

Research Article

A New Default Probability Calculation Formula and Its Application under Uncertain Environments

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In the real world, corporate defaults will be affected by both external market shocks and counterparty risks. With this in mind, we propose a new default intensity model with counterparty risks based on both external shocks and the internal contagion effect. The effects of the external shocks and internal contagion on a company cannot, however, be observed, as uncertainty in the real world contains both randomness and fuzziness. This prevents us from determining the size of the shocks accurately. In this study, fuzzy set theory is utilized to study a looping default credit default swap (CDS) pricing model under uncertain environments. Following this, we develop a new fuzzy form pricing formula for CDS, the simulation analysis of which shows that all kinds of fuzziness in the market have a significant impact on credit spreads, and that the credit spreads, relative to the degree of external shock fuzziness, are much more sensitive. Nevertheless, for a certain degree of fuzziness in the market, credit spreads, relative to changes in counterparty risk, are much more sensitive. Using random analysis and fuzzy numbers, one can think of even more uncertain sources at play than the processes of looping default and investor subjective judgment on the financial markets, and this broadens the scope of possible credit spreads. Compared to the existing related literature, our new fuzzy form CDS pricing model with counterparty risk can consider more factors that influence default and is closer to the reality of the complexity of the dynamics of default. It can also employ the membership function to describe the fuzzy phenomenon, enable the fuzzy phenomenon to be estimated in two kinds of state, and can simultaneously reflect both the fuzziness and randomness in financial markets.

1. Introduction

Information disclosure plays a role as a bridge of information asymmetry between issuers and investors, for disclosure can reduce information asymmetry. Credit derivatives, as nonstandardized financial derivatives, have the following characteristics: the information of trading environment not open to the public, no guarantee of the performance from stock exchange, high risk of performance, and volatile price floating. As a result, the asymmetry of market information flow that acts as a decisive factor of the sensitive credit risk pricing of financial instruments appears. Moreover, it will lead to obvious fuzziness in the process of characterization of the counterparty risk. For example, when some emergencies are caused by financial institution as stock and stock markets, the adverse information dissemination will lead to counterparty default behavior. Meanwhile, the internal market credit

default will be convergent and infectious. The counterparty default is featured by synchronicity, aggregation, fuzziness, and uncertainty. In the credit default swap pricing of counterparty risk, the external market shocks and internal contagion effects on the company's size cannot be observed; i.e., the precise value cannot be gotten through a set of random values. Similarly, in the process of risk analysis and other derivatives pricing, people will encounter many similar fuzziness due to the lack of cognition on financial markets, which leads to incapability of seizing market information. For instance, in factor copulas model, correlation cannot be observed between the company's assets. That is to say, people will face the ubiquity of fuzziness in the process of financial risk analysis and its derivatives pricing. As a consequence, the study of fuzziness, hesitation, presentation, and evolution under the condition of asymmetric information is extremely important to control the outbreak, diffusion, and aggregation of the

credit risk. In addition, the pricing model by combining fuzziness and randomness under the condition of asymmetric information can strip, transfer, and hedge credit risk more efficiently. Thus, the credit risk analysis in the fuzzy uncertain environment and derivatives pricing model in this paper is of great realistic and theoretical significance.

As pointed out in [1–5], in the real world, corporate defaults will be affected by both external market shocks and counterparty risks. With this in mind, we proposed a new default intensity model with counterparty risks based on both external shocks and the internal contagion effect. However, the effects of external shocks and internal contagion on a company cannot be observed, because uncertainty in the real world contains randomness and fuzziness [6–13]. This prevents us from determining the size of shocks accurately. Therefore, inspired by [14–18], fuzzy set theory is adopted to study a looping default credit default swap (CDS) pricing model under uncertain environments. Following this, we set up a new fuzzy form pricing formula for CDS, the simulation analysis of which shows that all kinds of fuzziness in the market have a significant impact on credit spreads. Using random analysis and fuzzy numbers, one can think of more uncertain available sources than the processes of looping default and investor subjective judgment on the financial markets, which broadens the scope of possible credit spreads. Compared to the existing related references [19–23], our new fuzzy form CDS pricing model with counterparty risk can consider more factors that influence default and is closer to the reality of complex default dynamics. It can also employ the membership function to describe the fuzzy phenomenon, enable the fuzzy phenomenon to be estimated in two kinds of state, and simultaneously reflect the fuzziness and randomness in financial markets.

