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Research Article

Asymptotic Properties of Solutions to Second-Order Difference Equations of Volterra Type

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We consider the discrete Volterra type equation of the form $\Delta(r_n\Delta x_n) = b_n + \sum_{k=1}^{n} K(n,k) f(x_k)$. We present sufficient conditions for the existence of solutions with prescribed asymptotic behavior. Moreover, we study the asymptotic behavior of solutions. We use $o(n^s)$, for given nonpositive real s, as a measure of approximation.

1. Introduction

In this paper we consider the nonlinear Volterra sum-difference equation of nonconvolution type:

$$\Delta\left(r_n\Delta x_n\right) = b_n + \sum_{k=1}^{n} K\left(n,k\right) f\left(x_k\right), \tag{E}$$

$$r_n, b_n \in \mathbb{R},\ r_n > 0,\ f : \mathbb{R} \longrightarrow \mathbb{R},\ K : \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{R}. \tag{1}$$

Here $\mathbb{N}$, $\mathbb{R}$ denote the set of positive integers and the set of real numbers, respectively. By a solution of (E) we mean a sequence $x : \mathbb{N} \longrightarrow \mathbb{R}$ satisfying (E) for large n.

Discrete Volterra equations of different types are widely used in the process of modeling of some real phenomena or by applying a numerical method to a Volterra integral equation. Let $m \in \mathbb{N}$. The general form of a Volterra sum-difference autonomous equation is

$$\Delta^m x_n = a_n + \sum_{k=1}^{n} K\left(n,k\right) f\left(x_k\right). \tag{2}$$

Such equations can be regarded as the discrete analogue of Volterra integrodifferential equations of the form

$$x^{(m)}\left(t\right) = f\left(t\right) + \int_0^t K\left(t,s\right) f\left(x\left(s\right)\right) ds. \tag{3}$$

There are relatively few works devoted to the study of equations of type (2); see, for example, [1–4]. In [5], the asymptotic behaviors of nonoscillatory solutions of the higher-order integrodynamic equation on time scales are presented.

In most papers, the following special case of (2) is considered:

$$x_{n+1} = a_n x_n + b_n + \sum_{k=1}^{n} K\left(n,k\right) x_k; \tag{4}$$

see, e.g., [6–9], [10–13], [14], [15], or [16]. For some recent results devoted to nonlinear Volterra equations we refer to [5, 17–22] and references therein.

Note, that equation (E) generalizes the second-order discrete Volterra difference equation of type (2):

$$\Delta^2 x_n = b_n + \sum_{k=1}^{n} K\left(n,k\right) f\left(x_k\right). \tag{5}$$

On the other hand, if $K(n,k) = 0$ for $k \neq n$, then denoting $a_n = K(n,n)$ equation (E) takes the form

$$\Delta\left(r_n\Delta x_n\right) = a_n f\left(x_n\right) + b_n. \tag{6}$$

Hence second-order difference equation (6) is a special case of (E). The results on asymptotic properties and oscillation of equations of type (6) can be found, i.e., in [23–26].

Our main goal is to present sufficient conditions for the existence of a solution x to equation (E) such that

$$x_n = \sum_{k=1}^{n-1} \frac{c}{r_k} + d + \mathrm{o}\left(n^s\right), \tag{7}$$

where $c, d \in \mathbb{R}$ and $s \in (-\infty, 0]$. We give also sufficient conditions for a given solution x of equation (E) to have an asymptotic property (7). Moreover, in Section 5 we show applications of the obtained results to linear Volterra equation of type (E). We present also some results for the case when (r_n) is a potential sequence.

2. Preliminaries

We will denote by SQ the space of all sequences $x : \mathbb{N} \longrightarrow \mathbb{R}$. If x, y in SQ, then xy and $|x|$ denote the sequences defined by $xy(n) = x_n y_n$ and $|x|(n) = |x_n|$, respectively. Moreover,

$$\|x\| = \sup\left\{\left|x_n\right| : n \in \mathbb{N}\right\}. \tag{8}$$

If $x \in \mathrm{SQ}$, $s \in \mathbb{R}$, and $\lim_{n\longrightarrow\infty} n^{-s} x_n = 0$, then we write $x_n = \mathrm{o}(n^s)$. Analogously, $x_n = \mathrm{O}(n^s)$ denotes the boundedness of the sequence $(n^{-s} x_n)$.

The following two lemmas will be useful in the proof of our main results.

Lemma 1. *Assume $u \in \mathrm{SQ}$, $n \in \mathbb{N}$, and*

$$\sum_{j=1}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \left|u_i\right| < \infty. \tag{9}$$

Then

$$\sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \left|u_i\right| \le \sum_{j=n}^{\infty} \sum_{i=1}^{j} \frac{\left|u_j\right|}{r_i} < \infty. \tag{10}$$

