Research Article

Efficient Two-Derivative Runge-Kutta-Nyström Methods for Solving General Second-Order Ordinary Differential Equations

\[ y''(x) = f(x, y, y') \]

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Received 17 September 2017; Revised 2 January 2018; Accepted 5 February 2018; Published 20 March 2018

1. Introduction

In this article, we are interested in initial value problems (IVPs) of second-order ordinary differential equations (ODEs):

\[ y''(x) = f(x, y(x), y'(x)), \]

\[ y(x_0) = \alpha, \]

\[ y'(x_0) = \beta, \]

\[ x \in [x_0, x_{\text{end}}], \]

where \( y \in \mathbb{R}^N, f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) are continuous vector valued functions. This type of problems arises naturally in many applied science fields such as the Kepler problems in celestial mechanics, quantum physics, and Newton's second law in classical mechanics (see Dormand [1], Hairer et al. [2], and Kristensson [3]).

Problems (1) in which the first derivative does not appear explicitly are an important subclass of second order (ODEs). Thus, several numerical methods for directly solving this subclass have been presented (see Dormand [1], Hairer et al. [2], Butcher [4], Lambert [5], and Senu [6]). In the case of direct solutions for the general second order (IVPs), some numerical methods have been proposed (see Chen et al. [7], Franco [8], Jator [9], Awoyemi [10], Wu et al. [11], Wu and Wang [12], and Chawla and Sharma [13]). The objective of this paper is to design STDRKN methods with a minimal number of function evaluation. This paper is organized as follows: In Section 3, we construct STDRKN methods; the stability analysis of STDRKN methods is discussed in Section 4; and numerical results are given in Section 5.

2. The Formulation of STDRKN Methods

In many problems in applications the third derivative

\[ y'''(x) = g(x, y, y') \]

\[ = f_x(x, y, y') + f_y(x, y, y') y' \]

\[ + f_{y'}(x, y, y') f(x, y, y') \]

is available and easy to obtain. This derivative can be computed but the evaluation of \( g(x, y, y') \) requires the evaluation
of \( f(x, y, y') \), \( f_x(x, y, y') \), \( f_y(x, y, y') \) \( f_y(x, y, y') \). Therefore, in the scalar case (differential systems of dimension one), an evaluation of the third derivative \( g(x, y, y') \) can be as expensive as four evaluations of the second derivative \( f(x, y, y') \) and at least as two \( f \)-evaluations. An \( s \)-stage two-derivative Runge-Kutta-Nyström (TDRKN) method for (1) is defined by the formula (see Chen et al. [7])

\[
y_{n+1} = y_n + hf_n + h^2 \sum_{i=1}^{s} b_i f_n (x_n + c_i h, y_n, y_n')
\]

\[
y_n' = y_n' + hf_n \sum_{j=1}^{s} a_{ij} f_n (x_n + c_i h, y_n, y_n')
\]

(3)

where

\[
y_j = y_n + c_j h y_n' + h^2 \sum_{j=1}^{s} a_{ij} f_n (x_n + c_i h, y_n, y_n')
\]

\[
y_n' = y_n' + h \sum_{j=1}^{s} a_{ij} f_n (x_n + c_i h, y_n, y_n')
\]

(4)

where

\[
y_j = y_n + c_j h y_n' + h^2 \sum_{j=1}^{s} a_{ij} f_n (x_n + c_i h, y_n, y_n')
\]

\[
y_1' = y_1' + h \sum_{j=1}^{s} a_{ij} f_n (x_n + c_i h, y_n, y_n')
\]

(5)

where

\[
y_j = y_n + c_j h y_n' + h^2 \sum_{j=1}^{s} a_{ij} f_n (x_n + c_i h, y_n, y_n')
\]

\[
y_1' = y_1' + h \sum_{j=1}^{s} a_{ij} f_n (x_n + c_i h, y_n, y_n')
\]

(6)

An alternative expression of formula (5) is given as follows:

\[
y_{n+1} = y_n + hf_n + h^2 \sum_{i=1}^{s} b_i k_i
\]

\[
y_{n+1}' = y_n' + h f_n \sum_{i=1}^{s} a_{ij} k_i
\]

