Asymptotic Properties of Solutions to Second-Order Difference Equations of Volterra Type

Janusz Migda,¹ Małgorzata Migda,² and Magdalena Nockowska-Rosiak ³

Faculty of Mathematics and Computer Science, A. Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland
Institute of Mathematics, Poznań University of Technology, Piotrowo 3A, 60-965 Poznań, Poland
Institute of Mathematics, Lodz University of Technology, Ul. Wólczańska 215, 90-924 Łódź, Poland

Correspondence should be addressed to Małgorzata Migda; malgorzata.migda@put.poznan.pl

Received 26 April 2018; Accepted 21 June 2018; Published 9 July 2018

1. Introduction

In this paper we consider the nonlinear Volterra sum-difference equation of nonconvolution type:

\[ \Delta (r_n \Delta x_n) = b_n + \sum_{k=1}^n K(n,k) f(x_k), \tag{E} \]

Here \( \mathbb{N}, \mathbb{R} \) denote the set of positive integers and the set of real numbers, respectively. By a solution of (E) we mean a sequence \( x : \mathbb{N} \to \mathbb{R} \) satisfying (E) for large \( n \).

Discrete Volterra equations of different types are widely used in the process of modeling of some real phenomena or by applying a numerical method to a Volterra integral equation. Let \( m \in \mathbb{N}. \) The general form of a Volterra sum-difference autonomous equation is

\[ \Delta^m x_n = a_n + \sum_{k=1}^n K(n,k) f(x_k). \tag{2} \]

Such equations can be regarded as the discrete analogue of Volterra integrodifferential equations of the form

\[ x^{(m)}(t) = f(t) + \int_0^t K(t,s) f(x(s)) ds. \tag{3} \]

There are relatively few works devoted to the study of equations of type (2); see, for example, [1–4], [5], [17–22] and references therein.

Note, that equation (E) generalizes the second-order discrete Volterra difference equation of type (2):

\[ \Delta^2 x_n = b_n + \sum_{k=1}^n K(n,k) f(x_k). \tag{5} \]

On the other hand, if \( K(n,k) = 0 \) for \( k \neq n, \) then denoting \( a_n = K(n,n) \) equation (E) takes the form

\[ \Delta (r_n \Delta x_n) = a_n f(x_n) + b_n. \tag{6} \]

Hence second-order difference equation (6) is a special case of (E). The results on asymptotic properties and oscillation of equations of type (6) can be found, i.e., in [23–26].
Our main goal is to present sufficient conditions for the existence of a solution \( x \) to equation (E) such that
\[
x_n = \sum_{k=1}^{n-1} \frac{c}{r_k} + d + o(n'),
\] (7)
where \( c, d \in \mathbb{R} \) and \( s \in (-\infty, 0] \). We give also sufficient conditions for a given solution \( x \) of equation (E) to have an asymptotic property (7). Moreover, in Section 5 we show applications of the obtained results to linear Volterra equation of type (E). We present also some results for the case when \( (r_n) \) is a potential sequence.

2. Preliminaries

We will denote by \( SQ \) the space of all sequences \( x : \mathbb{N} \rightarrow \mathbb{R} \). If \( x, y \) in \( SQ \), then \( xy \) and \( |x| \) denote the sequences defined by \( xy(n) = x_n y_n \) and \( |x|(n) = |x_n| \), respectively. Moreover,
\[
\|x\| = \sup \{|x_n| : n \in \mathbb{N}\}.
\] (8)

If \( x \in SQ \), \( s \in \mathbb{R} \), and \( \lim_{n \to \infty} x_n^s = 0 \), then we write \( x_n = o(n') \). Analogously, \( x_n = O(n') \) denotes the boundedness of the sequence \( (n^s x_n) \).

The following two lemmas will be useful in the proof of our main results.

**Lemma 1.** Assume \( u \in SQ, n \in \mathbb{N} \), and
\[
\sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |u_i| < \infty.
\] (9)

Then
\[
\sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |u_i| \leq \sum_{i=j}^{\infty} \frac{|u_i|}{r_j} < \infty.
\] (10)

**Proof.** We have
\[
\sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |u_i| = \frac{1}{r_n} (|u_n| + |u_{n+1}| + |u_{n+2}| + \cdots) + \frac{1}{r_{n+1}} (|u_{n+1}| + |u_{n+2}| + \cdots) + \cdots
\]
\[
= \frac{1}{r_n} |u_n| + \left( \frac{1}{r_n} + \frac{1}{r_{n+1}} \right) |u_{n+1}| + \cdots + \frac{1}{r_{n+1}} \left( \frac{1}{r_n} + \frac{1}{r_{n+1}} + \frac{1}{r_{n+2}} \right) |u_{n+2}| + \cdots
\]
\[
\leq \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |u_i| \leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \frac{|u_i|}{r_j}
\]
\[
= \frac{1}{r_1} |u_1| + \left( \frac{1}{r_1} + \frac{1}{r_2} \right) |u_2| + \cdots
\]
\[
= \frac{1}{r_1} (|u_1| + |u_2| + \cdots)
\]

