tau r_2^*\left(1-\frac{x_2^*}{k_{2*}}\right)\right]\\&\quad\cdot\exp\left[r_1^*\left(1-\frac{\exp\left[-\tau r_2^*\left(1-x_2^*/k_{2*}\right)\right]}{k_{2*}}x_2^*\right)\right]\end{aligned} \tag{61}$$

*is a constant independent of any positive solution of system (1); i.e., the second species $x_2$ of system (1) is permanent.*

*Proof.* By Lemma 3 and Theorem 8,

$$\begin{aligned}&\lim_{n\longrightarrow+\infty}x_1(n)=0,\\&\limsup_{n\longrightarrow+\infty}x_2(n)\le x_2^*,\end{aligned} \tag{62}$$

for arbitrarily small positive constant $\varepsilon(0<\varepsilon<1/\mu_2^*)$, there exists an $N_5>0$ such that

$$\begin{aligned}&x_1(n)<\varepsilon,\\&x_2(n)<x_2^*+\varepsilon\end{aligned} \tag{63}$$

for all $n>N_5$.

From the second equation of (1), for $n > N_5 + \tau$,

$$\begin{aligned} x_2(n+1) &= x_2(n)\exp\left[r_2(n)\left(1-\mu_1(n)x(n)-\frac{x_2(n-\tau)}{k_2(n)}\right)\right] \\ &> x_2(n)\exp\left[r_2(n)\left(1-\mu_1^*\varepsilon-\frac{x_2^*+\varepsilon}{k_{2*}}\right)\right]. \end{aligned} \tag{64}$$

Let $\varepsilon \longrightarrow 0$, and then

$$x_2(n+1) \geq x_2(n)\exp\left[r_2(n)\left(1-\frac{x_2^*}{k_{2*}}\right)\right]. \tag{65}$$

Noting the fact that $\exp(x-1)/x \geq 1$ for $x > 0$, we obtain

$$1-\frac{x_2^*}{k_{2*}} = 1-\frac{k_2^*}{k_{2*}}\frac{\exp\left(r_2^*(\tau+1)-1\right)}{r_2^*} \leq 0. \tag{66}$$

Therefore, from (65), we have

$$x_2(n+1) \geq x_2(n)\exp\left[r_2^*\left(1-\frac{x_2^*}{k_{2*}}\right)\right]. \tag{67}$$

Using (67), one could easily obtain that

$$x_2(n-\tau) \leq x_2(n)\exp\left[-\tau r_2^*\left(1-\frac{x_2^*}{k_{2*}}\right)\right]. \tag{68}$$

Substituting (68) into the second equation (1), for $n > N_5 + \tau$, deduces

$$\begin{aligned} x_2(n+1) > x_2(n)\exp\Bigg[&r_2(n) \\ &\cdot\left(1-\mu_1^*\varepsilon-\frac{\exp\left[-\tau r_2^*\left(1-x_2^*/k_{2*}\right)\right]}{k_{2*}}x_2(n)\right)\Bigg]. \end{aligned} \tag{69}$$

Let $\varepsilon \longrightarrow 0$, and then

$$\begin{aligned} x_2(n+1) \geq x_2(n)\exp\Bigg[&r_2(n) \\ &\cdot\left(1-\frac{\exp\left[-\tau r_2^*\left(1-x_2^*/k_{2*}\right)\right]}{k_{2*}}x_2(n)\right)\Bigg]. \end{aligned} \tag{70}$$

It is easy to check that the inequality

$$\frac{\exp\left[-\tau r_2^*\left(1-x_2^*/k_{2*}\right)\right]}{k_{2*}}x_2^* > 1 \tag{71}$$

holds. By Lemma 1, we have

$$\begin{aligned} \liminf_{n\longrightarrow+\infty} x_2(n) &\geq k_{2*}\exp\left[\tau r_2^*\left(1-\frac{x_2^*}{k_{2*}}\right)\right] \\ &\quad\cdot\exp\left[r_2^*\left(1-\frac{\exp\left[-\tau r_2^*\left(1-x_2^*/k_{2*}\right)\right]}{k_{2*}}x_2^*\right)\right] \\ &\triangleq x_{2*}. \end{aligned} \tag{72}$$