(7)

where

\[
k_i = g \left( x_n + c_i, y_n + c_i h y_n' + \frac{1}{2} c_i^2 h^2 f \left( x_n, y_n', y_n \right) \right)
\]

\[
+ h^2 \sum_{j=1}^{s} a_{ij} k_j + c_i h f_n (x_n, y_n, y_n') + h^2 \sum_{j=1}^{s} r_{ij} k_j
\]

(8)

This STDRKN method can be written in Butcher’s tableau as shown in Table 2. The STDRKN methods are explicit methods if \( a_{ij} = 0 \), \( r_{ij} = 0 \) for \( i \leq j \) and are implicit method if \( a_{ij} \neq 0 \), \( r_{ij} \neq 0 \) for \( i \leq j \). STDRKN methods involve only one evaluation of \( f \) and many evaluations of \( g \) per step.

### 3. Construction of STDRKN Methods

In this section, our effort is to determine the coefficients of the STDRKN methods as given in (7). Hence, using the Taylor series expansion in (3) with the Taylor series expansion of \( y_n \) and \( y_n' \) and by comparing the coefficients of the power of \( h \), we obtained the order conditions of STDRKN methods for \( y \) and \( y' \) as in (10)–(18), while the rooted trees for STDRKN methods up to order five based on [7] are given in Table 3.

The following simplifying assumption is suggested in practice:

\[
\sum_{j=1}^{s} r_{ij} = \frac{c_i^2}{2}, \quad i = 2, \ldots, s
\]

(9)

The following are the order conditions for explicit STDRKN. The order conditions for \( y' \):
Table 3: Root trees for STDRKN methods up to order five.

<table>
<thead>
<tr>
<th>Order $\rho(t)$</th>
<th>Tree $t$</th>
<th>$\alpha(t)$</th>
<th>Density $\Upsilon(t)$</th>
<th>Elementary weight $\Phi(t)$</th>
<th>Elementary differential $F(t)(y, y')$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\phi$</td>
<td>1</td>
<td>1</td>
<td>$y$</td>
<td>$y'$</td>
</tr>
<tr>
<td>1</td>
<td>$t$</td>
<td>1</td>
<td>1</td>
<td>$e$</td>
<td>$f$</td>
</tr>
<tr>
<td>2</td>
<td>$t^2$</td>
<td>1</td>
<td>2</td>
<td>$c$</td>
<td>$f_y' y'$</td>
</tr>
<tr>
<td>3</td>
<td>$t^3$</td>
<td>1</td>
<td>6</td>
<td>$c^2$</td>
<td>$f_{yy}''(y', y')$</td>
</tr>
<tr>
<td>3</td>
<td>$t^3$</td>
<td>1</td>
<td>6</td>
<td>$c^2$</td>
<td>$f_{yy}''(y', f)$</td>
</tr>
<tr>
<td>4</td>
<td>$t^4$</td>
<td>1</td>
<td>12</td>
<td>$c^2$</td>
<td>$f_{yy}''(y', y')$</td>
</tr>
<tr>
<td>4</td>
<td>$t^4$</td>
<td>1</td>
<td>12</td>
<td>$c^2$</td>
<td>$f_{yy}''(f, f)$</td>
</tr>
<tr>
<td>4</td>
<td>$t^4$</td>
<td>1</td>
<td>24</td>
<td>$c^2/2$</td>
<td>$f_{y}' f_{y}' (y', f)$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>1</td>
<td>20</td>
<td>$c^3$</td>
<td>$f_{y}' f_{y}' (y', y')$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>3</td>
<td>20</td>
<td>$c^3$</td>
<td>$f_{y}' f_{y}' (y', f)$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>3</td>
<td>20</td>
<td>$c^3$</td>
<td>$f_{y}' f_{y}' (y', f, f)$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>1</td>
<td>20</td>
<td>$(1/2)c^2$</td>
<td>$f_{y}' f_{y}' (y', y, f)$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>3</td>
<td>40</td>
<td>$R_c$</td>
<td>$f_{y}' f_{y}' (y', f)$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>1</td>
<td>40</td>
<td>$(1/2)c^3$</td>
<td>$f_{y}' f_{y}' (y', y')$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
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<td>$f_{y}' f_{y}' (y', f')$</td>
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<tr>
<td>5</td>
<td>$t^5$</td>
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<td>$R_c$</td>
<td>$f_{y}' f_{y}' (f, f')$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>1</td>
<td>40</td>
<td>$R_c$</td>
<td>$f_{y}' f_{y}' (f, y')$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>1</td>
<td>40</td>
<td>$R_c$</td>
<td>$f_{y}' f_{y}' (y, y')$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>1</td>
<td>60</td>
<td>$c^2$</td>
<td>$f_{y}' f_{y}' (y', y')$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>1</td>
<td>60</td>
<td>$c^2$</td>
<td>$f_{y}' f_{y}' (y', f)$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>1</td>
<td>60</td>
<td>$c^2$</td>
<td>$f_{y}' f_{y}' (f, f)$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>2</td>
<td>120</td>
<td>$c$</td>
<td>$f_{y}' f_{y}' y'$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>1</td>
<td>120</td>
<td>$c$</td>
<td>$f_{y}' f_{y}' y'$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>1</td>
<td>120</td>
<td>$c$</td>
<td>$f_{y}' f_{y}' y'$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>2</td>
<td>120</td>
<td>$c$</td>
<td>$f_{y}' f_{y}' f_{y}' y'$</td>
</tr>
<tr>
<td>5</td>
<td>$t^5$</td>
<td>2</td>
<td>120</td>
<td>$c$</td>
<td>$f_{y}' f_{y}' f_{y}' y'$</td>
</tr>
</tbody>
</table>
Third Order
\[ \sum_{i=1}^{s} b_i = \frac{1}{6}. \]  

