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Research Article

Controllability and Observability Analysis of Nonlinear Positive Discrete Systems

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This article studies the controllability and observability of nonlinear positive discrete systems. These properties play a fundamental role in system analysis before controller and observer design is engaged. We solve these problems by a technique based on the fixed point theory.

1. Introduction

Controllability and observability are two fundamental concepts in the mathematical control theory. Systematic studies on these topics in the linear case were started at the beginning of 1960s [1, 2] and in nonlinear one in 1970s [3]. Controllability property plays an important role in the existence of solutions to many control problems, for example, stabilization of unstable system by feedback, optimal control [4]. Observability plays a crucial role in the study of canonical forms of dynamical systems or observer synthesis [5]. Basically a system is controllable if it is possible to transfer it from an arbitrary initial state to an arbitrary final state using only certain admissible controls; it is observable if the initial state can be determined using the information given by an output over a finite time. There exist many papers in which these two properties for classical discrete and continues systems are studied. A meaningful fact in practice, also in classical systems, is to investigate both properties in their local formulations for nonlinear systems through global notions of controllability and observability by linearisation of the considered systems.

Positive systems are a wide class of systems in which state variables and outputs are constrained to be positive, or at least nonnegative, for all time whenever the initial state and inputs are nonnegative. Since the state variables and outputs of many real-world processes represent quantities that may not have meaning unless they are nonnegative because they measure concentrations, temperatures, cell birth, losses, etc., positive systems arise frequently in mathematical modeling of engineering problems, management sciences, economics, social sciences, chemistry, biology, ecology, medicine, and other areas.

The mathematical theory of positive linear systems is based on the theory of nonnegative matrices developed by Perron and Frobenius; see, for example, [6, 7]. An excellent survey of positive systems with an emphasis on their applications in the areas of management and social sciences is given by Luenberger in [6]. The more recent monographs by Farina and Rinaldi in [8] and Kaczorek in [9] are devoted entirely to positive linear systems and some of their applications. Since positive systems are not defined on linear spaces but on cones [10, 11], their analysis and synthesis are more complicated and more challenging.

Since late 1980s controllability and reachability of both discrete and continuous positive linear systems have been a subject of much research [12–19]. Therefore, it was discovered that controllability of continuous positive systems requires very restrictive conditions to be satisfied. Thus criteria for controllability of discrete systems and continuous systems are essentially different. Observability of positive linear systems has been addressed in [20, 21]. The reachability of nonlinear positive systems for continuous and discrete systems has been formulated and solved, respectively, in [22, 23].

This paper deals with the class of nonlinear positive discrete systems. The problems of controllability and observability for this class of systems are considered. First, we present a characterization of the positivity of such systems, and then necessary and sufficient conditions for checking the controllability and observability properties are established. Because of the nonlinearity of the proposed system, we characterize these properties by two different methods that are mainly based on fixed point technique. We show that controllability is equivalent to existence of at least one fixed point and observability is equivalent to existence of at most one fixed point of some functions. Furthermore, we characterize the set of nonnegative controls which steer the state of the positive system from a nonnegative initial state to a nonnegative desired final state. The set of all nonnegative states which correspond to a given nonnegative output is also characterized. To the best of the author's knowledge, the controllability and observability problems of nonlinear positive discrete systems have not been investigated yet by means of similar techniques as those presented in this work.

2. Preliminaries

First we introduce some notations. $\mathbb{N}$ is the set of nonnegative integers, $\mathbb{N}_+$ the set of positive integers, $\sigma_s^k = \{s, s+1, \dots, k\}$ the finite subset of $\mathbb{N}$ with $s \le k$, $\mathbb{R}^n$ the set of real vectors with n components, $\mathbb{R}_+^n$ the set of all vectors in $\mathbb{R}^n$ with nonnegative components, i.e.,

$$\mathbb{R}_+^n := \left\{ x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n : x_i \ge 0,\ i \in \sigma_1^n \right\}, \quad (1)$$

where T denotes the transpose, $(z_i)_j$ the jth component of a vector z_i, $\mathbb{R}^{n\times m}$ the set of real constant matrices of size $n \times m$, and I_n the identity matrix in $\mathbb{R}^{n\times n}$. In addition, if F is a mapping, then S_F denotes the set of all its fixed points.

In this work, we consider the discrete nonlinear system described by

$$\begin{aligned} x_{i+1} &= Ax_i + f(x_i) + Bu_i, \quad i \in \mathbb{N}, \\ x_0 &\in \mathbb{R}^n. \end{aligned} \quad (2)$$

Information on system (2) is given by the output equation

$$y_i = Cx_i, \quad i \in \mathbb{N}, \quad (3)$$

where $x_i \in \mathbb{R}^n$ is the system state, $u_i \in \mathbb{R}^m$ is the input (or control), $y_i \in \mathbb{R}^r$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a nonlinear function, and x_0 represents the initial state.

Now, we define the positivity of system (2)-(3) using the following.

Definition 1. System (2)-(3) is said to be positive if for any nonnegative initial state $x_0 \in \mathbb{R}_+^n$ and all nonnegative inputs $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{N}$, we have $x_i \in \mathbb{R}_+^n$ and $y_i \in \mathbb{R}_+^r$ for any $i \in \mathbb{N}$, where x_i is the solution corresponding to x_0 and u_i, and, similarly, y_i is the output corresponding to x_0 and u_i.

Definition 2. A matrix $A = (a_{ij})_{i\in\sigma_1^n, j\in\sigma_1^m}$ is said to be nonnegative and denoted by $A \in \mathbb{R}_+^{n\times m}$, if all of its elements are nonnegative, i.e., $a_{ij} \ge 0$ for all $i \in \sigma_1^n$, $j \in \sigma_1^m$.