This completes the proof. □

Consider the following discrete logistic equation:

$$x(n+1) = x(n)\exp\left[r_2(n)\left(1-\frac{x(n-\tau)}{k_2(n)}\right)\right]. \tag{73}$$

**Lemma 10** (see [22]). *Assume that $\{r_2(n)\}$ and $\{k_2(n)\}$ satisfy (2), and then any positive solution $x(n)$ of (73) satisfies*

$$x_{2*} < \liminf_{n\longrightarrow+\infty} x(n) \leq \limsup_{n\longrightarrow+\infty} x(n) \leq x_2^*. \tag{74}$$

**Theorem 11.** *Under the assumptions of Theorem 8 and Lemmas 9 and 10, furthermore, suppose that*

$$\frac{k_2^*}{k_{2*}}\exp\left(r_2^*(1+\tau)-1\right) \leq \frac{2}{1+2\tau}. \tag{75}$$

*Let $\tilde{x}(n) = (x_1(n), x_2(n))^T$ be any positive solution of system (1), and then the species $x_1$ will be driven to extinction; that is, $x_1(n) \longrightarrow 0$ as $n \longrightarrow +\infty$ and $x_2(n) \longrightarrow x(n)$ as $n \longrightarrow +\infty$, where $x(n)$ is any positive solution of (73).*

*Proof.* Let $\tilde{x}(n) = (x_1(n), x_2(n))^T$ be a solution of system (1) with initial conditions (3). To finish the proof of Theorem 11, it is enough to show that $x_2(n) \longrightarrow x(n)$ as $n \longrightarrow +\infty$, where $x(n)$ is any positive solution of (73).

From Lemmas 3 and 9, Theorem 8, and (74), for arbitrarily small positive constant $\varepsilon > 0$, there exists an $N_6 > 0$ such that, for all $n > N_6$,

$$\begin{aligned} x_{2*}-\varepsilon &< x_2(n), \\ x(n) &< x_2^*+\varepsilon, \\ x_1(n) &< \varepsilon. \end{aligned} \tag{76}$$

Let

$$\begin{aligned} u(n) &= \ln x_2(n) - \ln x(n) \\ &\quad - \sum_{s=n-\tau}^{n-1}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(s)-x(s)\right). \end{aligned} \tag{77}$$

From the second equation of (1) and (73), we have

$$\begin{aligned} &\ln x_2(n+1) - \ln x_2(n) \\ &\quad = r_2(n)\left(1-\mu_1(n)x_1(n)-\frac{x_2(n-\tau)}{k_2(n)}\right), \\ &\ln x(n+1) - \ln x(n) = r_2(n)\left(1-\frac{x(n-\tau)}{k_2(n)}\right), \end{aligned} \tag{78}$$

and then

$$\begin{aligned} u(n+1) &= \ln x_2(n) - \ln x(n) - \frac{r_2(n)}{k_2(n)}\left(x_2(n-\tau)\right. \\ &\quad \left.-x(n-\tau)\right) - \sum_{s=n+1-\tau}^{n}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(s)-x(s)\right) \\ &\quad - r_2(n)\mu_1(n)x_1(n). \end{aligned} \tag{79}$$

Therefore, we have

$$\begin{aligned}\Delta u(n) &= u(n+1) - u(n) \\ &= -\frac{r_2(n+\tau)}{k_2(n+\tau)}\left(x_2(n) - x(n)\right) \\ &\quad - r_2(n)\mu_1(n)x_1(n),\end{aligned} \tag{80}$$

and

$$\begin{aligned}&u(n+1) + u(n) \\ &\quad = 2\left(\ln x_2(n) - \ln x(n)\right) \\ &\qquad - \frac{r_2(n)}{k_2(n)}\left(x_2(n-\tau) - x(n-\tau)\right) \\ &\qquad - \sum_{s=n-\tau}^{n-1}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(s) - x(s)\right) \\ &\qquad - \sum_{s=n+1-\tau}^{n}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(s) - x(s)\right) \\ &\qquad - r_2(n)\mu_1(n)x_1(n) \\ &\quad = 2\left(\ln x_2(n) - \ln x(n)\right) \\ &\qquad - 2\frac{r_2(n)}{k_2(n)}\left(x_2(n-\tau) - x(n-\tau)\right) \\ &\qquad - \frac{r_2(n+\tau)}{k_2(n+\tau)}\left(x_2(n) - x(n)\right) \\ &\qquad - 2\sum_{s=n+1-\tau}^{n-1}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(s) - x(s)\right) \\ &\qquad - r_2(n)\mu_1(n)x_1(n).\end{aligned} \tag{81}$$