Fourth Order
\[ \sum_{i=1}^{s} b_i c_i = \frac{1}{24}. \]  

Fifth Order
\[ \frac{1}{2} \sum_{i=1}^{s} b_i c_i^2 = \frac{1}{120}. \]  

Sixth Order
\[ \sum_{i=1}^{s} b_i c_i^3 = \frac{1}{120}. \]
\[ \sum_{i=1}^{s} \left( \sum_{j=1}^{i-1} b_i a_{i,j} \right) = \frac{1}{720}. \]  

\[ \sum_{i=1}^{s} \left( \sum_{j=1}^{i-1} b_i r_{i,j} c_j \right) = \frac{1}{360}. \]

3.1. Two-Stage STDRKN Method of Order Four. To derive the fourth-order STDRKN method, we use the algebraic order conditions up to order four in the equations \( y \) and \( y' \), that is, (10)-(11) and (14)-(16), respectively. However, we get a system of equations which consists of 5 nonlinear equations with 6 unknown variables. Solving these equations with the simplifying assumption \( r_{2,1} = c_2^2/2 \), the resulting system has one free parameter \( a_{2,1} \); this free parameter can be selected by minimizing error equations. The truncation error norms and global error of the fifth-order condition are defined as follows:

Second Order
\[ \sum_{i=1}^{s} d_i = \frac{1}{2}. \]  

Third Order
\[ \sum_{i=1}^{s} d_i c_i = \frac{1}{6}. \]  

Fourth Order
\[ \sum_{i=1}^{s} d_i c_i^2 = \frac{1}{12}. \]  

Fifth Order
\[ \sum_{i=1}^{s} d_i c_i^3 = \frac{1}{20}. \]
\[ \sum_{i=1}^{s} \left( \sum_{j=1}^{i-1} d_i a_{i,j} \right) = \frac{1}{120}. \]  

\[ \sum_{i=1}^{s} \left( \sum_{j=1}^{i-1} d_i r_{i,j} c_j \right) = \frac{1}{120}. \]

Sixth Order
\[ \sum_{i=1}^{s} d_i c_i^4 = \frac{1}{30}. \]  

\[ \sum_{i=1}^{s} \left( \sum_{j=1}^{i-1} d_i a_{i,j} \right) = \frac{1}{180}. \]  

\[ \sum_{i=1}^{s} \left( \sum_{j=1}^{i-1} d_i c_{i,j} \right) = \frac{1}{720}. \]  

\[ \sum_{i=1}^{s} \left( \sum_{j=1}^{i-1} d_i r_{i,j} c_{j}^2 \right) = \frac{1}{360}. \]  

\[ \sum_{i=1}^{s} \left( \sum_{j=1}^{i-1} d_i r_{i,j} r_{i,j} c_{j}^2 \right) = \frac{1}{180}. \]  