Proof. We have

$$\begin{aligned}
\sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \left|u_i\right| &= \frac{1}{r_n}\left(\left|u_n\right| + \left|u_{n+1}\right| + \left|u_{n+2}\right| + \cdots\right) \\
&\quad + \frac{1}{r_{n+1}}\left(\left|u_{n+1}\right| + \left|u_{n+2}\right| + \cdots\right) + \cdots \\
&= \frac{1}{r_n}\left|u_n\right| + \left(\frac{1}{r_n} + \frac{1}{r_{n+1}}\right)\left|u_{n+1}\right| \\
&\quad + \left(\frac{1}{r_n} + \frac{1}{r_{n+1}} + \frac{1}{r_{n+2}}\right)\left|u_{n+2}\right| + \cdots \\
&\le \sum_{j=n}^{\infty} \sum_{i=1}^{j} \frac{\left|u_j\right|}{r_i} \le \sum_{j=1}^{\infty} \sum_{i=1}^{j} \frac{\left|u_j\right|}{r_i} \\
&= \frac{1}{r_1}\left|u_1\right| + \left(\frac{1}{r_1} + \frac{1}{r_2}\right)\left|u_2\right| + \cdots \\
&= \frac{1}{r_1}\left(\left|u_1\right| + \left|u_2\right| + \cdots\right) \\
&\quad + \frac{1}{r_2}\left(\left|u_2\right| + \left|u_3\right| + \cdots\right) + \cdots \\
&= \sum_{j=1}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \left|u_i\right| < \infty.
\end{aligned} \tag{11}$$

□

Lemma 2 ([27, Lemma 4.7]). *Assume $y, \rho \in SQ$, and $\lim_{n\longrightarrow\infty} \rho_n = 0$. In the set*

$$X = \left\{x \in SQ : \left|x - y\right| \le \left|\rho\right|\right\} \tag{12}$$

we define a metric by the formula

$$d(x, z) = \|x - z\|. \tag{13}$$

Then any continuous map $H : X \longrightarrow X$ has a fixed point.

3. Solutions with Prescribed Asymptotic Behavior

In this section we present sufficient conditions for the existence of a solution x to equation (E) such that

$$x_n = \sum_{k=1}^{n-1} \frac{c}{r_k} + d + \mathrm{o}\left(n^s\right), \tag{14}$$

where $c, d \in \mathbb{R}$ and $s \in (-\infty, 0]$.

Theorem 3. *Assume $s \in (-\infty, 0]$, $t \in [s, \infty)$, $c, d \in \mathbb{R}$, $y : \mathbb{N} \longrightarrow \mathbb{R}$, $q \in \mathbb{N}$, $\alpha \in (0, \infty)$,*

$$\begin{gathered}
r_n^{-1} = \mathrm{O}\left(n^t\right), \\
\sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} \left|K(n, i)\right| < \infty, \\
\sum_{n=1}^{\infty} n^{1+t-s} \left|b_n\right| < \infty, \\
y_n = d + c \sum_{k=1}^{n-1} \frac{1}{r_k}, \\
U = \bigcup_{n=q}^{\infty} \left[y_n - \alpha, y_n + \alpha\right],
\end{gathered} \tag{15}$$

and f is continuous and bounded on U. Then there exists a solution x of (E) *such that*

$$x_n = y_n + \mathrm{o}\left(n^s\right). \tag{16}$$

Proof. For $n \in \mathbb{N}$ and $x \in \mathrm{SQ}$ let

$$F(x)(n) = b_n + \sum_{k=1}^{n} K(n, k) f\left(x_k\right). \tag{17}$$

There exists $L > 0$ such that

$$\left|f(u)\right| \le L \quad \text{for any } u \in U. \tag{18}$$

Let

$$Y = \{x \in \mathrm{SQ} : |x - y| \le \alpha\}. \tag{19}$$

If $x \in Y$ and $n \ge q$, then

$$x_n \in [y_n - \alpha, y_n + \alpha] \subset U. \tag{20}$$

Choose a positive number Q such that $r_n^{-1} \le Qn^t$ for any n. Then

$$\sum_{n=1}^{\infty} \frac{1}{n^s r_n} \sum_{j=n}^{\infty} \sum_{k=1}^{j} |K(j,k)| \le Q \sum_{n=1}^{\infty} n^{t-s} \sum_{j=n}^{\infty} \sum_{k=1}^{j} |K(j,k)| \tag{21}$$

Since $t - s \ge 0$, we have

$$\sum_{n=1}^{\infty} n^{t-s} \sum_{j=n}^{\infty} \sum_{k=1}^{j} |K(j,k)| \le \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} j^{t-s} \sum_{k=1}^{j} |K(j,k)|. \tag{22}$$

For $j \in \mathbb{N}$ let

$$z_j = j^{t-s} \sum_{k=1}^{j} |K(j,k)|. \tag{23}$$

Then we have

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} z_j &= (z_1 + z_2 + \cdots) + (z_2 + z_3 + \cdots) + \cdots \\ &= z_1 + 2z_2 + 3z_3 + \cdots = \sum_{n=1}^{\infty} n z_n \\ &= \sum_{n=1}^{\infty} n n^{t-s} \sum_{k=1}^{n} |K(n,k)| \\ &= \sum_{n=1}^{\infty} n^{1+t-s} \sum_{k=1}^{n} |K(n,k)| < \infty. \end{aligned} \tag{24}$$

Hence, using (21) and (22), we get

$$\sum_{n=1}^{\infty} \frac{1}{n^s r_n} \sum_{j=n}^{\infty} \sum_{k=1}^{j} |K(j,k)| < \infty. \tag{25}$$

Analogously, replacing $\sum_{k=1}^{j} |K(j,k)|$ by $|b_j|$, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^s r_n} \sum_{j=n}^{\infty} |b_j| < \infty. \tag{26}$$