Define

$$V_1(n) = u^2(n), \tag{82}$$

and then

$$\begin{aligned}&\Delta V_1(n) = u^2(n+1) - u^2(n) = \Delta u(n)\left(u(n+1)\right. \\ &\quad \left. + u(n)\right) = \left[-\frac{r_2(n+\tau)}{k_2(n+\tau)}\left(x_2(n) - x(n)\right)\right. \\ &\quad \left. - r_2(n)\mu(n)x_1(n)\right]\left(u(n+1) + u(n)\right) \\ &\quad = -\frac{r_2(n+\tau)}{k_2(n+\tau)}\left(x_2(n) - x(n)\right)\left(u(n+1) + u(n)\right) \\ &\quad - r_2(n)\mu_1(n)x_1(n)\left(u(n+1) + u(n)\right) = -2 \\ &\quad \cdot \frac{r_2(n+\tau)}{k_2(n+\tau)}\left(x_2(n) - x(n)\right)\left(\ln x_2(n) - \ln x(n)\right) \\ &\quad + 2\frac{r_2(n+\tau)}{k_2(n+\tau)}\frac{r_2(n)}{k_2(n)}\left(x_2(n) - x(n)\right)\left(x_2(n-\tau)\right. \\ &\quad \left. - x(n-\tau)\right) + \frac{r_2^2(n+\tau)}{k_2^2(n+\tau)}\left(x_2(n) - x(n)\right)^2 + 2 \\ &\quad \cdot \frac{r_2(n+\tau)}{k_2(n+\tau)}\sum_{s=n+1-\tau}^{n-1}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(s) - x(s)\right)\left(x_2(n)\right. \\ &\quad \left. - x(n)\right) + \left[\frac{r_2(n+\tau)}{k_2(n+\tau)}\left(x_2(n) - x(n)\right)\right. \\ &\quad \left. - \left(u(n+1) + u(n)\right)\right] r_2(n)\mu_1(n)x_1(n).\end{aligned} \tag{83}$$

Using the mean value theorem, then

$$\ln x_2(n) - \ln x(n) = \frac{1}{\zeta(n)}\left(x_2(n) - x(n)\right), \tag{84}$$

where $\zeta(n)$ lies between $x_2(n)$ and $x(n)$.

Noting the fact that $2ab \le a^2 + b^2$, it follows from (83) that

$$\begin{aligned}&\Delta V_1(n) \le -2\frac{r_2(n+\tau)}{k_2(n+\tau)}\frac{1}{\zeta(n)}\left(x_2(n) - x(n)\right)^2 \\ &\quad + \frac{r_2(n+\tau)}{k_2(n+\tau)}\frac{r_2(n)}{k_2(n)}\left(x_2(n) - x(n)\right)^2 + \frac{r_2(n+\tau)}{k_2(n+\tau)} \\ &\quad \cdot \frac{r_2(n)}{k_2(n)}\left(x_2(n-\tau) - x(n-\tau)\right)^2 \\ &\quad + \frac{r_2^2(n+\tau)}{k_2^2(n+\tau)}\left(x_2(n) - x(n)\right)^2 + \frac{r_2(n+\tau)}{k_2(n+\tau)} \\ &\quad \cdot \sum_{s=n+1-\tau}^{n-1}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(s) - x(s)\right)^2 + \frac{r_2(n+\tau)}{k_2(n+\tau)} \\ &\quad \cdot \sum_{s=n+1-\tau}^{n-1}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(n) - x(n)\right)^2 \\ &\quad + \left[\frac{r_2(n+\tau)}{k_2(n+\tau)}\left(x_2(n) - x(n)\right)\right. \\ &\quad \left. - \left(u(n+1) + u(n)\right)\right] r_2(n)\mu_1(n)x_1(n).\end{aligned} \tag{85}$$