The error equations \( \| r^{(5)} \|_2 \) have a minimum value at \( a_{2,1} = 1/40 \) which produces

\[ \| r^{(5)} \|_2 = 2.08 \times 10^{-3}, \]
\[ \| r^{(5)}' \|_2 = 8.3 \times 10^{-3}, \]
\[ \| r^{(5)}_g \|_2 = 8.5 \times 10^{-3}, \]

where \( r^{(5)} \) and \( r^{(5)}' \) are error terms of the fifth-order conditions for \( y_n \) and \( y'_n \), respectively. The coefficients of the new method can be given in Butcher tableau and denoted by STDRKN4(2) as seen in Table 4.
3.2. Three-Stage STDRKN Method of Order Five. According to the order conditions in (10)–(12) and (14)–(17), we obtain system of equations that consists of 9 nonlinear equations with 14 unknowns that need to be solved. Solving the system simultaneously with simplifying assumption conditions \( r_{2,1} = c_2^2/2, \) \( r_{3,1} = c_3^2/2 - r_{3,2} \) yields a solution with three free parameters \( a_{2,1}, a_{3,2}, \) and \( c_3 \) as follows:

\[
a_{3,1} = -\frac{1}{10} \left( \frac{1500a_{2,1}c_2^2 - 3000a_{2,1}c_3^3}{-3 + 5c_3} \right) \left( 1500a_{2,1}c_2^2 - 3000a_{2,1}c_3^3 \right)
+ 2000a_{2,1}c_3^2 - 250a_{2,1}c_3 - 30a_{3,2} + 50a_{3,2}c_3 + 9c_3
- 45c_3^2 + 800c_3^3 - 50c_4^4,
\]

\[
b_1 = \frac{10c_3^2 - 8c_2 + 1}{24 (3 + 5c_3)} c_3^3,
\]

\[
b_2 = \frac{5 \left( 2 - 9c_3 + 10c_3^2 \right) (2c_3 - 1)}{24 (3 - 10c_3 + 10c_3^2) (-3 + 5c_3)} c_3^3,
\]

\[
b_3 = -\frac{c_3 - 1}{24c_3 (3 - 10c_3 + 10c_3^2)} c_3^3,
\]

\[
c_2 = -\frac{3 + 5c_3}{5 (2c_3 - 1)},
\]

\[
d_1 = \frac{10c_3^2 - 8c_3 + 1}{12 (-3 + 5c_3)} c_3^3,
\]

\[
d_2 = \frac{25 \left( 1 - 4c_3 + 4c_3^2 \right) (2c_3 - 1)}{12 (3 - 10c_3 + 10c_3^2) (-3 + 5c_3)} c_3^3.
\]

\[
d_3 = \frac{12c_3 (3 - 10c_3 + 10c_3^2)}{2 (-3 + 5c_3)} c_3^3.
\]

\[
r_{3,2} = \frac{3 - 10c_3 + 10c_3^2}{2 (-3 + 5c_3)} c_3 (2c_3 - 1).
\]

Substituting the above solution into \( \|r^{(6)}\|_2, \|r^{(6)}\|_2, \) and \( \|r_g^{(6)}\|_2 \) gives us

\[
\|r^{(6)}\|_2 = \frac{1}{3600 (-3 + 5c_3) (2c_3 - 1)} \sqrt{2} \left( -900000a_{2,1}c_6^6 + 1800000a_{2,1}^2c_3^6 + 337500c_3^6 - 115500c_3^5 
- 5400000a_{2,1}^2c_3^5 + 2940000a_{2,1}c_3^5 + 67500000a_{2,1}^2c_3^4 
+ 163525c_3^4 - 39900000a_{2,1}c_3^4 - 11460c_3 + 2880000a_{2,1}c_3^3 
- 122650c_3^3 + 51445c_3^2 + 16875000a_{2,1}^2c_3^2 - 1166250a_{2,1}c_3^2 
+ 251250a_{2,1}c_3 - 3375000a_{2,1}^2c_3^3 + 1062 + 281250a_{2,1}^2c_3^3 
- 22500a_{2,1} - 4500000a_{2,1}^2c_3^3)^{1/2}.
\]