Using (25) and (26) we get

$$\sum_{j=1}^{\infty} \frac{1}{j^s r_j} \sum_{i=j}^{\infty} \left(|b_i| + L \sum_{k=1}^{i} |K(i,k)| \right) < \infty. \tag{27}$$

Since $s \le 0$, we have

$$\sum_{j=1}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \left(|b_i| + L \sum_{k=1}^{i} |K(i,k)| \right) < \infty. \tag{28}$$

Define a sequence $\rho \in \mathrm{SQ}$ by

$$\rho_n = \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \left(|b_i| + L \sum_{k=1}^{i} |K(i,k)| \right). \tag{29}$$

Define $w, g \in \mathrm{SQ}$ by

$$\begin{aligned} w_n &= |b_n| + L \sum_{k=1}^{n} |K(n,k)|, \\ g_n &= \sum_{j=n}^{\infty} \frac{1}{j^s r_j} \sum_{i=j}^{\infty} w_i. \end{aligned} \tag{30}$$

By (27), $g_n = \mathrm{o}(1)$. We have

$$\begin{aligned} n^{-s} \rho_n = n^{-s} \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} w_i &= \sum_{j=n}^{\infty} \frac{1}{n^s r_j} \sum_{i=j}^{\infty} w_i \le \sum_{j=n}^{\infty} \frac{1}{j^s r_j} \sum_{i=j}^{\infty} w_i \\ &= g_n. \end{aligned} \tag{31}$$

Hence $n^{-s} \rho_n = \mathrm{o}(1)$ and we get

$$\rho_n = n^s \mathrm{o}(1) = \mathrm{o}(n^s). \tag{32}$$

Hence there exists an index $p \ge q$ such that

$$\rho_n \le \alpha \tag{33}$$

for $n \ge p$. Let

$$\begin{aligned} &X = \{x \in \mathrm{SQ} : |x - y| \le \rho,\ x_n = y_n \text{ for } n < p\}, \\ &H : Y \longrightarrow \mathrm{SQ}, \\ &H(x)(n) = \begin{cases} y_n & \text{for } n < p \\ y_n + \displaystyle\sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} F(x)(i) & \text{for } n \ge p. \end{cases} \end{aligned} \tag{34}$$

We define a metric on X by formula (13). Note that $X \subset Y$. Let $x \in X$. By (33) and (20) we have $x_i \in U$ for any $i \ge p$. Hence, by (18), $|f(x_i)| \le L$ for $i \ge p$. Using (17) and (29) we obtain

$$\begin{aligned} |H(x)(n) - y_n| &= \left| \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} F(x)(i) \right| \\ &\le \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |F(x)(i)| \le \rho_n \end{aligned} \tag{35}$$

for $n \ge p$. Therefore $HX \subset X$. Now we show that the map H is continuous. Using (25) and the assumption $s \le 0$, we have

$$\sum_{n=1}^{\infty} \frac{1}{r_n} \sum_{j=n}^{\infty} \sum_{k=1}^{j} |K(j,k)| < \infty. \tag{36}$$

Hence, by Lemma 1, we get

$$\sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{1}{r_j} \sum_{k=1}^{n} |K(n,k)| < \infty. \tag{37}$$

Let $\varepsilon > 0$. Choose an index $m \geq p$ and a positive constant γ such that

$$L\sum_{n=m}^{\infty}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{n}|K(n,k)| < \varepsilon,$$
$$\gamma\sum_{n=1}^{m}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{n}|K(n,k)| < \varepsilon. \tag{38}$$

Let

$$C = \bigcup_{n=1}^{m}[y_n - \alpha, y_n + \alpha]. \tag{39}$$

Choose a positive δ such that if $t_1, t_2 \in C$ and $|t_1 - t_2| < \delta$, then

$$|f(t_1) - f(t_2)| < \gamma. \tag{40}$$

Choose $x, z \in X$ such that $\|x - z\| < \delta$. Then we have

$$\begin{aligned}\|Hx - Hz\| &= \sup_{n\geq p}\left|\sum_{j=n}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}(F(x)(i) - F(z)(i))\right| \\ &\leq \sup_{n\geq p}\sum_{j=n}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}|F(x)(i) - F(z)(i)| \\ &= \sum_{j=p}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}|F(x)(i) - F(z)(i)| \\ &\leq \sum_{j=p}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}\sum_{k=1}^{i}|K(i,k)|\,|f(x_k) - f(z_k)|.\end{aligned} \tag{41}$$

Using Lemma 1 we obtain

$$\|Hx - Hz\| \leq \sum_{j=p}^{\infty}\sum_{i=1}^{j}\frac{1}{r_i}\sum_{k=1}^{j}|K(j,k)|\,|f(x_k) - f(z_k)|. \tag{42}$$

Note that $|f(x_j) - f(z_j)| \leq 2L$ for $j \geq p$ and

$$|f(x_j) - f(z_j)| \leq \gamma \quad \text{for } j \in \{p, p+1, \ldots, m\}. \tag{43}$$

Hence we obtain

$$\begin{aligned}\|Hx - Hz\| &\leq \gamma\sum_{n=1}^{m}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{n}|K(n,k)| \\ &\quad + 2L\sum_{n=m}^{\infty}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{n}|K(n,k)| < 3\varepsilon.\end{aligned} \tag{44}$$