From (76) and (85), let $\varepsilon$ be a sufficient small positive constant, for $n > N_6$, and we have

$$\begin{aligned}&\Delta V_1(n) \\ &\quad \le -2\frac{r_2(n+\tau)}{k_2(n+\tau)}\frac{1}{\zeta(n)}\left(x_2(n) - x(n)\right)^2 \\ &\qquad + \frac{r_2(n+\tau)}{k_2(n+\tau)}\frac{r_2(n)}{k_2(n)}\left(x_2(n) - x(n)\right)^2 \\ &\qquad + \frac{r_2(n+\tau)}{k_2(n+\tau)}\frac{r_2(n)}{k_2(n)}\left(x_2(n-\tau) - x(n-\tau)\right)^2 \\ &\qquad + \frac{r_2^2(n+\tau)}{k_2^2(n+\tau)}\left(x_2(n) - x(n)\right)^2\end{aligned}$$

$$+\frac{r_2(n+\tau)}{k_2(n+\tau)}\sum_{s=n+1-\tau}^{n-1}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(s)-x(s)\right)^2$$
$$+\frac{r_2(n+\tau)}{k_2(n+\tau)}\sum_{s=n+1-\tau}^{n-1}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(n)-x(n)\right)^2. \tag{86}$$

Let

$$V_2(n)=\sum_{s=n-\tau}^{n-1}\frac{r_2(s+2\tau)}{k_2(s+2\tau)}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(s)-x(s)\right)^2,$$
$$V_3(n)=\sum_{l=n}^{n-2+\tau}\frac{r_2(l+\tau)}{k_2(l+\tau)}\sum_{s=l+1-\tau}^{n-1}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(s)-x(s)\right)^2, \tag{87}$$

and then

$$\Delta V_2(n)=\frac{r_2(n+2\tau)}{k_2(n+2\tau)}\frac{r_2(n+\tau)}{k_2(n+\tau)}\left(x_2(n)-x(n)\right)^2-\frac{r_2(n+\tau)}{k_2(n+\tau)}\frac{r_2(n)}{k_2(n)}\left(x_2(n-\tau)-x(n-\tau)\right)^2,$$
$$\Delta V_3(n)=\frac{r_2(n+\tau)}{k_2(n+\tau)}\sum_{s=n+1}^{n-1+\tau}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(n)-x(n)\right)^2-\frac{r_2(n+\tau)}{k_2(n+\tau)}\sum_{s=n+1-\tau}^{n-1}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(n)-x(n)\right)^2. \tag{88}$$

Define

$$V(n)=V_1(n)+V_2(n)+V_3(n). \tag{89}$$

It follows from (86) and (88) that

$$\begin{aligned}\Delta V(n)\le{}&-2\frac{r_2(n+\tau)}{k_2(n+\tau)}\frac{1}{\zeta(n)}\left(x_2(n)-x(n)\right)^2\\&+\frac{r_2(n+\tau)}{k_2(n+\tau)}\frac{r_2(n)}{k_2(n)}\left(x_2(n)-x(n)\right)^2\\&+\frac{r_2^2(n+\tau)}{k_2^2(n+\tau)}\left(x_2(n)-x(n)\right)^2+\frac{r_2(n+\tau)}{k_2(n+\tau)}\\&\cdot\sum_{s=n+1-\tau}^{n-1}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(n)-x(n)\right)^2+\frac{r_2(n+2\tau)}{k_2(n+2\tau)}\\&\cdot\frac{r_2(n+\tau)}{k_2(n+\tau)}\left(x_2(n)-x(n)\right)^2+\frac{r_2(n+\tau)}{k_2(n+\tau)}\\&\cdot\sum_{s=n+1}^{n-1+\tau}\frac{r_2(s+\tau)}{k_2(s+\tau)}\left(x_2(n)-x(n)\right)^2\\={}&-\frac{r_2(n+\tau)}{k_2(n+\tau)}\left(\frac{2}{\zeta(n)}-\frac{r_2(n)}{k_2(n)}-\frac{r_2(n+\tau)}{k_2(n+\tau)}\right.\\&-\sum_{s=n+1-\tau}^{n-1}\frac{r_2(s+\tau)}{k_2(s+\tau)}-\frac{r_2(n+\tau)}{k_2(n+\tau)}\\&\left.-\sum_{s=n+1}^{n-1+\tau}\frac{r_2(s+\tau)}{k_2(s+\tau)}\right)\left(x_2(n)-x(n)\right)^2.\end{aligned} \tag{90}$$