The following result provides a necessary and sufficient condition for positivity of system (2)-(3).

Proposition 3. *System (2)-(3) is positive if and only if*

$$\begin{aligned} (A + f)(\mathbb{R}_+^n) &\subset \mathbb{R}_+^n, \\ B &\in \mathbb{R}_+^{n\times m}, \\ C &\in \mathbb{R}_+^{r\times n}. \end{aligned} \quad (4)$$

Proof. See Appendix A. □

In the remainder of this paper, we assume that system (2)-(3) is positive.

3. Controllability of Nonlinear Positive Discrete Systems

The precise definition of the controllability of system (2) is given as follows.

Definition 4. System (2) is said to be controllable in $N \in \mathbb{N}_+$ steps if for any initial state $x_0 \in \mathbb{R}_+^n$ and any desired final state $x_f \in \mathbb{R}_+^n$, there exists an input sequence $u_i \in \mathbb{R}_+^m, i \in \sigma_0^{N-1}$, which steers the state of the system from x_0 to x_f, i.e., $x_N = x_f$.

The explicit solution of (2) is given by

$$x_i = A^i x_0 + \sum_{j=0}^{i-1} A^{i-1-j} f(x_j) + \sum_{j=0}^{i-1} A^{i-1-j} B u_j, \quad i \in \mathbb{N}_+. \quad (5)$$

In this section, we will use the following notation: $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \in \mathbb{R}^{nN}$, $u = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} \in \mathbb{R}^{Nm}$, with $N \in \mathbb{N}_+$, $x_i \in \mathbb{R}^n$ for $i \in \sigma_1^N$ and $u_i \in \mathbb{R}^m$ for $i \in \sigma_0^{N-1}$.

We define the matrix

$$\widetilde{A} = \begin{pmatrix} A \\ A^2 \\ \vdots \\ A^N \end{pmatrix} \in \mathbb{R}^{nN\times n}. \quad (6)$$

Let us consider the nonlinear mapping G defined by

$$G : x \in \mathbb{R}_+^{nN} \longmapsto G(x) \in \mathbb{R}^{nN}, \quad (7)$$

with

$$(G(x))_i = \sum_{j=0}^{i-1} A^{i-1-j} f(x_j), \quad i \in \sigma_1^N, \quad (8)$$

and let D denote the linear mapping defined by

$$D : u \in \mathbb{R}^{Nm} \longmapsto Du \in \mathbb{R}^{nN}, \quad (9)$$

with

$$(Du)_i = \sum_{j=0}^{i-1} A^{i-1-j} B u_j, \quad i \in \sigma_1^N. \quad (10)$$

Then, solution (5) over N steps can be rewritten as

$$x = \widetilde{A} x_0 + G(x) + Du. \quad (11)$$

3.1. Characterization of Controllability: First Mapping. The aim of this subsection is to establish a necessary and sufficient condition for the controllability of system (2) based on fixed points of a function appropriately chosen. Also, we characterize the set $\mathcal{U}_+$ of nonnegative controls which steer the state of system (2) from an initial state $x_0 \in \mathbb{R}_+^n$ at $i = 0$ to a desired final state $x_f \in \mathbb{R}_+^n$ at $i = N \in \mathbb{N}_+$, i.e.,

$$\mathcal{U}_+ = \left\{ u \in \mathbb{R}_+^{Nm} : x_N(u) = x_f \right\}, \quad (12)$$

where $x_N(u)$ is the state of system (2) in step N corresponding to the control u.

Definition 5. The positive image of a matrix $E \in \mathbb{R}^{p\times q}$ is

$$\mathrm{Im}_+ E = \left\{ Eu \in \mathbb{R}^p : u \in \mathbb{R}_+^q \right\}. \quad (13)$$

Let $P : \mathbb{R}^{nN} \longrightarrow \mathrm{Im}_+ D$ be any projection on $\mathrm{Im}_+ D$ (i.e., any mapping $P : \mathbb{R}^{nN} \longrightarrow \mathrm{Im}_+ D$ that satisfies $Pz = z$ if and only if $z \in \mathrm{Im}_+ D$) and $\overline{x}$ be any fixed element of $\mathrm{Im}_+ D$ different from zero.

We define

$$\zeta : x \in \mathbb{R}_+^{nN} \longmapsto x - \widetilde{A} x_0 - G(x) \in \mathbb{R}^{nN}, \quad (14)$$

and we consider the mapping

$$F : x \in \mathbb{R}_+^{nN} \longmapsto \widetilde{A} x_0 + G(x) + P\zeta(x) + \left\| x_N - x_f \right\| \overline{x} \in \mathbb{R}^{nN}, \quad (15)$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$. It is to be noted that the mapping F depends on the states x_0 and x_f.

A necessary and sufficient condition for the controllability of system (2) is given by the following.

Proposition 6. *The nonlinear system (2) is controllable in N steps if and only if for all $x_0, x_f \in \mathbb{R}_+^n$, F has a fixed point.*

Proof. See Appendix B. □

Remark 7. The fixed points of F are independent of the choice of the projection operator P and the element $\overline{x}$. Indeed, let P_1 and P_2 be two projections on $\mathrm{Im}_+ D$ and $\overline{x}_1$ and $\overline{x}_2$ be two any elements not equal to zero of $\mathrm{Im}_+ D$. Let us consider the mappings

$$F_1 : x \in \mathbb{R}_+^{nN} \longmapsto \widetilde{A} x_0 + G(x) + P_1\zeta(x) + \left\| x_N - x_f \right\| \overline{x}_1 \in \mathbb{R}^{nN}, \quad (16)$$

and

$$F_2 : x \in \mathbb{R}_+^{nN} \longmapsto \widetilde{A} x_0 + G(x) + P_2\zeta(x) + \left\| x_N - x_f \right\| \overline{x}_2 \in \mathbb{R}^{nN}. \quad (17)$$

Let x be a fixed point of F_1. By proof of Proposition 6, we have $x_N = x_f$ and $\zeta(x) \in \mathrm{Im}_+ D$. Consequently $P_2\zeta(x) = \zeta(x)$ and $\|x_N - x_f\| = 0$, and then

$$F_2(x) = \widetilde{A} x_0 + G(x) + \zeta(x) = x. \quad (18)$$

Hence, if x is a fixed point of F_1, then it is also a fixed point of F_2.