3. Solutions with Prescribed Asymptotic Behavior

In this section we present sufficient conditions for the existence of a solution \( x \) to equation (E) such that
\[
x_n = \sum_{k=1}^{n-1} \frac{c}{r_k} + d + o(n'),
\] (14)
where \( c, d \in \mathbb{R} \) and \( s \in (-\infty, 0] \).

**Theorem 3.** Assume \( s \in (-\infty, 0], t \in [s, \infty), c, d \in \mathbb{R}, y : \mathbb{N} \rightarrow \mathbb{R}, q \in \mathbb{N}, \alpha \in (0, \infty),
\]
\[
r_n^{-1} = O(n'),
\]
\[
\sum_{j=n}^{\infty} j^{1-s-t} |K(n, j)| < \infty,
\]
\[
\sum_{j=n}^{\infty} j^{1-s-t} |b_n| < \infty,
\] (15)

\[
y_n = \frac{d + c \sum_{k=1}^{n-1} \frac{1}{r_k}}{n},
\]
\[
U = \bigcup_{n=1}^{\infty} [y_n - \alpha, y_n + \alpha],
\]
and \( f \) is continuous and bounded on \( U \). Then there exists a solution \( x \) of (E) such that
\[
x_n = y_n + o(n').
\] (16)

**Proof.** For \( n \in \mathbb{N} \) and \( x \in SQ \) let
\[
F(x)(n) = b_n + \sum_{k=1}^{n} K(n, k) f(x_k).
\] (17)

There exists \( L > 0 \) such that
\[
|f(u)| \leq L \quad \text{for any } u \in U.
\] (18)
Let
\[ Y = \{ x \in \text{SQ} : |x - y| \leq \alpha \}. \] (19)

If \( x \in Y \) and \( n \geq q \), then
\[ x_n \in [y_n - \alpha, y_n + \alpha] \subset U. \] (20)

Choose a positive number \( Q \) such that \( r_n^{-1} \leq Qn^r \) for any \( n \).
Then
\[ \sum_{n=1}^{\infty} \frac{1}{n^r} \sum_{j=1}^{n} \sum_{k=1}^{j} |K(j,k)| \leq Q \sum_{n=1}^{\infty} n^{r-1} \sum_{j=1}^{n} \sum_{k=1}^{j} |K(j,k)| \] (21)

Since \( t - s \geq 0 \), we have
\[ \sum_{n=1}^{\infty} n^{r-1} \sum_{j=1}^{n} \sum_{k=1}^{j} |K(j,k)| \leq \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \sum_{k=1}^{j} |K(j,k)| \] (22)

For \( j \in \mathbb{N} \) let
\[ z_j = j^{-s} \sum_{k=1}^{j} |K(j,k)|. \] (23)

Then we have
\[ \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} z_j = (z_1 + z_2 + \cdots) + (z_2 + z_3 + \cdots) + \cdots = \sum_{n=1}^{\infty} nz_n \]
\[ = \sum_{n=1}^{\infty} n^3 \sum_{k=1}^{n} |K(n,k)| \]
\[ = \sum_{n=1}^{\infty} n^{3r-1} \sum_{k=1}^{n} |K(n,k)| < \infty. \] (24)

Hence, using (21) and (22), we get
\[ \sum_{n=1}^{\infty} \frac{1}{n^r} \sum_{j=1}^{n} \sum_{k=1}^{j} |K(j,k)| < \infty. \] (25)

Analogously, replacing \( \sum_{k=1}^{j} |K(j,k)| \) by \( |b_j| \), we obtain
\[ \sum_{n=1}^{\infty} \frac{1}{n^r} \sum_{j=1}^{n} |b_j| < \infty. \] (26)

Using (25) and (26) we get
\[ \sum_{j=1}^{\infty} \frac{1}{n^r} \sum_{j=1}^{n} \sum_{k=1}^{j} |K(j,k)| < \infty. \] (27)

Since \( s \leq 0 \), we have
\[ \sum_{n=1}^{\infty} \frac{1}{n^r} \sum_{j=1}^{n} \sum_{k=1}^{j} |K(j,k)| < \infty. \] (28)

Define a sequence \( \rho \in \text{SQ} \) by
\[ \rho_n = \sum_{j=1}^{\infty} \frac{1}{n^r} \sum_{j=1}^{n} \sum_{k=1}^{j} |K(i,k)|. \] (29)