\[
\|r^{(6)}\|_2 = \frac{1}{3600 (-3 + 5c_3) (2c_3 - 1)} \sqrt{2} \left( -900000a_{2,1}c_6^6 + 1800000a_{2,1}^2c_3^6 + 337500c_3^6 - 115500c_3^5 
- 5400000a_{2,1}^2c_3^5 + 2940000a_{2,1}c_3^5 + 67500000a_{2,1}^2c_3^4 
+ 163525c_3^4 - 39900000a_{2,1}c_3^4 - 11460c_3 + 2880000a_{2,1}c_3^3 
- 122650c_3^3 + 51445c_3^2 + 16875000a_{2,1}^2c_3^2 - 1166250a_{2,1}c_3^2 
+ 251250a_{2,1}c_3 - 3375000a_{2,1}^2c_3^3 + 1062 + 281250a_{2,1}^2c_3^3 
- 22500a_{2,1} - 4500000a_{2,1}^2c_3^3)^{1/2}.
\]

\[
\|r_g^{(6)}\|_2 = \sqrt{\sum_{i=1}^{n} \left( \frac{r_{i}^{(6)}}{r_{i}^{(6)}} \right)^2}.
\]

Since the expression error norm (22) is so complicated and not suitable to write and minimizing (22) with respect to the free parameters \( a_{2,1}, a_{3,2}, c_3, \) and \( r_{3,2} \), we get \( a_{2,1} = 3/1000, a_{3,2} = 1/25, c_3 = 18/25, \) and \( r_{3,2} = 4059/15625. \) These values give us

\[
\|r^{(6)}\|_2 = 2.2 \times 10^{-4},
\]

\[
\|r^{(6)}\|_2 = 6.8 \times 10^{-4},
\]

\[
\|r_g^{(6)}\|_2 = 7.2 \times 10^{-4}.
\]

The coefficients of the new method can be given in Butcher tableau and denoted by STDRKN5(3) as seen in Table 5.
Table 5: The STDRKN5(3) method.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>0</th>
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<th>9/242</th>
<th>0</th>
<th>0</th>
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<td></td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>18/25</td>
<td>36221/1562500</td>
<td>1/25</td>
<td></td>
<td>−9/15625</td>
<td>409/15625</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>53/1296</td>
<td>121/1107</td>
<td>875/53136</td>
<td>53/648</td>
<td>1331/4428</td>
<td>3125/26568</td>
</tr>
</tbody>
</table>

4. Stability of the STDRKN Methods

In this part, we study the linear stability of the STDRKN methods. We use the test problem (see [7, 14])

\[ y'' = -\omega^2 y + \varepsilon y', \]

(24)

Applying STDRKN method (5) to test problem (24), we obtain

\[
\begin{bmatrix}
    y_{n+1} \\
    h y'_{n+1}
\end{bmatrix} = M(v, z) \begin{bmatrix}
    y_n \\
    h y'_n
\end{bmatrix},
\]

(25)

where

\[
M(v, z) = \begin{bmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{bmatrix}
\]

(26)

with

\[
m_{1,1} = 1 - \frac{v^2}{2} + (z^2 - v^2) b^T (N_1^{-1} e - v^2 z (z^2 - v^2)) N_3 - v^2 z b^T ((z^2 - v^2) N_1^{-1} R N_2^{-1} - N_2^{-1}) N_4,
\]

\[
m_{1,2} = 1 + \frac{z}{2} + (z^2 - v^2) b^T (N_1^{-1} e - v^2 z (z^2 - v^2)) N_3 - v^2 z b^T ((z^2 - v^2) N_1^{-1} R N_2^{-1} - N_2^{-1}) N_4,
\]

\[
m_{2,1} = -v^2 + (z^2)
\]