Remark 8. Proposition 6 is still true if the expression $\|x_N - x_f\|$ is substituted by the term $g(x_N)$, where $g : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ is any function which satisfies $g(z) = 0$ if and only if $z = x_f$.

Now, a characterization of the set $\mathcal{U}_+$ is given by the following result.

Proposition 9. *We have*

$$\mathcal{U}_+ = \left\{ D^\dagger \zeta(x) + \Psi \in \mathbb{R}_+^{Nm} : x \in S_F \text{ and } \Psi \in \ker D \right\}, \quad (19)$$

with $D^\dagger$ being the pseudoinverse of the matrix D (see Appendix C).

Proof. See Appendix D. □

Example 10. Consider the positive system

$$\begin{aligned} x_{i+1} &= A x_i + f(x_i) + B u_i, \quad i \in \mathbb{N}, \\ x_0 &= 0, \end{aligned} \quad (20)$$

with

$$\begin{aligned} A &= \mathrm{diag}\{a_1 \ a_2 \ \cdots \ a_n\}, \quad a_i > 0 \text{ for } i \in \sigma_1^n, \\ f &: \widetilde{x} = \begin{pmatrix} \widetilde{x}_1 \\ \vdots \\ \widetilde{x}_n \end{pmatrix} \in \mathbb{R}_+^n \longmapsto \begin{pmatrix} \widetilde{x}_1^2 \\ \vdots \\ \widetilde{x}_n^2 \end{pmatrix} = \sum_{k=1}^n \widetilde{x}_k^2 e_k \in \mathbb{R}_+^n, \\ B &= \alpha e_1, \quad \alpha > 0, \end{aligned} \quad (21)$$

where e_k is the kth column of I_n. The desired final state x_f is assumed to be of the form $x_f = \beta e_1$, $\beta > 0$.

In this example, the mappings G and ζ are given by

$$\begin{aligned} G &: x \in \mathbb{R}_+^{Nn} \longmapsto G(x) \in \mathbb{R}_+^{Nn} \\ (G(x))_i &= \sum_{k=1}^n \sum_{j=0}^{i-1} a_k^{i-1-j} \left(x_j\right)_k^2 e_k, \quad i \in \sigma_1^N, \\ \zeta &: x \in \mathbb{R}_+^{Nn} \longmapsto \zeta(x) \in \mathbb{R}^{Nn} \\ (\zeta(x))_i &= x_i - \sum_{k=1}^n \sum_{j=0}^{i-1} a_k^{i-1-j} \left(x_j\right)_k^2 e_k, \quad i \in \sigma_1^N. \end{aligned} \quad (22)$$

The application D is given by

$$D : u \in \mathbb{R}^N \longmapsto Du \in \mathbb{R}^{Nn}$$
$$(Du)_i = \sum_{j=0}^{i-1} a_1^{i-1-j} \alpha u_j e_1, \quad i \in \sigma_1^N. \tag{23}$$

The *pseudo inverse* of D is given by

$$D^\dagger : z \in \mathbb{R}^{Nn} \longmapsto D^\dagger z = u \in \mathbb{R}^N \tag{24}$$

where

$$u_0 = \frac{(z_1)_1}{\alpha},$$
$$u_i = \frac{(z_{i+1})_1 - a_1 (z_i)_1}{\alpha}, \quad i \in \sigma_1^{N-1}. \tag{25}$$

Set $\overline{x} = \begin{pmatrix} e_1 \\ a_1 e_1 \\ \vdots \\ a_1^{N-1} e_1 \end{pmatrix}$ and let P be the projection

$$P : \mathbb{R}^{Nn} \longrightarrow \mathrm{Im}_+ D$$
$$x \longmapsto \begin{cases} \left((x_1)_1 e_1 \ \ a_1 (x_1)_1 e_1 \ \ \cdots \ \ a_1^{N-1} (x_1)_1 e_1\right)^T, & \text{if } (x_1)_1 \in \mathbb{R}_+, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

The mapping F is given by

$$(F(x))_i = \sum_{k=1}^{n} \sum_{j=0}^{i-1} a_k^{i-1-j} (x_j)_k^2 e_k + a_1^{i-1} (x_1)_1 e_1$$
$$+ \left\| x_N - x_f \right\| \overline{x}_i, \quad i \in \sigma_1^N. \tag{27}$$

In the case of $N = 2$, if x is a fixed point of F, then

$$x_1 = (x_1)_1 e_1 + \left\| x_2 - x_f \right\| e_1,$$
$$x_2 = \sum_{k=1}^{n} (x_1)_k^2 e_k + a_1 (x_1)_1 e_1. \tag{28}$$

Hence $(x_1)_1 = (x_1)_1 + \|x_2 - x_f\|$ and $(x_1)_i = 0$ for $i \in \sigma_2^n$. Thus $x_1 = (x_1)_1 e_1$ and $x_f = x_2$. Then the set of fixed points of F is given by

$$S_F = \left\{ \left(\frac{-a_1 + \sqrt{a_1^2 + 4\beta}}{2} e_1 \quad \beta e_1 \right)^T \right\}. \tag{29}$$