Therefore $H : X \longrightarrow X$ is continuous. By Lemma 2 there exists a point $x \in X$ such that $x = Hx$. Then, for $n \geq p$, we have

$$x_n = y_n + \sum_{j=n}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}F(x)(i). \tag{45}$$

Note that

$$\begin{aligned}\Delta(r_n\Delta y_n) &= \Delta\left(r_n\Delta\left(d + c\sum_{k=1}^{n-1}\frac{1}{r_k}\right)\right) \\ &= c\Delta\left(r_n\Delta\left(\sum_{k=1}^{n-1}\frac{1}{r_k}\right)\right) = c\Delta 1 = 0\end{aligned} \tag{46}$$

for any n. Hence, for $n \geq p$, we get

$$\begin{aligned}\Delta(r_n\Delta x_n) &= \Delta\left(r_n\Delta\left(\sum_{j=n}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}F(x)(i)\right)\right) \\ &= -\Delta\left(r_n\frac{1}{r_n}\sum_{i=n}^{\infty}F(x)(i)\right) = F(x)(n) \\ &= b_n + \sum_{k=1}^{n}K(n,k)f(x_k).\end{aligned} \tag{47}$$

Therefore x is a solution of (E). Since $x \in X$ we have $x_n = y_n + o(n^s)$. □

If the function f is continuous, then from Theorem 3 we get the following two results.

Corollary 4. *Assume $s \in (-\infty, 0]$, $t \in [s, \infty)$, f is continuous, and*

$$r_n^{-1} = O(n^t),$$
$$\sum_{n=1}^{\infty}n^{1+t-s}\sum_{i=1}^{n}|K(n,i)| < \infty,$$
$$\sum_{n=1}^{\infty}n^{1+t-s}|b_n| < \infty. \tag{48}$$

Then for any $d \in \mathbb{R}$ there exists a solution x of (E) *such that $x_n = d + o(n^s)$.*

Proof. Taking $c = 0$, $q = 1$, and $\alpha = 1$ in Theorem 3, we obtain the result. □

Corollary 5. *Assume $t \in (-\infty, -1)$, $s \in (-\infty, t]$, f is continuous, and*

$$r_n^{-1} = O(n^t),$$
$$\sum_{n=1}^{\infty}n^{1+t-s}\sum_{i=1}^{n}|K(n,i)| < \infty,$$
$$\sum_{n=1}^{\infty}n^{1+t-s}|b_n| < \infty. \tag{49}$$

Then for any $c, d \in \mathbb{R}$ there exists a solution x of (E) *such that*

$$x_n = d + c\sum_{k=1}^{n-1}\frac{1}{r_k} + o(n^s). \tag{50}$$

Proof. Assume $c, d \in \mathbb{R}$ and a sequence $y \in SQ$ is defined by

$$y_n = d + c\sum_{k=1}^{n-1}\frac{1}{r_k}. \tag{51}$$

Since $r_n^{-1} = O(n^t)$ and $t < -1$, we see that y is bounded. Now, taking $q = 1$ and $\alpha = 1$ in Theorem 3, we obtain the result. □

Note that Corollaries 4 and 5 concern convergent solutions. However, Theorem 3 includes also divergent solutions. For example, if $f(x) = x^{-1}$ for $x \neq 0$, $s \in (-\infty, 0]$, $t = 0$, $r_n^{-1} = O(1)$, and

$$\begin{gathered}\sum_{k=1}^{\infty} r_k^{-1} = \infty,\\ \sum_{n=1}^{\infty} n^{1-s}\sum_{i=1}^{n}|K(n,i)| < \infty,\\ \sum_{n=1}^{\infty} n^{1-s}|b_n| < \infty.\end{gathered} \tag{52}$$

then, by Theorem 3, for any nonzero $c \in \mathbb{R}$ and any $d \in \mathbb{R}$ there exists a solution x of (E) such that

$$x_n = d + c\sum_{k=1}^{n-1}\frac{1}{r_k} + o(n^s). \tag{53}$$

Now we present an example that proves the assumption

$$\sum_{n=1}^{\infty} n^{1+t-s}\sum_{i=1}^{n}|K(n,i)| < \infty, \tag{54}$$

in Theorem 3, is essential.

Example 6. Assume $r_n = n$, $b_n = 0$,

$$\begin{gathered}K(n,k) = \frac{1}{n^2},\\ f(x) = \frac{1}{|x|+1} + 1,\end{gathered} \tag{55}$$

$s = 0$, and $t = 0$. Then equation (E) takes the form

$$\Delta(n\Delta x_n) = \frac{1}{n^2}\sum_{k=1}^{n}\left(\frac{1}{|x_k|+1} + 1\right). \tag{56}$$

Let $c, d \in \mathbb{R}$ and

$$y_n = d + c\sum_{k=1}^{n-1}\frac{1}{r_k} = d + c\sum_{k=1}^{n-1}\frac{1}{k}. \tag{57}$$

Notice that f is continuous and bounded on $\mathbb{R}$. Moreover,

$$\sum_{n=1}^{\infty} n^{1+t-s}\sum_{k=1}^{n}K(n,k) = \sum_{n=1}^{\infty} n\frac{1}{n} = \infty, \tag{58}$$

and

$$\begin{gathered}\Delta y_n = \frac{c}{n},\\ \Delta(n\Delta y_n) = 0.\end{gathered} \tag{59}$$