From (75), we have

$$\frac{2}{x_2^*}-(1+2\tau)\frac{r_2^*}{k_{2*}}>0, \tag{91}$$

and then, for the above small enough $\varepsilon$, there exists a positive constant $q>0$ such that

$$\frac{2}{x_2^*+\varepsilon}-(1+2\tau)\frac{r_2^*}{k_{2*}}>q>0. \tag{92}$$

From (76), (90), and (92), for $n>N_6$, we have

$$\begin{aligned}\Delta V(n)&\le-\frac{r_2(n+\tau)}{k_2(n+\tau)}\left(\frac{2}{x_1^*+\varepsilon}-(1+2\tau)\frac{r_2^*}{k_{2*}}\right)\\&\cdot\left(x_2(n)-x(n)\right)^2<-q\frac{r_{2*}}{k_2^*}\left(x_2(n)-x(n)\right)^2.\end{aligned} \tag{93}$$

Summating both sides of (93) from $N_6+\tau$ to $n$, we have

$$\begin{aligned}&\sum_{j=N_6+\tau}^{n}\left(V(j+1)-V(j)\right)\\&\quad\le-q\frac{r_{2*}}{k_2^*}\sum_{j=N_6+\tau}^{n}\left(x_2(j)-x(j)\right)^2,\end{aligned} \tag{94}$$

then

$$\begin{aligned}&V(n+1)+q\frac{r_{2*}}{k_2^*}\sum_{j=N_6+\tau}^{n}\left(x_2(j)-x(j)\right)^2\\&\quad\le V(N_6+\tau),\end{aligned} \tag{95}$$

and so,

$$\sum_{j=N_6+\tau}^{n}\left(x_2(j)-x(j)\right)^2\le\frac{V(N_6+\tau)}{q}\frac{k_2^*}{r_{2*}}. \tag{96}$$

It follows from (76) that $V_i(N_6+\tau), i=1,2,3$ are all bounded. Hence,

$$\sum_{j=N_6+\tau}^{n}\left(x_2(j)-x(j)\right)^2\le\frac{V(N_6+\tau)}{q}\frac{k_2^*}{r_{2*}}<+\infty, \tag{97}$$

and then

$$\begin{aligned}\limsup_{n\longrightarrow+\infty}\sum_{j=N_6+\tau}^{n}\left(x_2(j)-x(j)\right)^2&\le\frac{V(N_6+\tau)}{q}\frac{k_2^*}{r_{2*}}\\&<+\infty.\end{aligned} \tag{98}$$

This implies that

$$\lim_{n\longrightarrow+\infty}\left(x_2(n)-x(n)\right)^2=0. \tag{99}$$

This completes the proof. □

## 5. Numerical Examples and Simulations

In this section, we give two examples to illustrate the feasibility of our results.

*Example 1.* Consider the following system:

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left[(0.5-0.2\sin(n))\right.\\ &\quad\left.\cdot\left(1-\frac{x_1(n)}{2+0.2\cos(n)}-x_2(n-2)\right)\right],\\ x_2(n+1) &= x_2(n)\exp\left[(0.2-0.1\cos(n))\right.\\ &\quad\left.\cdot\left(1-x_1(n)-\frac{x_2(n-2)}{0.5+0.1\sin(n)}\right)\right]; \end{aligned} \tag{100}$$

that is

$$\begin{aligned} r_1(n) &= 0.5-0.2\sin(n),\\ k_1(n) &= 2+0.2\cos(n),\\ \mu_2(n) &= 1,\\ r_2(n) &= 0.2-0.1\cos(n),\\ \mu_1(n) &= 1,\\ k_2(n) &= 0.5+0.1\sin(n). \end{aligned} \tag{101}$$