We have

$$(\zeta(x))_1 = \frac{-a_1 + \sqrt{a_1^2 + 4\beta}}{2} e_1$$
$$\text{and} \quad (\zeta(x))_2 = \frac{-a_1^2 + a_1 \sqrt{a_1^2 + 4\beta}}{2} e_1, \tag{30}$$

and therefore

$$\mathscr{U}_+ = \left\{ D^\dagger \zeta(x) \in \mathbb{R}_+^2 : x \in S_F \right\}$$
$$= \left\{ \left(\frac{-a_1 + \sqrt{a_1^2 + 4\beta}}{2\alpha} \quad 0 \right)^T \right\}. \tag{31}$$

3.2. Characterization of Controllability: Second Mapping. In this subsection, we shall characterize the controllability of system (2) and the set $\mathscr{U}_+$ using another mapping. For this, we put

$$\mathscr{A} = \begin{pmatrix} \widetilde{A} \\ A^N \end{pmatrix} \in \mathbb{R}^{n(N+1)\times n}, \tag{32}$$

and we introduce the following mappings

$$\mathscr{G} : x \in \mathbb{R}_+^{nN} \longmapsto \begin{pmatrix} G(x) \\ (G(x))_N \end{pmatrix} \in \mathbb{R}^{nN} \times \mathbb{R}^n,$$
$$\mathscr{D} : u \in \mathbb{R}^{Nm} \longmapsto \begin{pmatrix} Du \\ (Du)_N \end{pmatrix} \in \mathbb{R}^{nN} \times \mathbb{R}^n. \tag{33}$$

Also, we define the following applications

$$\xi : \mathbb{R}_+^{nN} \times \mathbb{R}_+^n \longrightarrow \mathbb{R}^{nN} \times \mathbb{R}^n$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} - \mathscr{A} x_0 - \mathscr{G}(x),$$
$$\mathscr{F} : \mathbb{R}_+^{nN} \times \mathbb{R}_+^n \longrightarrow \mathbb{R}^{nN} \times \mathbb{R}^n$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \mathscr{A} x_0 + \mathscr{G}(x) + \mathscr{P} \xi \begin{pmatrix} x \\ x_f \end{pmatrix}$$
$$+ \begin{pmatrix} 0 \\ y - x_f \end{pmatrix}, \tag{34}$$

where $\mathscr{P} : \mathbb{R}^{nN} \times \mathbb{R}^n \longrightarrow \mathrm{Im}_+ \mathscr{D}$ is any projection on $\mathrm{Im}_+ \mathscr{D}$. Then we have the following result.

Lemma 11. *For $x \in \mathbb{R}_+^{nN}$ and $y \in \mathbb{R}_+^n$, if $\binom{x}{y} \in S_{\mathscr{F}}$, then $\xi \binom{x}{x_f} \in \mathrm{Im}_+ \mathscr{D}$.*

Proof. See Appendix E. □

Proposition 12. *The nonlinear system (2) is controllable in N steps if and only if for all $x_0, x_f \in \mathbb{R}^n_+$, $\mathcal{F}$ has a fixed point.*

Proof. See Appendix F. ☐

Proposition 13. *We have*

$$\mathcal{U}_+ = \left\{ \mathcal{D}^\dagger \xi \begin{pmatrix} x \\ x_f \end{pmatrix} + \Psi \in \mathbb{R}^{Nm}_+ : x \in \widetilde{\mathcal{S}}_{\mathcal{F}} \text{ and } \Psi \in \ker \mathcal{D} \right\}, \tag{35}$$

with

$$\widetilde{\mathcal{S}}_{\mathcal{F}} := \left\{ x \in \mathbb{R}^{nN}_+ : \text{there exists } y \in \mathbb{R}^n_+ \text{ such that } \begin{pmatrix} x \\ y \end{pmatrix} \in S_{\mathcal{F}} \right\}. \tag{36}$$

Proof. See Appendix G. ☐

4. Observability of Nonlinear Positive Discrete Systems

As the corresponding notion of controllability, observability is obviously an important concept. In this section we discuss the problem of observability for nonlinear positive discrete systems.

Consider the nonlinear systems (2)-(3) with $u_i = 0$, $i \in \mathbb{N}$, and x_0 is assumed to be unknown.

System (2)-(3) is said to be observable in $N \in \mathbb{N}_+$ steps if the information of the output sequence $y_i \in \mathbb{R}^r_+$ for $i \in \sigma_0^{N-1}$ is sufficient to determine uniquely the nonnegative initial state $x_0 \in \mathbb{R}^n_+$.

In this section, we will use the notations $x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix} \in \mathbb{R}^{nN}$ and $y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{pmatrix} \in \mathbb{R}^{rN}$, with $N \in \mathbb{N}_+$ and $x_i \in \mathbb{R}^n$, $y_i \in \mathbb{R}^r$ for $i \in \sigma_0^{N-1}$.

To define observability more precisely, let

$$\begin{aligned} \Gamma : \mathbb{R}^n_+ &\longrightarrow \mathbb{R}^{rN}_+ \\ x_0 &\longmapsto y \end{aligned} \tag{37}$$

where y is the output over N steps.

Definition 14. System (2)-(3) is said to be observable in $N \in \mathbb{N}_+$ steps if Γ is injective.