2. Introducing the Fuzziness to the CDS Pricing Model with Counterparty Risk

In this section, we study the CDS pricing model under fuzzy random environments based on the new looping default model, and the basic concept of fuzzy set theory can refer to literature [24–27].

2.1. The New Looping Default Intensity Model with Counterparty Risk. Firstly, we introduce some basic concepts. Let $(\Omega, \mathcal{F}, (F_t)_{0 \leq t \leq T^*}, Q)$ be a complete probability space. Here, $(F_t)_{0 \leq t \leq T^*}$ is a càdlàg, $F = F_{T^*}$, T^* is the time limit for uncertain economic system, and Q is the equivalent martingale measure. Suppose that there are $i = 1, 2, \dots, n$ companies in the market, and stochastic default time of each company is expressed as τ^i ($i = 1, 2, \dots, n$); the external shock arrival time is expressed as τ^E . According to [1], we define the default time as $\tau^i := \inf\{t \geq 0 : \int_0^t \lambda_s^i ds \geq E_i\}$, where E_i is a unit exponential random variable. Meanwhile, the external shock arrival time is assumed to obey uniform distribution in the time interval $[0, T]$. Let $F_t = \sigma(1_{\{\tau^E \leq s\}}, 0 \leq s \leq t) \vee \sigma(1_{\{\tau^1 \leq s\}}, 0 \leq s \leq t) \vee \dots \vee \sigma(1_{\{\tau^n \leq s\}}, 0 \leq s \leq t)$ represent

the information sets. So, the conditional survival probability of company i can be expressed as $F^i(t) = P(\tau^i \geq t \mid F_t) = \exp(-\int_0^t \lambda_s^i ds)$.

For considering more factors that influence default and that are closer to the reality of the complexity of the dynamics of default, we propose a new model as follows:

$$\lambda_t^i = a_0^i + a_1^i (a_1^i - 1) 1_{\{\tau^E \leq t\}} + \sum_{k=1, k \neq i}^n a^{i,k} 1_{\{\tau^k \leq t\}} \quad (1)$$

where $a_0^i > 0$, $a_1^i > 1$, and $a^{i,k}$ represent the initial default intensity, the upward jump ratio, and the counterparty risk, respectively, and $1_{\{\tau^i \leq t\}}$ is an indicator function of company i ; i.e., if the company i defaults, then the function value is 1; otherwise it is 0.

To simplify the discussion, we discuss the following:

$$\lambda_t^B = b_0 + b_0 (b_1 - 1) 1_{\{\tau^E \leq t\}} + b_2 1_{\{\tau^c \leq t\}} \quad (2)$$

$$\lambda_t^c = c_0 + c_0 (c_1 - 1) 1_{\{\tau^E \leq t\}} + c_2 1_{\{\tau^B \leq t\}} \quad (3)$$

Without loss of generality, we assume that before moment t , shock events happen, so (2) and (3) can be written as

$$\lambda_t^B = b_0 b_1 + b_2 1_{\{\tau^c \leq t\}} \quad (4)$$

$$\lambda_t^c = c_0 c_1 + c_2 1_{\{\tau^B \leq t\}} \quad (5)$$

In order to begin the operation in the new measure P^i , we have the following lemmas.

Lemma 1 ([28]). *Suppose the parameter λ is the intensity process of default time relative to the probability space (F, P) ; if there exist a constant $p \in [1, \infty)$ and a nonrandom time θ , which meets the integrability condition, $E[\exp(p \int_0^\theta \lambda_\mu d\mu)] < \infty$; at the same time, we set a nonrandom time $T \leq \theta$; then the nonnegative process*

$$Z_t := 1_{\{\tau > t \wedge T\}} \exp\left(\int_0^{t \wedge T} \lambda_\mu d\mu\right) \quad (6)$$

is a uniformly integrable martingale relative to the probability space (F, P) , and the process is almost always positive in the time interval $[0, \tau)$.