\[
- v^2 d^T (N_1^{-1} e - v^2 z (z^2 - v^2) N_1^{-1} R N_2^{-1} A N_1^{-1} e) N_3
- v^2 z d^T ((z^2 - v^2) N_1^{-1} R N_2^{-1} - N_2^{-1}) N_4,
\]

\[
m_{2,2} = 1 + z + (z^2)
\]

\[
- v^2 d^T (N_1^{-1} e - v^2 z (z^2 - v^2) N_1^{-1} R N_2^{-1} A N_1^{-1} e) N_3
- v^2 z d^T ((z^2 - v^2) N_1^{-1} R N_2^{-1} - N_2^{-1}) N_4,
\]

where \( N_1 = I - (z^2 - v^2) R \)

\[
N_2 = I + v^2 z A + v^2 z (z^2 - v^2) A N_1^{-1} R,
\]

\[
S_R = \{ (v, z) : |\lambda_i(M)| < 1, i = 1, 2 \}.
\]

(29)

\( \lambda_i \) are eigenvalues of \( M(v, z) \). The stability regions for STDRKN4(2) and STDRKN5(3) are shown in Figures 1 and 2, respectively.

5. Numerical Experiments

In this section, we test the effectiveness of the new methods of orders four and five on the same problems for comparison.
Problem 1. Consider the linear homogeneous problem

\[ y'' = x \left( y - y' \right), \]
\[ y(0) = 1, \]
\[ y'(0) = -1, \]
\[ x_{\text{end}} = 12, \]

whose analytic solution is \( y(x) = e^{-x^2/(2x^2)}. \)

Problem 2. Consider the linear nonhomogeneous problem

\[
y'' = -y' + \cos(x),
\]
\[ y(0) = -\frac{1}{2}, \]
\[ y'(0) = \frac{1}{2}, \]
\[ x_{\text{end}} = 10 \]

whose analytic solution is \( y(x) = (1/2)(\sin(x) - \cos(x)). \)

Problem 3. Consider the nonlinear problem

\[
y'' = \frac{1}{40} \left( 10 - y \right) y',
\]
\[ y(0) = 1, \]
\[ y'(0) = \frac{19}{80}, \]
\[ x_{\text{end}} = 10 \]

whose analytic solution is \( y(x) = 20/(1 + 19e^{-x^4/4}). \)

Problem 4. Consider the famous Van der Pol equation (see [15])

\[
y'' = -y + \delta \left( 1 - y^2 \right) y',
\]
with the initial values

\[
y(0) = 2 + \frac{1}{96} \delta^2 + \frac{1033}{553296} \delta^4 + \frac{1019689}{55738368000} \delta^6,
\]
\[ y'(0) = 0. \]

This is a nonlinear equation. Here we take \( \delta = 0.8 \times 10^{-4}. \) The problem is integrated on the interval \([0, 100]\) with stepsizes \( h = 1/2^i, \) \( i = 1, 2, 3, 4. \) Since the exact solution of the problem is not available, when estimating the error of each method, we use RKNG4 method given in [2] as a reference numerical solution with a very small stepsize.

Problem 5. Consider the linear homogeneous system (see [7])

\[ y'' = -My + Ky', \]
where

\[
M = \begin{bmatrix}
1 & -\frac{9}{40} & \frac{27}{40} \\
0 & \frac{9}{2} & -\frac{3}{2} \\
0 & 3 & -63 \\
0 & 8 & -8
\end{bmatrix},
\]
\[
K = \begin{bmatrix}
\frac{21}{40} & 3 & 10 \\
0 & \frac{1}{2} & 1 \\
0 & -\frac{7}{8} & -\frac{1}{2}
\end{bmatrix}
\]
with the initial values
\[ y(0) = \left(\frac{-1}{2}, \frac{1}{6}, \frac{-1}{6}\right)^T, \quad (37) \]
\[ y'(0) = (1, 2, 1)^T. \]

The problem is integrated on the interval \([0, 12]\) with
stepsizes \(h = 0.2/2^i, i = 0, 1, 2, 3\).