The state of (2) is given by

$$x_i = A^i x_0 + \sum_{j=0}^{i-1} A^{i-1-j} f(x_j), \quad i \in \mathbb{N}_+. \tag{38}$$

We put

$$\widetilde{A} = \begin{pmatrix} I_n \\ A \\ \vdots \\ A^{N-1} \end{pmatrix} \in \mathbb{R}^{nN\times n}, \tag{39}$$

and

$$S = \begin{pmatrix} C & 0 & \cdots & \cdots & 0 \\ 0 & C & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & C \end{pmatrix} \in \mathbb{R}^{rN\times nN}_+. \tag{40}$$

Let us consider the nonlinear mapping G defined by

$$G : x \in \mathbb{R}^{nN}_+ \longmapsto G(x) \in \mathbb{R}^{nN}, \tag{41}$$

with $(G(x))_i = \sum_{j=0}^{i-1} A^{i-1-j} f(x_j)$, $i \in \sigma_0^{N-1}$.

Then, solution (38) over N steps can be rewritten as

$$x = \widetilde{A} x_0 + G(x), \tag{42}$$

and the output has the form

$$y = Sx = S\widetilde{A} x_0 + SG(x). \tag{43}$$

4.1. Necessary and Sufficient Criteria for Observability. The goal of this subsection is to give a characterization of the set $\mathcal{O}_+$ of states of system (2) such that $y_g = Sx$ where y_g is the given output over N steps, i.e.,

$$\mathcal{O}_+ = \left\{ x \in \mathbb{R}^{nN}_+ : x = \widetilde{A} x_0 + G(x) \text{ and } y_g = Sx \right\}, \tag{44}$$

and consequently we shall establish a necessary and sufficient condition for the observability of system (2)-(3).

Let $P : \mathbb{R}^{nN} \longrightarrow \operatorname{Im}_+ \widetilde{A}$ be any projection on $\operatorname{Im}_+ \widetilde{A}$ and $\overline{x}$ be any fixed element of $\operatorname{Im}_+ \widetilde{A}$ different from zero.

We define

$$\zeta : x \in \mathbb{R}^{nN}_+ \longmapsto x - G(x) \in \mathbb{R}^{nN}, \tag{45}$$

and we consider the mapping

$$H : x \in \mathbb{R}^{nN}_+ \longmapsto G(x) + P\zeta(x) + \left\| Sx - y_g \right\| \overline{x} \in \mathbb{R}^{nN}. \tag{46}$$

The coming result gives a characterization of the set $\mathcal{O}_+$.

Proposition 15. *Let $x \in \mathbb{R}^{nN}_+$. Then x is an element of $\mathcal{O}_+$ if and only if x is a fixed point of H.*

Proof. It is similar to that of Proposition 6. ☐

Example 16. Consider the positive system

$$\begin{aligned} x_{i+1} &= Ax_i + f(x_i), \quad i \in \mathbb{N}, \\ x_0 &\in \mathbb{R}_+^n, \\ y_i &= Cx_i, \end{aligned} \tag{47}$$

with

$$\begin{aligned} A &= \operatorname{diag}\{a_1 \ \ a_2 \ \ \cdots \ \ a_n\}, \quad a_i > 0 \text{ for } i \in \sigma_1^n, \\ C &= (1 \ \ 0 \ \ \cdots \ \ 0), \end{aligned} \tag{48}$$

and

$$f : \tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{pmatrix} \in \mathbb{R}_+^n \longmapsto \begin{pmatrix} \tilde{x}_1^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \tilde{x}_1^2 e_1 \in \mathbb{R}_+^n. \tag{49}$$

The mappings G and ζ for this example are given by

$$\begin{aligned} &G : x \in \mathbb{R}_+^{nN} \longmapsto G(x) \in \mathbb{R}_+^{nN} \\ &(G(x))_i = \sum_{j=0}^{i-1} a_1^{i-1-j} \left(x_j\right)_1^2 e_1, \quad i \in \sigma_0^{N-1}, \\ &\zeta : x \in \mathbb{R}_+^{nN} \longmapsto \zeta(x) \in \mathbb{R}^{nN} \\ &(\zeta(x))_i = x_i - \sum_{j=0}^{i-1} a_1^{i-1-j} \left(x_j\right)_1^2 e_1, \quad i \in \sigma_0^{N-1}. \end{aligned} \tag{50}$$

Set $\overline{x} = \begin{pmatrix} e_1 \\ a_1 e_1 \\ \vdots \\ a_1^{N-1} e_1 \end{pmatrix}$ and introduce the projection P

$$\begin{aligned} &P : \mathbb{R}^{nN} \longrightarrow \operatorname{Im}_+ \widetilde{A} \\ &x \\ &\longmapsto \begin{cases} \left(x_0^T \ \ (Ax_0)^T \ \ \cdots \ \ \left(A^{N-1}x_0\right)^T\right)^T, & \text{if } x_0 \in \mathbb{R}_+^n, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{51}$$

The mapping H is given by

$$\begin{aligned} (H(x))_i = {} & \sum_{j=0}^{i-1} a_1^{i-1-j} \left(x_j\right)_1^2 e_1 + \sum_{j=1}^{n} a_j^i \left(x_0\right)_j e_j \\ & + \left\|Sx - y_g\right\| \overline{x}_i, \quad i \in \sigma_0^{N-1}. \end{aligned} \tag{52}$$

For $N = 2$ and $y_g = \left(1 \ \ 1 + a_1\right)^T$, the set of fixed points of H, and hence $\mathcal{O}_+$, are

$$\begin{aligned} &\mathcal{O}_+ \\ &= \left\{\left(1 \ \ (x_0)_2 \ \ \cdots \ \ (x_0)_n \quad 1 + a_1 \ \ a_2 (x_0)_2 \ \ \cdots \ \ a_n (x_0)_n\right)^T : \right. \\ &\left. (x_0)_i \in \mathbb{R}_+, i \in \sigma_2^n \right\}. \end{aligned} \tag{53}$$

The following proposition gives a necessary and sufficient condition for the observability of our system.