Lemma 2 ([29]). *Suppose there exists a probability space $(\Omega, \mathcal{G}, \{G_t\}_{t \geq 0}, P)$, and Q is a new absolutely continuous probability measure relative to the measure P , $\{Z_t\}_{t \geq 0}$ is the derivation process of the corresponding Radon-Nikodym, and if X is G_t -measurable, then we have the following conclusion: $E^Q[X] = E^P[X \cdot Z_t]$.*

Theorem 3. *The joint survival probability of different companies and the marginal survival probability of a single company for formulas (4) and (5), are, respectively, as follows: if $0 \leq t_2 \leq t_1 \leq T^*$, we have the following conclusion:*

$$P(\tau^B > t_1, \tau^C > t_2) = \begin{cases} \frac{c_0 c_1}{(b_2 - c_0 c_1)} e^{-(b_0 b_1 + b_2) t_1} [e^{(b_2 - c_0 c_1) t_1} - e^{(b_2 - c_0 c_1) t_2}] + e^{-(c_0 c_1 + b_0 b_1) t_1}, & b_2 \neq c_0 c_1 \\ c_0 c_1 e^{-(b_0 b_1 + b_2) t_1} [t_1 - t_2] + e^{-(c_0 c_1 + b_0 b_1) t_1}, & b_2 = c_0 c_1 \end{cases} \quad (7)$$

and if $0 \leq t_1 \leq t_2 \leq T^*$, we have

$$P(\tau^B > t_1, \tau^C > t_2) = \begin{cases} \frac{b_0 b_1}{(c_2 - b_0 b_1)} e^{-(c_0 c_1 + c_2) t_2} [e^{(c_2 - b_0 b_1) t_2} - e^{(c_2 - b_0 b_1) t_1}] + e^{-(c_0 c_1 + b_0 b_1) t_2}, & c_2 \neq b_0 b_1 \\ b_0 b_1 e^{-(c_0 c_1 + c_2) t_2} [t_2 - t_1] + e^{-(c_0 c_1 + b_0 b_1) t_2}, & c_2 = b_0 b_1 \end{cases} \quad (8)$$

The single company's marginal survival probability is as follows: if $0 \leq t_2 \leq t_1 \leq T^*$, we have

$$P(\tau^B > t) = \begin{cases} \frac{c_0 c_1}{(b_2 - c_0 c_1)} e^{-(b_0 b_1 + b_2) t} [e^{(b_2 - c_0 c_1) t} - 1] + e^{-(c_0 c_1 + b_0 b_1) t}, & b_2 \neq c_0 c_1 \\ c_0 c_1 e^{-(b_0 b_1 + b_2) t} + e^{-(c_0 c_1 + b_0 b_1) t}, & b_2 = c_0 c_1 \end{cases} \quad (9)$$

and if $0 \leq t_1 \leq t_2 \leq T^*$, we have

$$P(\tau^C > t) = \begin{cases} \frac{b_0 b_1}{(c_2 - b_0 b_1)} e^{-(c_0 c_1 + c_2) t} [e^{(c_2 - b_0 b_1) t} - 1] + e^{-(c_0 c_1 + b_0 b_1) t}, & c_2 \neq b_0 b_1 \\ b_0 b_1 e^{-(c_0 c_1 + c_2) t} + e^{-(c_0 c_1 + b_0 b_1) t}, & c_2 = b_0 b_1 \end{cases} \quad (10)$$

Proof. By Lemmas 1 and 2, we have the following: if $0 \leq t_2 \leq t_1 \leq T^*$, where $(\tau^B > t_1, \tau^C > t_2) \in F_{t_1}$ and $b_2 \neq c_0 c_1$, then we can derive the following:

$$\begin{aligned} P(\tau^B > t_1, \tau^C > t_2) &= E^B \left[1_{\{\tau^C > t_2\}} \exp \left\{ - \int_0^{t_1} (b_0 b_1 + b_2 1_{\{\tau^C \leq t\}}) dt \right\} \right] \\ &= E^B \left[1_{\{\tau^C > t_2\}} e^{-b_0 b_1 t_1} \exp \left\{ -1_{\{\tau^C \leq t_1\}} b_2 (t_1 - \tau^C) \right\} \right] \\ &= e^{-b_0 b_1 t_1} E^B \left[1_{\{t_2 < \tau^C \leq t_1\}} e^{-b_2 (t_1 - \tau^C)} + 1_{\{\tau^C > t_1\}} \right] \quad (11) \\ &= e^{-b_0 b_1 t_1} \left[\int_{t_2}^{t_1} c_0 c_1 e^{-c_0 c_1 t} e^{-b_2 (t_1 - t)} dt + e^{-c_0 c_1 t_1} \right] \\ &= \frac{c_0 c_1}{(b_2 - c_0 c_1)} e^{-(b_0 b_1 + b_2) t_1} [e^{(b_2 - c_0 c_1) t_1} - e^{(b_2 - c_0 c_1) t_2}] \\ &\quad + e^{-(c_0 c_1 + b_0 b_1) t_1}, \end{aligned}$$