By a direct calculation, we can get

$$\begin{aligned} x_1^* &= 2.3283,\\ x_{1*} &= 1.4657,\\ x_2^* &= 1.8097;\\ \limsup_{n\longrightarrow+\infty}\frac{r_2(n)}{r_1(n)} &= 0.3709<0.4078\\ &= \liminf_{n\longrightarrow+\infty}\left\{\frac{r_2(n)\mu_1(n)k_1(n)}{r_1(n)},\right.\\ &\quad\left.\frac{r_2(n)}{r_1(n)\mu_2(n)k_2(n)}\right\};\\ \frac{k_1^*}{k_{1*}}\exp\left(r_1^*-1\right) &= 0.9054<2; \end{aligned} \tag{102}$$

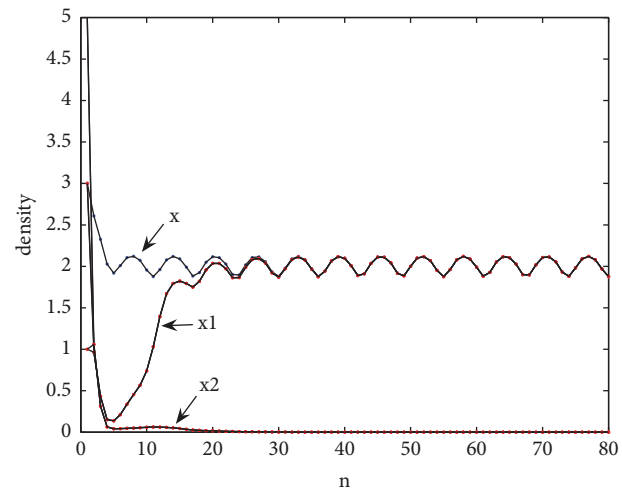


Figure 1: Dynamic behaviors of species $x_1$ and $x_2$ in (100) with initial values $x_1(0)=1,3,5$ and $x_2(\vartheta)=1,3,5$ for $\vartheta=-2,-1,0$, respectively; $x$ is a solution of (103).

that is, the conditions of Theorem 7 hold, and so species $x_2$ will be driven to extinction while species $x_1$ is asymptotic to any positive solution of

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left[(0.15+0.05\sin(n))\right.\\ &\quad\left.\cdot\left(1-\frac{x_1(n)}{1.2+0.2\cos(n)}\right)\right]. \end{aligned} \tag{103}$$

The solutions of systems (100) and (103) corresponding to initial values are displayed in Figure 1.

*Example 2.* Consider the following system:

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left[(0.15+0.05\sin(n))\right.\\ &\quad\left.\cdot\left(1-\frac{x_1(n)}{1.2+0.2\cos(n)}-x_2(n-1)\right)\right],\\ x_2(n+1) &= x_2(n)\exp\left[(0.17+0.07\cos(n))\right.\\ &\quad\left.\cdot\left(1-0.3x_1(n)-\frac{x_2(n-1)}{2.1+0.1\sin(n)}\right)\right]; \end{aligned} \tag{104}$$

that is,

$$\begin{aligned} r_1(n) &= 0.15+0.05\sin(n),\\ k_1(n) &= 1.2+0.2\cos(n),\\ \mu_2(n) &= 1,\\ r_2(n) &= 0.17+0.07\cos(n),\\ \mu_1(n) &= 0.3,\\ k_2(n) &= 2.1+0.1\sin(n). \end{aligned} \tag{105}$$

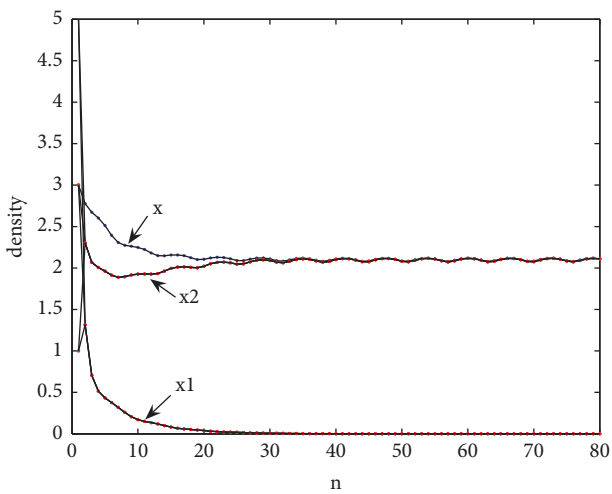


Figure 2: Dynamic behaviors of species $x_1$ and $x_2$ in (104) with initial values $x_1(0) = 1, 3, 5$ and $x_2(\vartheta) = 1, 3, 5$ for $\vartheta = -1, 0$, respectively; $x$ is a solution of (107).