Proposition 17. *System (2)-(3) is observable in N steps if and only if for every given output $y_g \in \mathbb{R}_+^{rN}$, H has at most one fixed point.*

Proof. See Appendix H. □

4.2. Another Characterization of the Observability. The aim of this subsection is to give a second characterization of the set $\mathcal{O}_+$ and of the observability of system (2)-(3) based on the fixed points of another function appropriately chosen.

We pose

$$\mathscr{A} = \begin{pmatrix} \widetilde{A} \\ S\widetilde{A} \end{pmatrix} \in \mathbb{R}^{N(r+n)\times n}, \tag{54}$$

and we introduce the following mappings

$$\begin{aligned} &\mathscr{G} : \mathbb{R}_+^{nN} \longrightarrow \mathbb{R}^{nN} \times \mathbb{R}^{rN} \\ &\qquad x \longmapsto \begin{pmatrix} G(x) \\ SG(x) \end{pmatrix}, \\ &\xi : \mathbb{R}_+^{nN} \times \mathbb{R}_+^{rN} \longrightarrow \mathbb{R}^{nN} \times \mathbb{R}^{rN} \\ &\qquad \begin{pmatrix} x \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x \\ z \end{pmatrix} - \mathscr{G}(x). \end{aligned} \tag{55}$$

We define

$$\begin{aligned} &\mathscr{H} : \mathbb{R}_+^{nN} \times \mathbb{R}_+^{rN} \longrightarrow \mathbb{R}^{nN} \times \mathbb{R}^{rN} \\ &\qquad \begin{pmatrix} x \\ z \end{pmatrix} \longmapsto \mathscr{G}(x) + \mathscr{P}\xi \begin{pmatrix} x \\ y_g \end{pmatrix} + \begin{pmatrix} 0 \\ z - y_g \end{pmatrix} \end{aligned} \tag{56}$$

where $\mathscr{P} : \mathbb{R}^{nN} \times \mathbb{R}^{rN} \longrightarrow \operatorname{Im}_+\mathscr{A}$ is any projection on $\operatorname{Im}_+\mathscr{A}$.

Lemma 18. *For $x \in \mathbb{R}_+^{nN}$ and $z \in \mathbb{R}_+^{rN}$, if $\binom{x}{z} \in S_{\mathscr{H}}$, then $\xi\binom{x}{y_g} \in \operatorname{Im}_+\mathscr{A}$.*

Proof. It is similar to that of Lemma 11. □

Proposition 19. *The set $\mathcal{O}_+$ is given by*

$$\begin{aligned} \mathcal{O}_+ = \widetilde{S}_{\mathscr{H}} := \Bigg\{ & x \in \mathbb{R}_+^{nN} : \text{there exists } z \\ & \in \mathbb{R}_+^{rN} \text{ such that } \begin{pmatrix} x \\ z \end{pmatrix} \in S_{\mathscr{H}} \Bigg\}. \end{aligned} \tag{57}$$

Proof. It is similar to that of Proposition 12. □

Proposition 20. *System (2)-(3) is observable in N steps if and only if for every given output $y_g \in \mathbb{R}_+^{rN}$, $\mathscr{H}$ has at most one fixed point.*

Proof. It is similar to that of Proposition 17. □

5. Conclusion

The fixed point technique is an important tool used in mathematics to treat different nonlinear problems [24–26]. In this work we have employed this tool for resolving the controllability and observability problem for nonlinear positive discrete systems. Necessary and sufficient conditions for the positivity of our discrete system have been established (Proposition 3). Criteria for the controllability (Propositions 6 and 12) and observability (Propositions 17 and 20) have been proved. A characterization of nonnegative controls which drives the state of the system from its initial value to a given desired final state is given (Propositions 9 and 13). The set of all nonnegative states which correspond to the given output is also characterized (Propositions 15 and 19). In our future work, we investigate the controllability and observability of positive nonlinear continuous systems.

Appendix

A.

Proof. (Sufficiency) From (2), for $i = 0$, we have

$$x_1 = (A + f)(x_0) + Bu_0 \in \mathbb{R}^n_+, \tag{A.1}$$

since (4) holds, $x_0 \in \mathbb{R}^n_+$ and $u_0 \in \mathbb{R}^m_+$. Similarly, for $i = 1$, we obtain $x_2 = (A + f)(x_1) + Bu_1 \in \mathbb{R}^n_+$ since (4) and (A.1) hold and $u_1 \in \mathbb{R}^m_+$. Repeating the procedure for $i = 2, 3, \ldots$, we have $x_i \in \mathbb{R}^n_+$ for every $i \in \mathbb{N}$. Consequently, if $C \in \mathbb{R}^{r\times n}_+$, then $y_i \in \mathbb{R}^r_+$ for any $i \in \mathbb{N}$. Thus system (2)-(3) is positive.

(Necessity) If system (2)-(3) is positive, then, in particular for $i = 0$, we have $x_1 = (A+f)(x_0)+Bu_0 \in \mathbb{R}^n_+$ and $y_0 = Cx_0 \in \mathbb{R}^r_+$. Then $C \in \mathbb{R}^{r\times n}_+$ since $x_0 \in \mathbb{R}^n_+$ is arbitrary.