if $b_2 = c_0 c_1$, we can derive the following:

$$\begin{aligned} P(\tau^B > t_1, \tau^C > t_2) &= E^B \left[1_{\{\tau^C > t_2\}} \exp \left\{ - \int_0^{t_1} (b_0 b_1 + b_2 1_{\{\tau^C \leq t\}}) dt \right\} \right] \\ &= E^B \left[1_{\{\tau^C > t_2\}} e^{-b_0 b_1 t_1} \exp \left\{ -1_{\{\tau^C \leq t_1\}} b_2 (t_1 - \tau^C) \right\} \right] \quad (12) \\ &= e^{-b_0 b_1 t_1} E^B \left[1_{\{t_2 < \tau^C \leq t_1\}} e^{-b_2 (t_1 - \tau^C)} + 1_{\{\tau^C > t_1\}} \right] \\ &= e^{-b_0 b_1 t_1} \left[c_0 c_1 e^{-b_2 t_1} \int_{t_2}^{t_1} e^{(b_2 - c_0 c_1) t} dt + e^{-c_0 c_1 t_1} \right] \\ &= c_0 c_1 e^{-(b_0 b_1 + b_2) t_1} [t_1 - t_2] + e^{-(c_0 c_1 + b_0 b_1) t_1}, \end{aligned}$$

if $0 \leq t_1 \leq t_2 \leq T^*$, where $(\tau^B > t_1, \tau^C > t_2) \in F_{t_2}$ and $c_2 \neq b_0 b_1$, then there are

$$\begin{aligned} P(\tau^B > t_1, \tau^C > t_2) &= E^C \left[1_{\{\tau^B > t_1\}} \exp \left\{ - \int_0^{t_2} (c_0 c_1 + c_2 1_{\{\tau^B \leq t\}}) dt \right\} \right] \\ &= E^C \left[1_{\{\tau^B > t_1\}} e^{-c_0 c_1 t_2} \exp \left\{ -1_{\{\tau^B \leq t_2\}} c_2 (t_2 - \tau^B) \right\} \right] \quad (13) \\ &= e^{-c_0 c_1 t_2} E^C \left[1_{\{t_1 < \tau^B \leq t_2\}} e^{-c_2 (t_2 - \tau^B)} + 1_{\{\tau^B > t_2\}} \right] \\ &= e^{-c_0 c_1 t_2} \left[\int_{t_1}^{t_2} b_0 b_1 e^{-b_0 b_1 t} e^{-c_2 (t_2 - t)} dt + e^{-b_0 b_1 t_2} \right] \\ &= \frac{b_0 b_1}{(c_2 - b_0 b_1)} e^{-(c_0 c_1 + c_2) t_2} [e^{(c_2 - b_0 b_1) t_2} - e^{(c_2 - b_0 b_1) t_1}] \\ &\quad + e^{-(c_0 c_1 + b_0 b_1) t_2}, \end{aligned}$$

and if $c_2 = b_0 b_1$, we can derive the following:

$$\begin{aligned}
& P(\tau^B > t_1, \tau^C > t_2) \\
&= E^C \left[1_{\{\tau^B > t_1\}} \exp \left\{ - \int_0^{t_2} (c_0 c_1 + c_2 1_{\{\tau^B \leq t\}}) dt \right\} \right] \\
&= E^C \left[1_{\{\tau^B > t_1\}} e^{-c_0 c_1 t_2} \exp \left\{ -1_{\{\tau^B \leq t_2\}} c_2 (t_2 - \tau^B) \right\} \right] \\
&= e^{-c_0 c_1 t_2} E^C \left[1_{\{t_1 < \tau^B \leq t_2\}} e^{-c_2 (t_2 - \tau^B)} + 1_{\{\tau^B > t_2\}} \right] \\
&= e^{-c_0 c_1 t_2} \left[b_0 b_1 e^{-c_2 t_2} \int_{t_1}^{t_2} e^{(c_2 - b_0 b_1)t} dt + e^{-b_0 b_1 t_2} \right] \\
&= b_0 b_1 e^{-(c_0 c_1 + c_2) t_2} [t_2 - t_1] + e^{-(c_0 c_1 + b_0 b_1) t_2}.
\end{aligned} \tag{14}$$