Define \( w, g \in \text{SQ} \) by
\[ w_n = |b_n| + \sum_{k=1}^{n} |K(n,k)|, \] (30)
\[ g_n = \sum_{j=1}^{\infty} \frac{1}{n^r} \sum_{j=1}^{n} w_j. \]

By (27), \( g_n = o(1) \). We have
\[ n^{-s} \rho_n = n^{-s} \sum_{j=1}^{\infty} \frac{1}{n^r} \sum_{j=1}^{n} w_j = \sum_{j=1}^{\infty} \frac{1}{n^r} \sum_{j=1}^{n} \sum_{i=1}^{j} w_i \leq \sum_{j=1}^{\infty} \frac{1}{n^r} \sum_{j=1}^{n} \sum_{i=1}^{j} w_i \] (31)

Hence \( n^{-s} \rho_n = o(1) \) and we get
\[ \rho_n = n^s o(1) = o(n^s). \] (32)

Hence there exists an index \( p \geq q \) such that
\[ \rho_n \leq \alpha \] (33)

for \( n \geq p \). Let
\[ X = \{ x \in \text{SQ} : |x - y| \leq \rho, \ x_n = y_n \ for \ n < p \}, \]
\[ H : Y \rightarrow \text{SQ}. \]

We define a metric on \( X \) by formula (13). Note that \( X \subset Y \). Let \( x \in X \). By (33) and (20) we have \( x_j \in U \) for any \( i \geq p \).
Hence, by (18), \( |f(x_i)| \leq L \) for \( i \geq p \). Using (17) and (29) we obtain
\[ |H(x)(n) - y_n| = \sum_{i=1}^{\infty} \frac{1}{n^r} \sum_{j=1}^{i} F(x)(i) \leq \rho_n \]
\[ \leq \sum_{j=1}^{\infty} \frac{1}{n^r} \sum_{j=1}^{i} F(x)(i) \leq \rho_n \]

for \( n \geq p \). Therefore \( HX \subset X \). Now we show that the map \( H \) is continuous. Using (25) and the assumption \( s \leq 0 \), we have
\[ \sum_{n=1}^{\infty} \frac{1}{n^r} \sum_{j=1}^{n} \sum_{k=1}^{j} |K(j,k)| < \infty. \]

Hence, by Lemma 1, we get
\[ \sum_{n=1}^{\infty} \frac{1}{n^r} \sum_{j=1}^{n} \sum_{k=1}^{j} |K(n,k)| < \infty. \] (37)
Let $\varepsilon > 0$. Choose an index $m \geq p$ and a positive constant $\gamma$ such that
\[
L \sum_{n=m}^{\infty} \sum_{j=1}^{n} \frac{1}{r_j} \sum_{k=1}^{n} |K(n, k)| < \varepsilon,
\]
\[
\gamma \sum_{n=1}^{m} \sum_{j=1}^{n} \frac{1}{r_j} \sum_{k=1}^{n} |K(n, k)| < \varepsilon.
\]
Let
\[C = \bigcup_{n=1}^{m} [y_n - \alpha, y_n + \alpha].\]  

Choose a positive $\delta$ such that if $t_1, t_2 \in C$ and $|t_1 - t_2| < \delta$, then
\[
|f(t_1) - f(t_2)| < \gamma.
\]
Choose $x, z \in X$ such that $\|x - z\| < \delta$. Then we have
\[
\|Hx - Hz\| = \left\|\sum_{n=p}^{\infty} \sum_{j=1}^{n} \frac{1}{r_j} \sum_{k=1}^{n} (F(x)(i) - F(z)(i))\right\|
\leq \sup_{n=p}^{\infty} \sum_{j=1}^{n} \sum_{i=1}^{j} |F(x)(i) - F(z)(i)|
\leq \sum_{j=p}^{\infty} \sum_{i=1}^{j} \sum_{k=1}^{j} |K(j, k)||f(x_k) - f(z_k)|.
\]

Using Lemma 1 we obtain
\[
\|Hx - Hz\| \leq \sum_{j=p}^{\infty} \sum_{i=1}^{j} \sum_{k=1}^{j} |K(j, k)||f(x_k) - f(z_k)|.
\]

Note that $|f(x_j) - f(z_j)| \leq 2L$ for $j \geq p$ and
\[
|f(x_j) - f(z_j)| \leq \gamma \quad \text{for} \quad j \in \{p, p + 1, \ldots, m\}.
\]