Assume x is a solution of (56) such that

$$\begin{gathered}x_n = y_n + z_n,\\ z_n = o(n^s) = o(1).\end{gathered} \tag{60}$$

Since $\Delta(n\Delta y_n) = 0$, we have

$$\Delta(n\Delta x_n) = \Delta(n\Delta y_n) + \Delta(n\Delta z_n) = \Delta(n\Delta z_n). \tag{61}$$

Hence

$$\Delta(n\Delta z_n) = \Delta(n\Delta x_n) = \sum_{k=1}^{n}\frac{1}{n^2}\left(\frac{1}{|x_k|+1} + 1\right) > 0 \tag{62}$$

for large n. Therefore, the sequence $n\Delta z_n$ is eventually increasing and there exists the limit

$$\lambda = \lim_{n\longrightarrow\infty} n\Delta z_n > -\infty. \tag{63}$$

If $\lambda < \infty$, then the sequence $n\Delta z_n$ is convergent in $\mathbb{R}$. Hence the series

$$\sum_{n=1}^{\infty}\Delta(n\Delta x_n) = \sum_{n=1}^{\infty}\Delta(n\Delta z_n) \tag{64}$$

is convergent. On the other hand

$$\Delta(n\Delta x_n) = \sum_{k=1}^{n}\frac{1}{n^2}\left(\frac{1}{|x_k|+1} + 1\right) > \frac{1}{n} \tag{65}$$

for large n. Hence $\lambda = \infty$. Therefore $n\Delta z_n > 1$ for large n and we get

$$\sum_{n=1}^{\infty}\Delta z_n \geq \sum_{n=1}^{\infty}\frac{1}{n} = \infty. \tag{66}$$

But since $z_n \longrightarrow 0$, the series $\sum_{n=1}^{\infty}\Delta z_n$ is convergent.

4. Asymptotic Behavior of Solutions

In this section we present sufficient conditions for a given solution x of equation (E) to have an asymptotic property

$$x_n = \sum_{k=1}^{n-1}\frac{c}{r_k} + d + o(n^s), \tag{67}$$

where $c, d \in \mathbb{R}$ and $s \in (-\infty, 0]$.

Theorem 7. *Assume $s \in (-\infty, 0]$, $t \in [s, \infty)$,*

$$\begin{gathered}r_n^{-1} = O(n^t),\\ \sum_{n=1}^{\infty} n^{1+t-s}\sum_{i=1}^{n}|K(n,i)| < \infty,\\ \sum_{n=1}^{\infty} n^{1+t-s}|b_n| < \infty,\end{gathered} \tag{68}$$

and x is a solution of (E) *such that the sequence* $(f(x_n))$ *is bounded. Then there exist* $c, d \in \mathbb{R}$ *such that*

$$x_n = \sum_{k=1}^{n-1} \frac{c}{r_k} + d + o(n^s). \tag{69}$$

Proof. We have

$$\Delta(r_n \Delta x_n) = b_n + \sum_{k=1}^{n} K(n,k) f(x_k) \tag{70}$$

for large n. Using boundedness of the sequence $(f(x_n))$ and (68) we get

$$\sum_{n=1}^{\infty} n^{1+t-s} |\Delta(r_n \Delta x_n)| < \infty. \tag{71}$$

Define $w, u \in \mathrm{SQ}$ by

$$\begin{aligned} w_n &= \Delta(r_n \Delta x_n), \\ u_n &= n^{t-s} |w_n|. \end{aligned} \tag{72}$$

Choose a positive L such that $r_n^{-1} \le L n^t$ for any n. Since $t - s \ge 0$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^s r_n} \sum_{j=n}^{\infty} |w_j| &\le L \sum_{n=1}^{\infty} n^{t-s} \sum_{j=n}^{\infty} |w_j| = L \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} n^{t-s} |w_j| \\ &\le L \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} j^{t-s} |w_j|. \end{aligned} \tag{73}$$

Moreover,

$$\begin{aligned} \sum_{n=1}^{\infty} n^{1+t-s} |w_n| &= \sum_{n=1}^{\infty} n u_n \\ &= u_1 + (u_2 + u_2) + (u_3 + u_3 + u_3) + \cdots \\ &= \sum_{j=1}^{\infty} u_j + \sum_{j=2}^{\infty} u_j + \sum_{j=3}^{\infty} u_j + \cdots = \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} u_j \\ &= \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} j^{t-s} |w_j|. \end{aligned} \tag{74}$$

Hence, by (73)

$$\sum_{n=1}^{\infty} \frac{1}{n^s r_n} \sum_{j=n}^{\infty} |w_j| < \infty. \tag{75}$$

Since $s \le 0$, we have

$$\sum_{n=1}^{\infty} \frac{1}{r_n} \sum_{j=n}^{\infty} |w_j| < \sum_{n=1}^{\infty} \frac{1}{n^s r_n} \sum_{j=n}^{\infty} |w_j| < \infty. \tag{76}$$

Let

$$z_n = \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} w_i. \tag{77}$$

Then

$$\begin{aligned} n^{-s} |z_n| &= n^{-s} \left| \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} w_i \right| \le n^{-s} \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |w_i| \\ &= \sum_{j=n}^{\infty} \frac{1}{n^s r_j} \sum_{i=j}^{\infty} |w_i| \le \sum_{j=n}^{\infty} \frac{1}{j^s r_j} \sum_{i=j}^{\infty} |w_i| = o(1). \end{aligned} \tag{78}$$