The proof of the marginal survival probability of a single company only needs to take the value $t_2 = 0, t_1 = 0$ in formulas (7) and (8), which completes the proof. \square

2.2. The CDS Pricing Model with Counterparty Risk under Uncertain Environments. Financial data may not be timely or accurately recorded, due to unforeseen circumstances or man-made errors. In order to reflect the influence of fuzziness on the interest rate, we assume that the interest rate is a triangular fuzzy random variable $\tilde{r}(t)$,

$$\mu_{\tilde{r}_t}(x) = \begin{cases} 1 - \frac{r-x}{\alpha^-(t)}, & r - \alpha^-(t) \leq x \leq r \\ 1 - \frac{x-r}{\alpha^+(t)}, & r \leq x \leq r + \alpha^+(t) \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

That is, the triangular fuzzy random variable $\tilde{r}(t)$ is represented by the center r , left-width $\alpha^-(t)$, and right-width $\alpha^+(t)$ (also known as fuzzy degree factor); see Figure 1. We can find that the fuzziness of investors on the market interest rate in the process increases as $\alpha^\pm(t)$ becomes larger. The γ -cut of $\tilde{r}(t)$ is

$$\begin{aligned}
\tilde{r}_\gamma(t) &= \{x \mid \mu_{\tilde{r}_t}(x) \geq \gamma\} \\
&= [r - (1 - \gamma)\alpha_t^-, r + (1 - \gamma)\alpha_t^+], \quad \gamma \in [0, 1].
\end{aligned} \tag{16}$$

Meanwhile, the influences of external market shocks b_1, c_1 and counterparty risks b_2, c_2 are assumed to be triangular fuzzy numbers, whose membership functions are defined as follows:

$$\begin{aligned}
b_1(x) &= \max \left\{ 1 - \frac{|x - b_1|}{\beta_1}, 0 \right\}, \\
c_1(x) &= \max \left\{ 1 - \frac{|x - c_1|}{\beta_2}, 0 \right\}, \\
b_2(x) &= \max \left\{ 1 - \frac{|x - b_2|}{\kappa_1}, 0 \right\}, \\
c_2(x) &= \max \left\{ 1 - \frac{|x - c_2|}{\kappa_2}, 0 \right\},
\end{aligned} \tag{17}$$

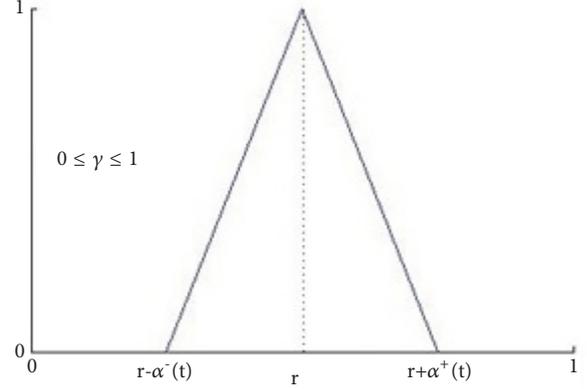


FIGURE 1: The triangular fuzzy risk-free interest rate.

that is, the fuzzy numbers \tilde{b}_1, \tilde{c}_1 and \tilde{b}_2, \tilde{c}_2 have a symmetric triangle-type shape, respectively, with centers b_1, c_1 and b_2, c_2 , and fuzzy degree factors β_1, β_2 and κ_1, κ_2 . The rationale behind fuzzy external market shocks and counterparty risk lies in the difficulty of getting a precise estimate of the actual external shock and counterparty risk of default intensity. By modeling the external shocks and counterparty risk as a fuzzy number, one can take the investors' subjective judgment into account.

According to [7], we introduce a reasonable assumption as follows to simplify the discussion.

Assumption 4. The stochastic process $\alpha^\pm(t)$ is specified by $\alpha^\pm(t) = c^\pm r$, where $0 < c^-, c^+ < 1$. Similarly, the values β_1, β_2 are specified by $\beta_1 = ab_1, \beta_2 = ac_1$, where $0 < a < 1$; the values κ_1, κ_2 are specified by $\kappa_1 = bb_2, \kappa_2 = bc_2$, where $0 < b < 1$.