Hence we obtain
\[
\|Hx - Hz\| \leq \gamma \sum_{n=1}^{m} \sum_{j=1}^{n} \sum_{i=1}^{j} |K(n, k)|
+ 2L \sum_{n=m}^{\infty} \sum_{j=1}^{n} \sum_{i=1}^{j} |K(n, k)| < 3\varepsilon.
\]

Therefore $H : X \rightarrow X$ is continuous. By Lemma 2 there exists a point $x \in X$ such that $x = Hx$. Then, for $n \geq p$, we have
\[
x_n = y_n + \sum_{j=n}^{\infty} \sum_{i=1}^{j} F(x)(i).
\]

Note that
\[
\Delta(r_n \Delta y_n) = \Delta \left(r_n \Delta \left(d + c \sum_{k=1}^{n-1} \frac{1}{r_k}\right)\right)
= c\Delta \left(r_n \Delta \left(\sum_{k=1}^{n-1} \frac{1}{r_k}\right)\right) = c\Delta 1 = 0
\]
for any $n$. Hence, for $n \geq p$, we get
\[
\Delta(r_n \Delta x_n) = \Delta \left[r_n \Delta \left(\sum_{j=1}^{\infty} \frac{1}{r_j} \sum_{i=1}^{j} F(x)(i)\right)\right]
= -\Delta \left(r_n \frac{1}{r_n} \sum_{i=1}^{\infty} F(x)(i)\right) = F(x)(n)
\]
\[
= b_n + \sum_{k=1}^{n} K(n, k) f(x_k).
\]

Therefore $x$ is a solution of (E). Since $x \in X$ we have $x_n = y_n + o(n^\varepsilon)$.

If the function $f$ is continuous, then from Theorem 3 we get the following two results.

**Corollary 4.** Assume $s \in (-\infty, 0], t \in [s, \infty), f$ is continuous, and
\[
r_{n}^{-1} = O(n^t),
\]
\[
\sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n, i)| < \infty,
\]
\[
\sum_{n=1}^{\infty} n^{1+t-s} |b_n| < \infty.
\]

Then for any $d \in \mathbb{R}$ there exists a solution $x$ of (E) such that $x_n = d + o(n^t)$.

**Proof.** Taking $c = 0, q = 1$, and $\alpha = 1$ in Theorem 3, we obtain the result.

**Corollary 5.** Assume $t \in (-\infty, -1), s \in (-\infty, t], f$ is continuous, and
\[
r_{n}^{-1} = O(n^t),
\]
\[
\sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n, i)| < \infty,
\]
\[
\sum_{n=1}^{\infty} n^{1+t-s} |b_n| < \infty.
\]

Then for any $c, d \in \mathbb{R}$ there exists a solution $x$ of (E) such that
\[
x_n = d + c \sum_{k=1}^{n-1} \frac{1}{r_k} + o(n^t).
\]
Proof. Assume \( c, d \in \mathbb{R} \) and a sequence \( y \in \text{SQ} \) is defined by
\[
y_n = d + c \sum_{k=1}^{n-1} \frac{1}{r_k}.
\] (51)
Since \( r_n^{-1} = O(n^t) \) and \( t < -1 \), we see that \( y \) is bounded. Now, taking \( q = 1 \) and \( \alpha = 1 \) in Theorem 3, we obtain the result.

Note that Corollaries 4 and 5 concern convergent solutions. However, Theorem 3 includes also divergent solutions. For example, if \( f(x) = x^{-1} \) for \( x \neq 0 \), \( s \in (-\infty, 0] \), \( t = 0 \), \( r_n^{-1} = O(1) \), and
\[
\sum_{n=1}^{\infty} n^{1-t-s} \sum_{i=1}^{n} |K(n,i)| < \infty,
\] (52)
then, by Theorem 3, for any nonzero \( c \in \mathbb{R} \) and any \( d \in \mathbb{R} \) there exists a solution \( x \) of (E) such that
\[
x_n = d + c \sum_{k=1}^{n-1} \frac{1}{r_k} + o(n^t).
\] (53)

Now we present an example that proves the assumption
\[
\sum_{n=1}^{\infty} n^{1-t-s} \sum_{i=1}^{n} |K(n,i)| < \infty,
\] (54)
in Theorem 3, is essential.