Thus $z_n = o(n^s)$. Let

$$y_n = x_n - z_n. \tag{79}$$

Then

$$\Delta y_n = \Delta x_n - \Delta z_n = \Delta x_n + \frac{1}{r_n} \sum_{i=n}^{\infty} w_i. \tag{80}$$

Hence

$$r_n \Delta y_n = r_n \Delta x_n + \sum_{i=n}^{\infty} w_i \tag{81}$$

and we get

$$\begin{aligned} \Delta(r_n \Delta y_n) &= \Delta(r_n \Delta x_n) - w_n \\ &= \Delta(r_n \Delta x_n) - \Delta(r_n \Delta x_n) = 0 \end{aligned} \tag{82}$$

for any $n \in \mathbb{N}$. Therefore, there exists a real constant c such that $r_n \Delta y_n = c$. Thus

$$y_n - y_1 = \Delta y_1 + \cdots + \Delta y_{n-1} = \frac{c}{r_1} + \cdots + \frac{c}{r_{n-1}}. \tag{83}$$

Hence

$$x_n = y_n + z_n = \sum_{k=1}^{n-1} \frac{c}{r_k} + d + o(n^s) \tag{84}$$

where $d = y_1$. □

Corollary 8. *Assume* $s \in (-\infty, 0]$, $t \in [s, \infty)$, $r_n^{-1} = O(n^t)$, f *is locally bounded,*

$$\begin{aligned} \sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)| &< \infty, \\ \sum_{n=1}^{\infty} n^{1+t-s} |b_n| &< \infty, \end{aligned} \tag{85}$$

and x is a bounded solution of (E). *Then there exist* $c, d \in \mathbb{R}$ *such that*

$$x_n = \sum_{k=1}^{n-1} \frac{c}{r_k} + d + o(n^s). \tag{86}$$

Proof. Since x is bounded and f is locally bounded, the sequence $(f(x_n))$ is bounded. Hence the assertion is a consequence of Theorem 7. □

Corollary 9. *Assume $t \in [0,\infty)$, $r_n^{-1} = O(n^t)$, f is locally bounded, and*

$$\sum_{n=1}^{\infty} n^{1+t} \sum_{i=1}^{n} |K(n,i)| < \infty, \qquad \sum_{n=1}^{\infty} n^{1+t} |b_n| < \infty. \tag{87}$$

Then any bounded solution of (E) *is convergent.*

Proof. Assume x is a bounded solution of (E). Let $s = 0$. By Corollary 8, there exist $c, d \in \mathbb{R}$ such that

$$x_n = c \sum_{k=1}^{n-1} \frac{1}{r_k} + d + o(1). \tag{88}$$

Define a sequence $u \in SQ$, by $u_n = r_1^{-1} + r_2^{-1} + \cdots + r_{n-1}^{-1}$. Then u is increasing and bounded. Hence u is convergent. Therefore $x_n = cu_n + d + o(1)$ is convergent. □

Corollary 10. *Assume $s \in (-\infty, 0]$, $t \in [s,\infty)$, $r_n^{-1} = O(n^t)$, f is bounded,*

$$\sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)| < \infty, \qquad \sum_{n=1}^{\infty} n^{1+t-s} |b_n| < \infty, \tag{89}$$

and x is an arbitrary solution of (E). *Then there exist $c, d \in \mathbb{R}$ such that*

$$x_n = \sum_{k=1}^{n-1} \frac{c}{r_k} + d + o(n^s). \tag{90}$$

Proof. The assertion is an immediate consequence of Theorem 7. □

5. Additional Results

In this section we present some additional results. First, we give some applications of our results to linear discrete Volterra equations of type (E). From Corollary 4 we get the following result.

Corollary 11. *Assume $s \in (-\infty, 0]$, $t \in [s,\infty)$,*

$$r_n^{-1} = O(n^t), \qquad \sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)| < \infty, \qquad \sum_{n=1}^{\infty} n^{1+t-s} |b_n| < \infty. \tag{91}$$

Then for any $d \in \mathbb{R}$ there exists a solution x of equation

$$\Delta(r_n \Delta x_n) = b_n + \sum_{k=1}^{n} K(n,k) x_k \tag{92}$$

such that $x_n = d + o(n^s)$.

From Corollary 5 we get the following.

Corollary 12. *Assume $t \in (-\infty, -1)$, $s \in (-\infty, t]$, and*

$$r_n^{-1} = O(n^t), \qquad \sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)| < \infty, \qquad \sum_{n=1}^{\infty} n^{1+t-s} |b_n| < \infty. \tag{93}$$

Then for any $c, d \in \mathbb{R}$ there exists a solution x of (92) such that

$$x_n = d + c \sum_{k=1}^{n-1} \frac{1}{r_k} + o(n^s). \tag{94}$$

Example 13. Assume $s = 0$, $t = 1$, and

$$r_n = \frac{1}{n-1}, \qquad b_n = -\frac{3}{(n+2)(n+1)n(n-1)} - \frac{1}{n^4}, \qquad K(n,k) = \frac{2k}{n^6}. \tag{95}$$