Assumption 4 is reasonable since $\alpha^\pm(t)$ and β_1, β_2 and κ_1, κ_2 are related to the fuzziness of the volatility in the financial environment. Thus,

$$\begin{aligned}
\tilde{r}_{t,\gamma} &= [(1 - (1 - \gamma)c^-)r, (1 + (1 - \gamma)c^+)r], \\
\tilde{b}_{1,\gamma} &= [(1 - (1 - \gamma)a)b_1, (1 + (1 - \gamma)a)b_1], \\
\tilde{c}_{1,\gamma} &= [(1 - (1 - \gamma)a)c_1, (1 + (1 - \gamma)a)c_1], \\
\tilde{b}_{2,\gamma} &= [(1 - (1 - \gamma)b)b_2, (1 + (1 - \gamma)b)b_2], \\
\tilde{c}_{2,\gamma} &= [(1 - (1 - \gamma)b)c_2, (1 + (1 - \gamma)b)c_2],
\end{aligned} \tag{18}$$

$\gamma \in [0, 1].$

For CDS pricing, we provide some propositions. Hypothesize that the market short-term interest rate $\tilde{r}(t)$ satisfies the CIR (Cox-Ingersoll-Ross) model; that is,

$$d\tilde{r}_t = \alpha(k - \tilde{r}_t)dt + \sigma\sqrt{\tilde{r}_t}dW_t. \tag{19}$$

Therefore, the default-free zero coupon bond price, whose the face value is \$1 and expiration date is $T < T^*$, can be calculated by the following formula:

$$\tilde{p}(t, T) = E^Q \left[\exp \left(- \int_t^T \tilde{r}_s ds \right) \right]. \quad (20)$$

According to Theorem 4.3.1 of [30], the conclusions are $\tilde{p}(t, T) = e^{-\tilde{r}_t C(t, T) - A(t, T)}$,

$$C(t, T) = \frac{e^{\rho(T-t)} - e^{-\rho(T-t)}}{(\rho + (1/2)\alpha)e^{\rho(T-t)} + (\rho - (1/2)\alpha)e^{-\rho(T-t)}}, \quad (21)$$

$$A(t, T) = -\alpha k \int_t^T C(s, T) ds,$$

$$\rho = \frac{1}{2} \sqrt{\alpha^2 + 2\sigma^2}.$$

Theorem 5. Suppose that the market short-term interest rate $\tilde{r}(t)$ satisfies the CIR model, then, the γ -cut of the price of default-free zero coupon bond, whose face value is \$1 and expiration date is T , can be written as $\tilde{p}_\gamma(t, T) = [\tilde{p}_\gamma^-, \tilde{p}_\gamma^+]$, where the left and right ends, respectively, are

$$\begin{aligned} \tilde{p}_\gamma^-(t, T) &= e^{-(1+(1-\gamma)c^+)rC(t, T) - A(t, T)}, \\ \tilde{p}_\gamma^+(t, T) &= e^{-(1-(1-\gamma)c^-)rC(t, T) - A(t, T)}, \end{aligned} \quad (22)$$

$\gamma \in [0, 1].$

Proof. According to the monotonicity of the function and fuzzy number operation rule, we have the conclusion.

At moment t , the price of defaultable zero coupon bonds is equal to the discounted future cash flow:

$$\begin{aligned} \tilde{V}^i(t, T) &= E^Q \left[\exp \left(- \int_t^T \tilde{r}_\mu d\mu \right) (\beta^i 1_{\{\tau^i \leq T\}} + 1_{\{\tau^i > T\}}) \right] \quad (23) \\ &\quad (i = B, C), \end{aligned}$$

where β^i is the default recovery rate of i company; to simplify the analysis, the following discussion assumes that $\beta^i = 0$. \square

Now, we develop a CDS pricing model under uncertain environments based on the new looping default models (4) and (5).