Example 6. Assume \( r_n = n, b_n = 0, K(n,k) = \frac{1}{n^2} \),
\[
f(x) = \frac{1}{|x|+1} + 1,
\] (55)
s = 0, and \( t = 0 \). Then equation (E) takes the form
\[
\Delta(n\Delta x_n) = \frac{1}{n^2} \sum_{k=1}^{n} \left( \frac{1}{|x_k|+1} + 1 \right).
\] (56)
Let \( c, d \in \mathbb{R} \) and
\[
y_n = d + c \sum_{k=1}^{n-1} \frac{1}{r_k} = d + c \sum_{k=1}^{n-1} \frac{1}{k}.
\] (57)
Notice that \( f \) is continuous and bounded on \( \mathbb{R} \). Moreover,
\[
\sum_{n=1}^{\infty} n^{1-t-s} \sum_{i=1}^{n} |K(n,i)| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,
\] (58)
and
\[
\Delta y_n = \frac{c}{n},
\] (59)
\[
\Delta(n\Delta y_n) = 0.
\]
Assume \( x \) is a solution of (56) such that
\[
x_n = y_n + z_n,
\] (60)
\[
z_n = o(n^t) = o(1).
\]
Since \( \Delta(n\Delta y_n) = 0 \), we have
\[
\Delta(n\Delta x_n) = \Delta(n\Delta y_n) + \Delta(n\Delta z_n) = \Delta(n\Delta z_n),
\] (61)
Hence
\[
\Delta(n\Delta z_n) = \Delta(n\Delta x_n) = \frac{n}{n^2} \left( \frac{1}{|x_k|+1} + 1 \right) > \frac{1}{n}
\] (62)
for large \( n \). Therefore, the sequence \( n\Delta z_n \) is eventually increasing and there exists the limit
\[
\lambda = \lim_{n \to \infty} n\Delta z_n > -\infty.
\] (63)
If \( \lambda < \infty \), then the sequence \( n\Delta z_n \) is convergent in \( \mathbb{R} \). Hence the series
\[
\sum_{n=1}^{\infty} \Delta(n\Delta x_n) = \sum_{n=1}^{\infty} \Delta(n\Delta z_n)
\] (64)
is convergent. On the other hand
\[
\Delta(n\Delta x_n) = \frac{n}{n^2} \left( \frac{1}{|x_k|+1} + 1 \right) > \frac{1}{n}
\] (65)
for large \( n \). Hence \( \lambda = \infty \). Therefore \( n\Delta z_n > 1 \) for large \( n \) and we get
\[
\sum_{n=1}^{\infty} \Delta z_n \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty
\] (66)
But since \( z_n \to 0 \), the series \( \sum_{n=1}^{\infty} \Delta z_n \) is convergent.

4. Asymptotic Behavior of Solutions

In this section we present sufficient conditions for a given solution \( x \) of equation (E) to have an asymptotic property
\[
x_n = \sum_{k=1}^{n-1} \frac{c}{r_k} + d + o(n^t),
\] (67)
where \( c, d \in \mathbb{R} \) and \( s \in (-\infty, 0] \).

Theorem 7. Assume \( s \in (-\infty, 0], t \in [s, \infty), \)
\[
r_n^{-1} = O\left(n^t\right),
\] (68)
\[
\sum_{n=1}^{\infty} n^{1-t-s} \sum_{i=1}^{n} |K(n,i)| < \infty,
\] (69)
\[
\sum_{n=1}^{\infty} n^{1-t-s} |b_n| < \infty.
\] (70)
and \( x \) is a solution of (E) such that the sequence \( (f(x_n)) \) is bounded. Then there exist \( c, d \in \mathbb{R} \) such that
\[
  x_n = \sum_{k=1}^{n} \frac{c}{r_k} + d + o(n^t).
\]  

(69)

Proof. We have
\[
  \Delta(r_n \Delta x_n) = b_n + \sum_{k=1}^{n} K(n,k) f(x_k)
\]

(70)

for large \( n \). Using boundedness of the sequence \( (f(x_n)) \) and (68) we get
\[
  \sum_{n=1}^{\infty} n^{1+s-t} \left| \Delta(r_n \Delta x_n) \right| < \infty.
\]

(71)

Define \( w, u \in SQ \) by
\[
  w_n = \Delta(r_n \Delta x_n),
\]

(72)

\[
  u_n = n^{1-s} \left| w_n \right|.
\]

Choose a positive \( L \) such that \( r_n^{-1} \leq Lt^s \) for any \( n \). Since \( t-s \geq 0 \), we have
\[
  \sum_{n=1}^{\infty} \frac{1}{n^{1-t}} \sum_{j=n}^{\infty} \left| w_j \right| \leq L \sum_{n=1}^{\infty} \frac{1}{n^{1+s-t}} \sum_{j=n}^{\infty} \left| w_j \right| = L \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} n^{1+t-s} \left| w_j \right|
\]

(73)

Moreover,
\[
  \sum_{n=1}^{\infty} n^{1+t-s} \left| w_n \right| = \sum_{n=1}^{\infty} n u_n
\]

\[
  = u_1 + (u_2 + u_2) + (u_3 + u_3 + u_3) + \cdots
\]

\[
  = \sum_{j=1}^{\infty} u_j + \sum_{j=2}^{\infty} u_j + \sum_{j=3}^{\infty} u_j + \cdots = \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} u_j
\]

\[
  = \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} j^{t-s} \left| w_j \right|.
\]