Then (92) takes the form

$$\Delta\left(\frac{1}{n-1}\Delta x_n\right) = -\frac{3}{(n+2)(n+1)n(n-1)} - \frac{1}{n^4} + \sum_{k=1}^{n} \frac{2k}{n^6} x_k. \tag{96}$$

It is easy to check that all assumptions of Corollary 11 hold. Indeed, we have

$$\sum_{n=1}^{\infty} n^2 \sum_{k=1}^{n} |K(n,k)| = \sum_{n=1}^{\infty} n^2 \sum_{k=1}^{n} \frac{2k}{n^6} = \sum_{n=1}^{\infty} \frac{n+1}{n^3} < \infty, \tag{97}$$

and

$$\sum_{n=1}^{\infty} n^2 |b_n| < \infty. \tag{98}$$

So, for every $d \in \mathbb{R}$, there exists a solution x of (96) such that $\lim_{n\to\infty} x_n = d$. One such solution is $x_n = 1 - 1/n$.

In our investigations the condition

$$\sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)| < \infty \tag{99}$$

plays an important role. In practice, this condition can be difficult to verify. In the following remark we present the condition, which is a little stronger but easier to check.

Remark 14. Assume $s \in (-\infty, 0]$, $t \in \mathbb{R}$, $\lambda \in (-\infty, s-t-2)$, and $u_n = O(n^\lambda)$. Let $\varepsilon = s - t - 2 - \lambda$, $L > 0$, $|u_n| \le Ln^\lambda$ for any n. Then $\lambda = s - t - 2 - \varepsilon$ and

$$\sum_{n=1}^{\infty} n^{1+t-s} |u_n| \le L \sum_{n=1}^{\infty} n^{1+t-s} n^\lambda = \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} < \infty. \tag{100}$$

Applying this remark to Corollaries 4, 5, 8, and 9, respectively, we obtain following results.

Corollary 15. *Assume $s \in (-\infty, 0]$, $t \in [s, \infty)$, $\lambda \in (-\infty, s - t - 2)$, f is continuous, and*

$$r_n^{-1} = O\left(n^t\right),$$
$$\sum_{i=1}^{n} |K(n,i)| = O\left(n^\lambda\right), \tag{101}$$
$$b_n = O\left(n^\lambda\right).$$

Then for any $d \in \mathbb{R}$ there exists a solution x of (E) *such that $x_n = d + o(n^s)$.*

Corollary 16. *Assume $t \in (-\infty, -1)$, $s \in (-\infty, t]$, $\lambda \in (-\infty, s - t - 2)$, f is continuous, and*

$$r_n^{-1} = O\left(n^t\right),$$
$$\sum_{i=1}^{n} |K(n,i)| = O\left(n^\lambda\right), \tag{102}$$
$$b_n = O\left(n^\lambda\right).$$

Then for any $c, d \in \mathbb{R}$ there exists a solution x of (E) *such that*

$$x_n = d + c\sum_{k=1}^{n-1} \frac{1}{r_k} + o\left(n^s\right). \tag{103}$$

Corollary 17. *Assume $s \in (-\infty, 0]$, $t \in [s, \infty)$, $\lambda \in (-\infty, s - t - 2)$, f is locally bounded,*

$$r_n^{-1} = O\left(n^t\right),$$
$$\sum_{i=1}^{n} |K(n,i)| = O\left(n^\lambda\right), \tag{104}$$
$$b_n = O\left(n^\lambda\right),$$

and x is a bounded solution of (E). *Then there exist $c, d \in \mathbb{R}$ such that*

$$x_n = \sum_{k=1}^{n-1} \frac{c}{r_k} + d + o\left(n^s\right). \tag{105}$$

Corollary 18. *Assume $t \in [0, \infty)$, $\lambda \in (-\infty, -t - 2)$, f is locally bounded, and*

$$r_n^{-1} = O\left(n^t\right),$$
$$\sum_{i=1}^{n} |K(n,i)| = O\left(n^\lambda\right), \tag{106}$$
$$b_n = O\left(n^\lambda\right).$$

Then any bounded solution of (E) *is convergent.*

Now we present some results for the case when the series

$$\sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)|,$$
$$\sum_{n=1}^{\infty} n^{1+t-s} |b_n| \tag{107}$$

are strongly convergent.

Remark 19. If $u : \mathbb{N} \longrightarrow \mathbb{R}$ and $\limsup_{n\longrightarrow\infty} \sqrt[n]{|u_n|} < 1$, then, by the root test,

$$\sum_{n=1}^{\infty} n^\lambda |u_n| < \infty \tag{108}$$

for any $\lambda \in \mathbb{R}$.

Corollary 20. *Assume $t \in \mathbb{R}$, f is continuous, and*

$$r_n^{-1} = O\left(n^t\right),$$
$$\limsup_{n\longrightarrow\infty} \sqrt[n]{\sum_{i=1}^{n} |K(n,i)|} < 1, \tag{109}$$
$$\limsup_{n\longrightarrow\infty} \sqrt[n]{|b_n|} < 1.$$

Then for any $d \in \mathbb{R}$ and any $\lambda \in (-\infty, 0]$ there exists a solution x of (E) *such that*

$$x_n = d + o\left(n^\lambda\right). \tag{110}$$

Proof. Let $d \in \mathbb{R}$. Choose $s \in (-\infty, \min\{t, \lambda\})$. By Remark 19, we have

$$\sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)| < \infty,$$
$$\sum_{n=1}^{\infty} n^{1+t-s} |b_n| < \infty. \tag{111}$$

By Corollary 4, there exists a solution x of (E) such that

$$x_n = d + o\left(n^s\right) = d + o\left(n^\lambda\right). \tag{112}$$

□

Analogously, using Corollary 5, we get the following.