The present fuzzy value of fixed leg is

$$\begin{aligned} PV(\text{fixed leg}) &= E^Q \left[\tilde{s} \int_0^T \exp \left(- \int_0^t \tilde{r}_\mu d\mu \right) 1_{\{\tau^B > t, \tau^C > t\}} dt \mid H_t \right]. \end{aligned} \quad (24)$$

The present fuzzy value of default leg is

$$\begin{aligned} PV(\text{default leg}) &= E^Q \left[1_{\{\tau^C \leq T\}} \exp \left(- \int_0^T \tilde{r}_\mu d\mu \right) 1_{\{\tau^B > T\}} \mid H_t \right]. \end{aligned} \quad (25)$$

Theorem 6. The present fuzzy value of a credit default swap \tilde{s} is given as

$$\tilde{s} = \frac{\tilde{V}^B(0, T) - e^{-b_0 \tilde{b}_1 T} \tilde{V}^C(0, T)}{\int_0^T \tilde{P}(0, \mu) e^{-(b_0 \tilde{b}_1 + c_0 \tilde{c}_1) \mu} d\mu} \quad (26)$$

The γ -cut of \tilde{s} is calculated as $\tilde{s}_\gamma = [\tilde{s}_\gamma^-, \tilde{s}_\gamma^+]$, where the left and right endpoints are

$$\tilde{s}_\gamma^- = \frac{\tilde{V}_\gamma^{B,-}(0, T) - e^{-b_0(1-(1-\gamma)a)b_1 T} \tilde{V}_\gamma^{C,+}(0, T)}{\int_0^T \tilde{P}_\gamma^+(0, \mu) e^{-(b_0(1-(1-\gamma)a)b_1 + c_0(1-(1-\gamma)a)c_1) \mu} d\mu}, \quad (27)$$

$$\tilde{s}_\gamma^+ = \frac{\tilde{V}_\gamma^{B,+}(0, T) - e^{-b_0(1+(1-\gamma)a)b_1 T} \tilde{V}_\gamma^{C,-}(0, T)}{\int_0^T \tilde{P}_\gamma^-(0, \mu) e^{-(b_0(1+(1-\gamma)a)b_1 + c_0(1+(1-\gamma)a)c_1) \mu} d\mu}, \quad (28)$$

and where

$$\begin{aligned} \tilde{V}_\gamma^{B,-}(0, T) &= \begin{cases} \tilde{p}_\gamma^-(0, T) \left\{ \frac{c_0(1-(1-\gamma)a)c_1}{((1+(1-\gamma)b)b_2 - c_0(1-(1-\gamma)a)c_1)} e^{-(b_0(1+(1-\gamma)a)b_1 + (1+(1-\gamma)b)b_2)T} \times \right. \\ \left. \left[e^{((1-(1-\gamma)b)b_2 - c_0(1+(1-\gamma)a)c_1)T} - 1 \right] + e^{-(c_0(1+(1-\gamma)a)c_1 + b_0(1+(1-\gamma)a)b_1)T} \right\}, & \tilde{b}_2 \neq c_0 \tilde{c}_1 \\ \tilde{p}_\gamma^-(0, T) \left\{ c_0(1-(1-\gamma)a)c_1 e^{-(b_0(1+(1-\gamma)a)b_1 + (1+(1-\gamma)b)b_2)T} T + e^{-(c_0(1+(1-\gamma)a)c_1 + b_0(1+(1-\gamma)a)b_1)T} \right\}, & \tilde{b}_2 = c_0 \tilde{c}_1 \end{cases} \\ \tilde{V}_\gamma^{B,+}(0, T) &= \begin{cases} \tilde{p}_\gamma^+(0, T) \left\{ \frac{c_0(1+(1-\gamma)a)c_1}{((1-(1-\gamma)b)b_2 - c_0(1+(1-\gamma)a)c_1)} e^{-(b_0(1-(1-\gamma)a)b_1 + (1-(1-\gamma)b)b_2)T} \times \right. \\ \left. \left[e^{((1+(1-\gamma)b)b_2 - c_0(1-(1-\gamma)a)c_1)T} - 1 \right] + e^{-(c_0(1-(1-\gamma)a)c_1 + b_0(1-(1-\gamma)a)b_1)T} \right\}, & \tilde{b}_2 \neq c_0 \tilde{c}_1 \\ \tilde{p}_\gamma^+(0, T) \left\{ c_0(1+(1-\gamma)a)c_1 e^{-(b_0(1-(1-\gamma)a)b_1 + (1-(1-\gamma)b)b_2)T} T + e^{-(c_0(1-(1-\gamma)a)c_1 + b_0(1-(1-\gamma)a)b_1)T} \right\}, & \tilde{b}_2 = c_0 \tilde{c}_1 \end{cases} \end{aligned}$$