Hence, by (73)
\[
  \sum_{n=1}^{\infty} \frac{1}{n^{1-t}} \sum_{j=n}^{\infty} \left| w_j \right| < \infty.
\]

(75)

Since \( s \leq 0 \), we have
\[
  \sum_{n=1}^{\infty} \frac{1}{n^{1-t}} \sum_{j=n}^{\infty} \left| w_j \right| < \sum_{n=1}^{\infty} \frac{1}{n^{1-t}} \sum_{j=n}^{\infty} \left| w_j \right| < \infty.
\]

(76)

Let
\[
  z_n = \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} w_i.
\]

(77)

Then
\[
  n^{-s} \left| z_n \right| = n^{-s} \left| \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} w_i \right| \leq n^{-s} \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \left| w_i \right|
\]

(78)

\[
  = \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \left| w_i \right| \leq \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \left| w_i \right| = o(1).
\]

(79)

Thus \( z_n = o(n^t) \). Let
\[
  y_n = x_n - z_n.
\]

(80)

Then
\[
  \Delta y_n = \Delta x_n - \Delta z_n = \Delta x_n + \frac{1}{r_n} \sum_{i=n}^{\infty} w_i.
\]

Hence
\[
  r_n \Delta y_n = r_n \Delta x_n + \sum_{i=n}^{\infty} w_i
\]

(81)

and we get
\[
  \Delta(r_n \Delta y_n) = \Delta(r_n \Delta x_n) - w_n = \Delta(r_n \Delta x_n) - \Delta(r_n \Delta x_n) = 0
\]

(82)

for any \( n \in \mathbb{N} \). Therefore, there exists a real constant \( c \) such that \( r_n \Delta y_n = c \). Thus
\[
  y_n - y_1 = \Delta y_1 + \cdots + \Delta y_{n-1} = \frac{c}{r_1} + \cdots + \frac{c}{r_n-1}.
\]

(83)

Hence
\[
  x_n = y_n + z_n = \sum_{k=1}^{n} \frac{c}{r_k} + d + o(n^t)
\]

(84)

where \( d = y_1 \).

\[\square\]

Corollary 8. Assume \( s \in (-\infty,0] \), \( t \in [s,\infty) \), \( r_n^{-1} = O(n^t) \), \( f \) is locally bounded,
\[
  \sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)| < \infty,
\]

(85)

and \( x \) is a bounded solution of (E). Then there exist \( c, d \in \mathbb{R} \) such that
\[
  x_n = \sum_{k=1}^{n} \frac{c}{r_k} + d + o(n^t).
\]

(86)

Proof. Since \( x \) is bounded and \( f \) is locally bounded, the sequence \( (f(x_n)) \) is bounded. Hence the assertion is a consequence of Theorem 7.

\[\square\]
Corollary 9. Assume $t \in [0, \infty)$, $r_n^{-1} = O(n^t)$, $f$ is locally bounded, and
\[
\sum_{n=1}^{\infty} n^{1-t-s} \sum_{i=1}^{n} |K(n,i)| < \infty, \quad (87)
\]
\[
\sum_{n=1}^{\infty} n^{1-t-s} |b_n| < \infty.
\]
Then any bounded solution of (E) is convergent.

Proof. Assume $x$ is a bounded solution of (E). Let $s = 0$. By Corollary 8, there exist $c, d \in \mathbb{R}$ such that
\[
x_n = c \sum_{k=1}^{n-1} \frac{1}{r_k} + d + o(1).
\]
Define a sequence $u \in SQ$ by $u_n = r_1^{-1} + r_2^{-1} + \cdots + r_n^{-1}$. Then $u$ is increasing and bounded. Hence $u$ is convergent. Therefore $x_n = cu_n + d + o(1)$ is convergent. \qed

Corollary 10. Assume $s \in (-\infty, 0]$, $t \in [s, \infty)$, $r_n^{-1} = O(n^t)$, $f$ is bounded,
\[
\sum_{n=1}^{\infty} n^{1-t-s} \sum_{i=1}^{n} |K(n,i)| < \infty, \quad (89)
\]
\[
\sum_{n=1}^{\infty} n^{1-t-s} |b_n| < \infty.
\]
and $x$ is an arbitrary solution of (E). Then there exist $c, d \in \mathbb{R}$ such that
\[
x_n = \sum_{k=1}^{n-1} \frac{c}{r_k} + d + o(n^t). \quad (90)
\]
Proof. The assertion is an immediate consequence of Theorem 7. \qed

5. Additional Results

In this section we present some additional results. First, we give some applications of our results to linear discrete Volterra equations of type (E). From Corollary 4 we get the following result.