Corollary 21. *Assume $t \in (-\infty, -1)$, f is continuous, and*

$$r_n^{-1} = O\left(n^t\right),$$
$$\limsup_{n\longrightarrow\infty} \sqrt[n]{\sum_{i=1}^{n} |K(n,i)|} < 1, \tag{113}$$
$$\limsup_{n\longrightarrow\infty} \sqrt[n]{|b_n|} < 1.$$

Then for any $c, d \in \mathbb{R}$ and any $\lambda \in (-\infty, 0]$ there exists a solution x of (E) *such that*

$$x_n = d + \sum_{k=1}^{n-1} \frac{c}{r_k} + o\left(n^\lambda\right). \tag{114}$$

To the end we consider the case when (r_n) is a potential sequence.

Lemma 22. *If $\omega \in (1,\infty)$, then*

$$\sum_{k=1}^{n-1}\frac{1}{k^{\omega}} = \sum_{n=1}^{\infty}\frac{1}{n^{\omega}} + O\left(n^{1-\omega}\right). \tag{115}$$

Proof. Define $u \in \mathrm{SQ}$ and $\lambda \in \mathbb{R}$ by

$$u_n = \sum_{k=1}^{n-1}\frac{1}{k^{\omega}}, \qquad \lambda = \sum_{k=1}^{\infty}\frac{1}{k^{\omega}}. \tag{116}$$

By [28, Theorem 2.2], we have

$$\Delta n^{1-\omega} = (1-\omega)\, n^{-\omega} + o\left(n^{-\omega}\right). \tag{117}$$

Since $\Delta u_n = n^{-\omega}$, we get

$$\frac{\Delta\left(u_n-\lambda\right)}{\Delta n^{1-\omega}} = \frac{n^{-\omega}}{(1-\omega)\, n^{-\omega} + o\left(n^{-\omega}\right)} = \frac{1}{(1-\omega)+o(1)} \longrightarrow \frac{1}{1-\omega}. \tag{118}$$

Note that $n^{1-\omega} \longrightarrow 0$ and $(u_n-\lambda) \longrightarrow 0$. Hence, by discrete L'Hospital's Rule,

$$\frac{u_n-\lambda}{n^{1-\omega}} = \frac{1}{1-\omega} + o(1). \tag{119}$$

Therefore

$$u_n = \lambda + \frac{1}{1-\omega}n^{1-\omega} + o\left(n^{1-\omega}\right) = \sum_{k=1}^{\infty}\frac{1}{k^{\omega}} + O\left(n^{1-\omega}\right). \tag{120}$$

□

Corollary 23. *Assume $t \in (-\infty,-1)$, $s \in (-\infty,t]$, f is continuous, and*

$$r_n = n^{-t}, \qquad \sum_{n=1}^{\infty} n^{1+t-s}\sum_{i=1}^{n}|K(n,i)| < \infty, \qquad \sum_{n=1}^{\infty} n^{1+t-s}\left|b_n\right| < \infty. \tag{121}$$

Then for any $\mu \in \mathbb{R}$ there exists a solution x of (E) *such that*

$$x_n = \mu + O\left(n^{t+1}\right). \tag{122}$$

Proof. Assume $\mu \in \mathbb{R}$. Let

$$d = 0, \qquad \lambda = \sum_{n=1}^{\infty} n^{t}, \qquad c = \frac{\mu}{\lambda}. \tag{123}$$

By Corollary 5, there exists a solution x of (E) such that

$$x_n = d + c\sum_{k=1}^{n-1}k^{t} + o\left(n^{s}\right). \tag{124}$$

By Lemma 22

$$\sum_{k=1}^{n-1}k^{t} = \lambda + O\left(n^{t+1}\right). \tag{125}$$

Hence

$$x_n = \mu + O\left(n^{t+1}\right) + o\left(n^{s}\right) = \mu + O\left(n^{t+1}\right). \tag{126}$$

□

Corollary 24. *Assume $t \in (-\infty,-1)$, $s \in (-\infty,t]$, f is locally bounded, and*

$$r_n = n^{-t}, \qquad \sum_{n=1}^{\infty} n^{1+t-s}\sum_{i=1}^{n}|K(n,i)| < \infty, \qquad \sum_{n=1}^{\infty} n^{1+t-s}\left|b_n\right| < \infty. \tag{127}$$

Then for any bounded solution x of (E) *there exists a real number μ such that*

$$x_n = \mu + O\left(n^{t+1}\right). \tag{128}$$

Proof. By Corollary 8 there exist $c, d \in \mathbb{R}$ such that

$$x_n = d + c\sum_{k=1}^{n-1}k^{t} + o\left(n^{s}\right). \tag{129}$$

By Lemma 22 we obtain

$$x_n = d + c\left(\sum_{k=1}^{\infty}k^{t} + O\left(n^{t+1}\right)\right) + o\left(n^{s}\right) = \mu + O\left(n^{t+1}\right), \quad \text{where } \mu = d + c\sum_{k=1}^{\infty}k^{t}. \tag{130}$$

□

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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