By a direct calculation, we can get

$$\begin{aligned} x_1^* &= 3.1453, \\ x_2^* &= 5.4498, \\ x_{2*} &= 1.7240; \\ \liminf_{n\longrightarrow+\infty} \frac{r_2(n)}{r_1(n)} &= 1.0712 > 0.9552 \\ &= \limsup_{n\longrightarrow+\infty} \left\{ \frac{r_2(n)\mu_1(n)k_1(n)}{r_1(n)}, \right. \\ &\quad \left. \frac{r_2(n)}{r_1(n)\mu_2(n)k_2(n)} \right\}; \\ \frac{k_2^*}{k_{2*}} \exp\left(r_2^*(1+\tau)-1\right) &= 0.6540 < 0.6667 \\ &= \frac{2}{1+2\tau}; \end{aligned} \tag{106}$$

that is, the conditions of Theorem 11 hold, and so species $x_1$ will be driven to extinction while species $x_2$ is asymptotic to any positive solution of

$$\begin{aligned} x_2(n+1) = x_2(n) \exp\Bigg[ &(0.17 + 0.07\cos(n)) \\ &\cdot \left(1 - \frac{x_2(n-1)}{2.1 + 0.1\sin(n)}\right)\Bigg]. \end{aligned} \tag{107}$$

The solutions of systems (104) and (107) corresponding to initial values are displayed in Figure 2.

## 6. Conclusion

A nonautonomous discrete Lotka-Volterra competitive system with time delay has been studied in this paper. It is shown that if the coefficients are bounded above and below by positive constants and satisfy certain inequalities, then one of the species will be driven to extinction while the other one will stabilize at a certain solution of a nonlinear single species model.

This paper provided an effective method for the further study on permanence and extinction of population dynamic systems with time delay. In fact, our techniques in this paper are applicable to a pure delayed discrete $n$-species Lotka-Volterra competitive system. Furthermore, one may consider a discrete Lotka-Volterra competitive system with infinite delay, which we leave for future work.

As we know, system (1) is a basic model, and, based on system (1), we can establish different types of Lotka-Volterra competitive systems according to the ecological significance, such as plankton allelopathy systems and functional response systems, by using the same methods and analytical techniques, and similar results can be obtained. From the obtained results, we not only can reveal the inherent law of the system and predict the development of the population but also can control or adjust the ecological development of the population in a better way. Besides, the results obtained in this paper also can be applied to economic systems, such as systems of industrial clusters and financial ecology; one may see [23].

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Acknowledgments

This work is supported by the Key Project of Scientific Research in Colleges and Universities of Henan Province (no. 18A110005), the National College Students' Innovative Entrepreneurial Training Plan, the College Students' Innovative Entrepreneurial Training Plan Program of Henan Province (no. 201810479012), and the College Students' Innovation Fund Project of Anyang Normal University (no. ASCX/2018-Z108).

## References

[1] L. Chen, J. Sun, F. Chen, and L. Zhao, "Extinction in a Lotka-Volterra competitive system with impulse and the effect of toxic substances," *Applied Mathematical Modelling: Simulation and Computation for Engineering and Environmental Systems*, vol. 40, no. 3, pp. 2015–2024, 2016.

[2] L. Liu and Y. Shen, "New criteria on persistence in mean and extinction for stochastic competitive Lotka-Volterra systems with regime switching," *Journal of Mathematical Analysis and Applications*, vol. 430, no. 1, pp. 306–323, 2015.

[3] Z. Liu and C. Yang, "Permanence and periodic solutions for a two-patch impulsive migration periodic N-species Lotka-Volterra competitive System," *Discrete Dynamics in Nature and Society*, vol. 2015, Article ID 293050, 14 pages, 2015.

[4] J. D. Zhao, Z. C. Zhang, and J. Ju, "Necessary and sufficient conditions for permanence and extinction in a three dimensional competitive Lotka-Volterra system," *Applied Mathematics and Computation*, vol. 230, pp. 587–596, 2014.