The exact solution of the problem is given by
\[ y(x) = \left[ \begin{array}{c}
\sin(x) - \frac{1}{2} \cos(2x) \\
\sin(2x) + \frac{1}{2} \cos(2x) - \frac{1}{3} \cos(3x) \\
\frac{1}{3} \sin(3x) - \frac{1}{2} \cos(2x) + \frac{1}{3} \cos(3x)
\end{array} \right]. \quad (38) \]

**Problem 6.** Consider the linear nonhomogeneous system (see [8])
\[ y'' = -My + \frac{12\epsilon}{5}Ky' + \epsilon^2 L(x), \]
\[ M = \begin{bmatrix}
13 & -12 \\
-12 & 13
\end{bmatrix}, \]
\[ K = \begin{bmatrix}
3 & 2 \\
-2 & -3
\end{bmatrix}, \]
\[ L = \begin{bmatrix}
\frac{36}{5} \sin(x) + 24 \sin(5x) \\
\frac{-24}{5} \sin(x) - 36 \sin(5x)
\end{bmatrix}, \] with the initial values
\[ y(0) = (\epsilon, \epsilon)^T, \]
\[ y'(0) = (-4, 6)^T, \quad (40) \]
\[ \epsilon = 10^{-3}. \]

The problem is integrated on the interval \([0, 5]\) with stepsizes \(h = 0.1/2^i, i = 1, 2, 3, 4\).

The exact solution is
\[ y(x) = \left[ \begin{array}{c}
\sin(x) - \sin(5x) + \epsilon \cos(x) \\
\sin(x) + \sin(5x) + \epsilon \cos(5x)
\end{array} \right]. \quad (41) \]

**Problem 7.** Consider the damped wave equation with periodic boundary conditions (see [15])
\[ \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \quad -1 < x < 1, \quad t > 0, \]
\[ u(-1, t) = u(1, t). \quad (42) \]

A semidiscretization in the spatial variable by second-order symmetric differences leads to the following system of second-order ODEs in time:
\[ \ddot{U} + \dot{K}U = F(U, \dot{U}), \quad 0 < t < t_{end}, \quad (43) \]

where \(U(t) = (u_1(t), \ldots, u_N(t))^T\) with \(u_i(t) \approx u(x_i, t), i = 1, 2, \ldots, N, \) and
\[ F(U, \dot{U}) = (f(u_1) - \delta \dot{u}_1, \ldots, f(u_N) - \delta \dot{u}_N), \]
\[ K = \frac{1}{\Delta x^2} \begin{pmatrix}
2 & -1 & \cdots & 0 \\
-1 & 2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 2 -1
\end{pmatrix} \quad (44) \]

with \(\Delta x = 2/N\) and \(x_i = -1 + i\Delta x\) and \(F(U, \dot{U}) = (f(u_1) - \delta \dot{u}_1, \ldots, f(u_N) - \delta \dot{u}_N).\) In this experiment, we take \(f(u) = \sin u\) and \(\delta = 0.08\) and the initial conditions as
\[ U(0) = (\pi)^N_i, \]
\[ U_i(0) = \sqrt{N} \left(0.01 + \sin \left(\frac{2\pi i}{N}\right)\right)^N_{i=1}. \quad (45) \]

We choose \(N = 40\) and integrate the problem in the interval \([0, 100]\) with the stepsizes \(h = 1/2^i, i = 5, 6, 7, 8.\) The reference numerical solution is obtained by method RKNG4 with a very small stepsize.

**6. Conclusion**

In this study, the special class of explicit two-derivative Runge-Kutta-Nyström methods of order up to five that involve one \(f\)-evaluation and minimal number of \(g\)-evaluations was derived. Figures 3–9 display the efficiency
curves showing the common logarithm of the maximum global absolute error throughout the integration versus computational cost measured by time used by each method in the same computation machine. An advantage of the STDRKN methods over the general classical Runge-Kutta-Nyström methods and TDRKN methods is that they can reach higher order with fewer functions evaluations per step and also give us higher stage order than RKN. Some tested problems were performed. From Figures 3, 4, 5, 6, 7, 8, and 9, the numerical results showed that the new methods agreed very well with the existing methods in the literature and required less time compared to the existing methods.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.
Acknowledgments

The authors gratefully acknowledge the financial support of Fundamental Research Grant Scheme (Project no. 01-01-16-1866FR) and Universiti Putra Malaysia.

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