Suppose $u_0 = 0$. Hence $(A + f)(x_0) \in \mathbb{R}^n_+$, and consequently $(A + f)(\mathbb{R}^n_+) \subset \mathbb{R}^n_+$. Now assuming that $x_0 = 0$, then $f(0) + Bu_0 \in \mathbb{R}^n_+$. Suppose one of the components of B, b_{ij}, is negative. Then, for the nonnegative vector $u_0 = (0, \ldots, 0, \alpha, 0, \ldots, 0)^T$ with $\alpha > 0$ being the jth component, the ith component of x_1 would be $(f(0))_i + b_{ij}\alpha \geq 0$. But, if $\alpha \longrightarrow +\infty$, then $(f(0))_i + b_{ij}\alpha \longrightarrow -\infty$, and this is absurd, which completes the proof. □

B.

Proof. (Sufficiency) Let $x_0, x_f \in \mathbb{R}^n_+$. If x is a fixed point of F, then

$$x = \widetilde{A}x_0 + G(x) + P\zeta(x) + \left\|x_N - x_f\right\| \overline{x}. \tag{B.1}$$

Hence

$$\zeta(x) = P\zeta(x) + \left\|x_N - x_f\right\| \overline{x} \in \mathrm{Im}_+ D, \tag{B.2}$$

which implies that $\zeta(x) = P\zeta(x)$ and $\|x_N - x_f\|\overline{x} = 0$, which ensures that $x_N = x_f$ because $\overline{x} \neq 0$.

Since $\zeta(x) \in \mathrm{Im}_+ D$, then there exists an input $u \in \mathbb{R}^{Nm}_+$ such that

$$\zeta(x) = Du. \tag{B.3}$$

Consequently, (B.1) becomes

$$x = \widetilde{A}x_0 + G(x) + Du. \tag{B.4}$$

Thus

$$x_N = x_f = A^N x_0 + (G(x))_N + (Du)_N = x_N(u). \tag{B.5}$$

Hence system (2) is controllable in N steps.

(Necessity) Let x_0, $x_f \in \mathbb{R}^n_+$. Since system (2) is controllable in N steps, there exists an input $u \in \mathbb{R}^{Nm}_+$ such that $x_N := x_N(u) = x_f$. Then we have

$$\begin{aligned} x &= \widetilde{A}x_0 + G(x) + Du, \\ x_N &= x_f, \end{aligned} \tag{B.6}$$

with x being the solution of system (2) over N steps corresponding to the control u.

Consequently

$$\begin{aligned} \zeta(x) &= Du \in \mathrm{Im}_+ D, \\ \left\|x_N - x_f\right\| &= 0, \end{aligned} \tag{B.7}$$

and then

$$P\zeta(x) = \zeta(x). \tag{B.8}$$

Hence, we obtain

$$\begin{aligned} F(x) &= \widetilde{A}x_0 + G(x) + P\zeta(x) + \left\|x_N - x_f\right\| \overline{x} \\ &= \widetilde{A}x_0 + G(x) + Du = x. \end{aligned} \tag{B.9}$$

Thus x is a fixed point of the mapping F. The proposition is proved. □

C.

Definition C.1 (see [27]). A matrix $E^\dagger \in \mathbb{R}^{p\times q}$ is said to be the Moore-Penrose generalized inverse (pseudo inverse) of $E \in \mathbb{R}^{q\times p}$ if

(i) $EE^\dagger E = E$,

(ii) $E^\dagger EE^\dagger = E^\dagger$,

(iii) $(EE^\dagger)^T = EE^\dagger$,

(iv) $(E^\dagger E)^T = E^\dagger E$.

Lemma C.2 (see [28]). *For any matrix E, its Moore-Penrose generalized inverse matrix $E^\dagger$ is existent and unique.*

D.

Proof. If $u \in \mathscr{U}_+$, then by proof of Proposition 6, the trajectory x of system (2) corresponding to control u is a fixed point of F and $\zeta(x) = Du$. Moreover, we can write

$$u = D^\dagger \zeta(x) + \Psi, \tag{D.1}$$

with $\Psi = u - D^{\dagger}\zeta(x)$ and, by Definition C.1, we have

$$D\Psi = Du - DD^{\dagger}\zeta(x) = Du - DD^{\dagger}Du = 0, \tag{D.2}$$

i.e., $\Psi \in \ker D$.

Conversely, let $u = D^{\dagger}\zeta(x) + \Psi \in \mathbb{R}_+^{Nm}$ with $x \in S_F$ and $\Psi \in \ker D$; then

$$Du = DD^{\dagger}\zeta(x). \tag{D.3}$$

Since x is a fixed point of F, then by proof of Proposition 6, we have $x_N = x_f$ and $\zeta(x) \in \operatorname{Im}_+ D$. Consequently there exists an input $v \in \mathbb{R}_+^{Nm}$ such that

$$\zeta(x) = Dv. \tag{D.4}$$

Thus

$$Du = DD^{\dagger}\zeta(x) = DD^{\dagger}Dv = Dv = \zeta(x). \tag{D.5}$$

Hence $x = \widetilde{A}x_0 + G(x) + Du$; thus $u \in \mathscr{U}_+$. This finishes the proof. □

E.

Proof. Let $x \in \mathbb{R}_+^{nN}$ and $y \in \mathbb{R}_+^{n}$. If $\binom{x}{y} \in S_{\mathscr{F}}$, then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathscr{A}x_0 + \mathscr{G}(x) + \mathscr{P}\xi\begin{pmatrix} x \\ x_f \end{pmatrix} + \begin{pmatrix} 0 \\ y - x_f \end{pmatrix}, \tag{E.1}$$

and hence

$$\mathscr{P}\xi\begin{pmatrix} x \\ x_f \end{pmatrix} = \begin{pmatrix} x \\ x_f \end{pmatrix} - \mathscr{A}x_0 - \mathscr{G}(x) = \xi\begin{pmatrix} x \\ x_f \end{pmatrix}, \tag{E.2}$$

which implies that $\xi\binom{x}{x_f} \in \operatorname{Im}_+\mathscr{D}$. □

F.