[5] Z. Y. Hou, "On permanence of Lotka-Volterra systems with delays and variable intrinsic growth rates," *Nonlinear Analysis: Real World Applications*, vol. 14, no. 2, pp. 960–975, 2013.

[6] J. Zhao, X. Guo, Z. Han, and Z. Chen, "Average conditions for competitive exclusion in a nonautonomous two dimensional Lotka-Volterra system," *Mathematical and Computer Modelling*, vol. 57, no. 5-6, pp. 1131–1138, 2013.

[7] X. Feng, F. Zhang, K. Wang, and X. Li, "Permanence in multispecies nonautonomous Lotka-Volterra competitive systems with delays and impulses," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 125197, 12 pages, 2012.

[8] C. L. Shi, Z. Li, and F. D. Chen, "Extinction in a nonautonomous Lotka-Volterra competitive system with infinite delay and feedback controls," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 5, pp. 2214–2226, 2012.

[9] H. Hu, K. Wang, and D. Wu, "Permanence and global stability for nonautonomous N-species Lotka-Volterra competitive system with impulses and infinite delays," *Journal of Mathematical Analysis and Applications*, vol. 377, no. 1, pp. 145–160, 2011.

[10] Z. Hou, "Permanence and extinction in competitive Lotka-Volterra systems with delays," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 4, pp. 2130–2141, 2011.

[11] F. Montes de Oca and M. L. Zeeman, "Extinction in nonautonomous competitive Lotka-Volterra systems," *Proceedings of the American Mathematical Society*, vol. 124, no. 12, pp. 3677–3687, 1996.

[12] M. L. Zeeman, "Extinction in competitive Lotka-Volterra systems," *Proceedings of the American Mathematical Society*, vol. 123, no. 1, pp. 87–96, 1995.

[13] S. Ahmad, "Extinction of species in nonautonomous Lotka-Volterra systems," *Proceedings of the American Mathematical Society*, vol. 127, no. 10, pp. 2905–2910, 1999.

[14] Z. Teng, "On the non-autonomous Lotka-Volterra N-species competing systems," *Applied Mathematics and Computation*, vol. 114, no. 2-3, pp. 175–185, 2000.

[15] J. Zhao, L. Fu, and J. Ruan, "Extinction in a nonautonomous competitive Lotka-Volterra system," *Applied Mathematics Letters*, vol. 22, no. 5, pp. 766–770, 2009.

[16] X. Y. Liao, Z. G. Ouyang, and S. Zhou, "Permanence of species in nonautonomous discrete Lotka-Volterra competitive system with delays and feedback controls," *Journal of Computational and Applied Mathematics*, vol. 211, no. 1, pp. 1–10, 2008.

[17] D. Y. Wu, "Global attractivity of a discrete Lotka-Volterra competition system with infinite delays and feedback controls," *Advances in Difference Equations*, vol. 2013, article 14, 2013.

[18] J. B. Xu, Z. D. Teng, and H. J. Jiang, "Permanence and global attractivity for discrete nonautonomous two-species Lotka-Volterra competitive system with delays and feedback controls," *Periodica Mathematica Hungarica*, vol. 63, no. 1, pp. 19–45, 2011.

[19] R. M. May, "Time-delay versus stability in population models with two and three trophic levels," *Ecology*, vol. 54, no. 2, pp. 315–325, 1973.

[20] X. Yang, "Uniform persistence and periodic solutions for a discrete predator-prey system with delays," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 1, pp. 161–177, 2006.

[21] Z. Zhou and X. Zou, "Stable periodic solutions in a discrete periodic logistic equation," *Applied Mathematics Letters*, vol. 16, no. 2, pp. 165–171, 2003.

[22] Z. Li, M. A. Han, and F. D. Chen, "Almost periodic solutions of a discrete almost periodic logistic equation with delay," *Applied Mathematics and Computation*, vol. 232, pp. 743–751, 2014.

[23] G. I. Bischi and F. Tramontana, "Three-dimensional discrete-time Lotka-Volterra models with an application to industrial clusters," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 10, pp. 3000–3014, 2010.