Corollary 11. Assume $s \in (-\infty, 0]$, $t \in [s, \infty)$,
\[
r_n^{-1} = O(n^t),
\]
\[
\sum_{n=1}^{\infty} n^{1-t-s} \sum_{i=1}^{n} |K(n,i)| < \infty, \quad (91)
\]
\[
\sum_{n=1}^{\infty} n^{1-t-s} |b_n| < \infty.
\]
Then for any $d \in \mathbb{R}$ there exists a solution $x$ of equation
\[
\Delta (r_n \Delta x_n) = b_n + \sum_{k=1}^{n} K(n,k) x_k \quad (92)
\]
such that $x_n = d + o(n^t)$.

From Corollary 5 we get the following.

Corollary 12. Assume $t \in (-\infty, -1)$, $s \in (-\infty, t]$, and
\[
r_n^{-1} = O(n^t),
\]
\[
\sum_{n=1}^{\infty} n^{1-t-s} \sum_{i=1}^{n} |K(n,i)| < \infty, \quad (93)
\]
\[
\sum_{n=1}^{\infty} n^{1-t-s} |b_n| < \infty.
\]
Then for any $c, d \in \mathbb{R}$ there exists a solution $x$ of (92) such that
\[
x_n = d + c \sum_{k=1}^{n-1} \frac{1}{r_k} + o(n^t). \quad (94)
\]

Example 13. Assume $s = 0$, $t = 1$, and
\[
r_n = \frac{1}{n-1},
\]
\[
b_n = -\frac{3}{(n+2)(n+1)n(n-1)} - \frac{1}{n^3},
\]
\[
K(n,k) = \frac{2k}{n^6}.
\]
Then (92) takes the form
\[
\Delta \left( \frac{1}{n-1} \Delta x_n \right) = -\frac{3}{(n+2)(n+1)n(n-1)} - \frac{1}{n^3} + \sum_{k=1}^{n} \frac{2k}{n^5} x_k. \quad (96)
\]
It is easy to check that all assumptions of Corollary 11 hold. Indeed, we have
\[
\sum_{n=1}^{\infty} \sum_{k=1}^{n} |K(n,k)| = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{2k}{n^5} = \sum_{n=1}^{\infty} \frac{n+1}{n^3} < \infty, \quad (97)
\]
and
\[
\sum_{n=1}^{\infty} n^3 |b_n| < \infty. \quad (98)
\]
So, for every $d \in \mathbb{R}$, there exists a solution $x$ of (96) such that $\lim_{n \to \infty} x_n = d$. One such solution is $x_n = 1 - 1/n$.

In our investigations the condition
\[
\sum_{n=1}^{\infty} n^{1-t-s} \sum_{i=1}^{n} |K(n,i)| < \infty \quad (99)
\]
plays an important role. In practice, this condition can be difficult to verify. In the following remark we present the condition, which is a little stronger but easier to check.

Remark 14. Assume $s \in (-\infty, 0]$, $t \in \mathbb{R}$, $\lambda \in (-\infty, s-t-2)$, and $u_n = O(n^t)$. Let $\varepsilon = s-t-2-\lambda$, $L > 0$, $|u_n| \leq L n^\varepsilon$ for any $n$. Then $\lambda = s-t-2-\varepsilon$ and
\[
\sum_{n=1}^{\infty} n^{1-t-s} |u_n| \leq L \sum_{n=1}^{\infty} n^{1-t-s} n^\lambda = \sum_{n=1}^{\infty} \frac{1}{n^\varepsilon} < \infty. \quad (100)
\]
Applying this remark to Corollaries 4, 5, 8, and 9, respectively, we obtain following results.
Corollary 15. Assume \( s \in (-\infty, 0], t \in [s, \infty), \lambda \in (-\infty, s - t - 2) \), \( f \) is continuous, and
\[
\begin{align*}
\sum_{i=1}^{n} |K(n, i)| &= O\left(n^{3}\right), \\
\text{then for any } d \in \mathbb{R} &\text{ there exists a solution } x \text{ of (E) such that } x_n = d + o\left(n^{s}\right).
\end{align*}
\]

Remark 19. If \( u : \mathbb{N} \rightarrow \mathbb{R} \) and \( \limsup_{n \rightarrow \infty} \sqrt[n]{|u_n|} < 1 \), then, by the root test,
\[
\sum_{n=1}^{\infty} n^{\lambda} |u_n| < \infty
\] for any \( \lambda \in \mathbb{R} \).