Proof. (Sufficiency) Let $x_0, x_f \in \mathbb{R}_+^{n}$. If $\binom{x}{y} \in S_{\mathscr{F}}$, then by Lemma 11, we have $\xi\binom{x}{x_f} \in \operatorname{Im}_+\mathscr{D}$, so there exists an input $u \in \mathbb{R}_+^{Nm}$ such that

$$\xi\begin{pmatrix} x \\ x_f \end{pmatrix} = \mathscr{D}u. \tag{F.1}$$

On the other hand, by (34) we have

$$\begin{aligned} \begin{pmatrix} x \\ x_f \end{pmatrix} &= \mathscr{A}x_0 + \mathscr{G}(x) + \xi\begin{pmatrix} x \\ x_f \end{pmatrix} \\ &= \mathscr{A}x_0 + \mathscr{G}(x) + \mathscr{D}u, \end{aligned} \tag{F.2}$$

which implies that

$$\begin{aligned} x &= \widetilde{A}x_0 + G(x) + Du, \\ x_f &= x_N(u). \end{aligned} \tag{F.3}$$

(Necessity) Let $x_0, x_f \in \mathbb{R}_+^{n}$. Since the system (2) is controllable in N steps, there exists an input $u \in \mathbb{R}_+^{Nm}$ such that $x_N \coloneqq x_N(u) = x_f$.

Hence

$$x = \widetilde{A}x_0 + G(x) + Du \tag{F.4}$$

$$x_N = x_f, \tag{F.5}$$

where x is the solution of system (2) corresponding to the control u.

Then

$$\begin{pmatrix} x \\ x_f \end{pmatrix} = \mathscr{A}x_0 + \mathscr{G}(x) + \mathscr{D}u, \tag{F.6}$$

which implies that $\xi\binom{x}{x_f} = \mathscr{D}u \in \operatorname{Im}_+\mathscr{D}$, and consequently

$$\mathscr{P}\xi\begin{pmatrix} x \\ x_f \end{pmatrix} = \xi\begin{pmatrix} x \\ x_f \end{pmatrix}. \tag{F.7}$$

Therefore, we have

$$\mathscr{F}\begin{pmatrix} x \\ x_f \end{pmatrix} = \mathscr{A}x_0 + \mathscr{G}(x) + \xi\begin{pmatrix} x \\ x_f \end{pmatrix} = \begin{pmatrix} x \\ x_f \end{pmatrix}, \tag{F.8}$$

and hence $\binom{x}{x_f} \in S_{\mathscr{F}}$. This completes the proof. □

G.

Proof. If $u \in \mathscr{U}_+$, then by proof of Proposition 12, we have $\binom{x}{x_f} \in S_{\mathscr{F}}$ and $\xi\binom{x}{x_f} = \mathscr{D}u$, with x the solution of system (2) corresponding to the control u. Moreover, we can write

$$u = \mathscr{D}^{\dagger}\xi\begin{pmatrix} x \\ x_f \end{pmatrix} + \Psi, \tag{G.1}$$

with $\Psi = u - \mathscr{D}^{\dagger}\xi\binom{x}{x_f}$ and we have

$$\mathscr{D}\Psi = \mathscr{D}u - \mathscr{D}\mathscr{D}^{\dagger}\xi\begin{pmatrix} x \\ x_f \end{pmatrix} = \mathscr{D}u - \mathscr{D}\mathscr{D}^{\dagger}\mathscr{D}u = 0, \tag{G.2}$$

i.e., $\Psi \in \ker\mathscr{D}$.

Conversely, let $u = \mathscr{D}^{\dagger}\xi\binom{x}{x_f} + \Psi \in \mathbb{R}_+^{Nm}$ with $x \in \widetilde{\mathscr{S}}_{\mathscr{F}}$ and $\Psi \in \ker\mathscr{D}$; then

$$\mathscr{D}u = \mathscr{D}\mathscr{D}^{\dagger}\xi\begin{pmatrix} x \\ x_f \end{pmatrix}, \tag{G.3}$$

and by the proof of Proposition 12, we have $x_N = x_f$ and $\xi\binom{x}{x_f} \in \operatorname{Im}_+\mathscr{D}$. Consequently there exists an input $v \in \mathbb{R}_+^{Nm}$ such that

$$\xi\begin{pmatrix} x \\ x_f \end{pmatrix} = \mathscr{D}v, \tag{G.4}$$

then

$$\mathscr{D}u = \mathscr{D}\mathscr{D}^{\dagger}\xi\begin{pmatrix} x \\ x_f \end{pmatrix} = \mathscr{D}\mathscr{D}^{\dagger}\mathscr{D}v = \mathscr{D}v = \xi\begin{pmatrix} x \\ x_f \end{pmatrix}. \tag{G.5}$$

Hence $x = \widetilde{A}x_0 + G(x) + Du$; thus $u \in \mathscr{U}_+$. The proposition is proved. □

H.

Proof. System (2)-(3) is observable in N steps if and only if, for all $y_g \in \mathbb{R}_+^{rN}$, there exists at most one $x_0 \in \mathbb{R}_+^n$ such that

$$\begin{aligned} x &= \widetilde{A}x_0 + G(x), \\ y_g &= Sx, \end{aligned} \tag{H.1}$$

where x is the trajectory of system (2) corresponding to the initial state x_0. Consequently, system (2)-(3) is observable if and only if the set $\mathcal{O}_+$, and hence S_H, contains at most one element. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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