Corollary 20. Assume \( t \in \mathbb{R} \), \( f \) is continuous, and
\[
\begin{align*}
\sum_{i=1}^{n} |K(n, i)| &= O\left(n^{3}\right), \\
\text{then for any } d \in \mathbb{R} &\text{ there exists a solution } x \text{ of (E) such that } x_n = d + o\left(n^{s}\right).
\end{align*}
\]

Proof. Let \( d \in \mathbb{R} \). Choose \( s \in (-\infty, \min\{t, \lambda\}) \). By Remark 19, we have
\[
\sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n, i)| < \infty,
\]
and there exists a bounded solution of (E). Then there exist \( c, d \in \mathbb{R} \) such that
\[
x_n = d + c \sum_{k=1}^{n-1} \frac{1}{r_k} + o\left(n^{s}\right).
\]

Analogously, using Corollary 5, we get the following.

Corollary 21. Assume \( t \in (-\infty, -1) \), \( f \) is continuous, and
\[
\begin{align*}
\sum_{i=1}^{n} |K(n, i)| &= O\left(n^{3}\right), \\
\text{then for any } c, d \in \mathbb{R} &\text{ and any } \lambda \in (-\infty, 0] \text{ there exists a solution } x \text{ of (E) such that } x_n = d + o\left(n^{s}\right).
\end{align*}
\]
\[
\sum_{n=1}^{\infty} n^{1+t-s} |b_n| < \infty
\]
Lemma 22. If \( \omega \in (1, \infty) \), then
\[
\sum_{k=1}^{n-1} \frac{1}{k^{\omega}} = \sum_{m=1}^{\infty} \frac{1}{m^{\omega}} + O \left( n^{1-\omega} \right).
\] (115)

Proof. Define \( u \in \mathbb{Q} \) and \( \lambda \in \mathbb{R} \) by
\[
u_n = \sum_{k=1}^{n-1} \frac{1}{k^{\omega}}, \quad \lambda = \sum_{k=1}^{\infty} \frac{1}{k^{\omega}}.
\] (116)

By [28, Theorem 2.2], we have
\[
\Delta n^{1-\omega} = (1-\omega) n^{-\omega} + o \left( n^{-\omega} \right).
\] (117)

Since \( \Delta u_n = n^{-\omega} \), we get
\[
\frac{\Delta (u_n - \lambda)}{\Delta n^{1-\omega}} = \frac{n^{-\omega}}{(1-\omega) n^{-\omega} + o \left( n^{-\omega} \right)} = \frac{1}{1-\omega} + o(1) \rightarrow \frac{1}{1-\omega}.
\] (118)

Note that \( n^{1-\omega} \rightarrow 0 \) and \( (u_n - \lambda) \rightarrow 0 \). Hence, by discrete L'Hospital's Rule,
\[
u_n - \lambda = \frac{1}{1-\omega} + o(1).
\] (119)

Therefore
\[
u_n = \lambda + \frac{1}{1-\omega} n^{1-\omega} + o \left( n^{1-\omega} \right)
\] (120)

By Corollary 5, there exists a solution \( x \) of (E) such that
\[
x_n = d + c \sum_{k=1}^{n-1} k^i + o \left( n^i \right).
\] (124)

By Lemma 22
\[
\sum_{k=1}^{n-1} k^i = \lambda + O \left( n^{i+1} \right).
\] (125)

Hence
\[
x_n = \mu + O \left( n^{i+1} \right) + o \left( n^i \right) = \mu + O \left( n^{i+1} \right).
\] (126)

\[\square\]

Corollary 23. Assume \( t \in (-\infty, -1), s \in (-\infty, t], f \) is locally bounded, and
\[
r_n = n^{-t},
\] (127)

\[
\sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)| < \infty,
\] (128)

\[
\sum_{n=1}^{\infty} n^{1+t-s} |b_n| < \infty.
\] (129)

Then for any bounded solution \( x \) of (E) there exists a real number \( \mu \) such that
\[
x_n = \mu + O \left( n^{i+1} \right).
\] (130)

Proof. By Corollary 8 there exist \( c, d \in \mathbb{R} \) such that
\[
x_n = d + c \sum_{k=1}^{n-1} k^i + o \left( n^i \right).
\] (131)

By Lemma 22 we obtain
\[
x_n = d + c \left( \sum_{k=1}^{\infty} k^i + O \left( n^{i+1} \right) \right) + o \left( n^i \right) = \mu + O \left( n^{i+1} \right), \text{ where } \mu = d + c \sum_{k=1}^{\infty} k^i.
\] (132)

\[\square\]

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The second author was supported by the Ministry of Science and Higher Education of Poland (04/43/DSPB/0095).
References


Submit your manuscripts at
www.hindawi.com