$$\begin{aligned} \bar{V}_\gamma^{C,-}(0, T) &= \begin{cases} \bar{p}_\gamma^-(0, T) \left\{ \frac{b_0(1-(1-\gamma)a)b_1}{((1+(1-\gamma)b)c_2 - b_0(1-(1-\gamma)a)b_1)} e^{-(c_0(1+(1-\gamma)a)c_1 + (1+(1-\gamma)b)c_2)T} \times \right. \\ \left. \left[e^{((1-(1-\gamma)b)c_2 - b_0(1+(1-\gamma)a)b_1)T} - 1 \right] + e^{-(c_0(1+(1-\gamma)a)c_1 + b_0(1+(1-\gamma)a)b_1)T} \right\}, & \bar{c}_2 \neq b_0\bar{b}_1 \\ \bar{p}_\gamma^-(0, T) \left\{ b_0(1-(1-\gamma)a)b_1 e^{-(c_0(1+(1-\gamma)a)c_1 + (1+(1-\gamma)b)c_2)T} T + e^{-(c_0(1+(1-\gamma)a)c_1 + b_0(1+(1-\gamma)a)b_1)T} \right\}, & \bar{c}_2 = b_0\bar{b}_1 \end{cases} \\ \bar{V}_\gamma^{C,+}(0, T) &= \begin{cases} \bar{p}_\gamma^+(0, T) \left\{ \frac{b_0(1+(1-\gamma)a)b_1}{((1-(1-\gamma)b)c_2 - b_0(1+(1-\gamma)a)b_1)} e^{-(c_0(1-(1-\gamma)a)c_1 + (1-(1-\gamma)b)c_2)T} \times \right. \\ \left. \left[e^{((1+(1-\gamma)b)c_2 - b_0(1-(1-\gamma)a)b_1)T} - 1 \right] + e^{-(c_0(1-(1-\gamma)a)c_1 + b_0(1-(1-\gamma)a)b_1)T} \right\}, & \bar{c}_2 \neq b_0\bar{b}_1 \\ \bar{p}_\gamma^+(0, T) \left\{ b_0(1+(1-\gamma)a)b_1 e^{-(c_0(1-(1-\gamma)a)c_1 + (1-(1-\gamma)b)c_2)T} T + e^{-(c_0(1-(1-\gamma)a)c_1 + b_0(1-(1-\gamma)a)b_1)T} \right\}, & \bar{c}_2 = b_0\bar{b}_1 \end{cases} \end{aligned} \quad (29)$$

Proof. Based on the no arbitrage principle, as well as the conditional function $PV(\text{fixed leg}) = PV(\text{default leg})$, we have the following:

$$\begin{aligned} \bar{s} &= \frac{E^Q \left[1_{\{\tau^C \leq T\}} \exp \left(- \int_0^T \bar{r}_\mu d\mu \right) 1_{\{\tau^B > T\}} \mid H_t \right]}{E^Q \left[\int_0^T \exp \left(- \int_0^t \bar{r}_\mu d\mu \right) 1_{\{\tau^B > t, \tau^C > t\}} dt \mid H_t \right]} \\ &= \frac{E^Q \left[1_{\{\tau^B > T\}} \exp \left(- \int_0^T \bar{r}_\mu d\mu \right) \mid H_t \right] - E^Q \left[1_{\{\tau^C > T\}} \exp \left(- \int_0^T \bar{r}_\mu + \bar{\lambda}_\mu^B d\mu \right) \mid H_t \right]}{E^Q \left[\int_0^T \exp \left(- \int_0^t \bar{r}_\mu d\mu \right) 1_{\{\tau^B > t, \tau^C > t\}} dt \mid H_t \right]} \\ &= \frac{\bar{V}^B(0, T) - E^Q \left[1_{\{\tau^C > T\}} \exp \left(- \int_0^T \bar{r}_\mu + \bar{\lambda}_\mu^B d\mu \right) \mid H_t \right]}{\int_0^T \bar{p}(0, \mu) e^{-(b_0\bar{b}_1 + c_0\bar{c}_1)\mu} d\mu}, \end{aligned} \quad (30)$$

where

$$\begin{aligned} E^Q \left[1_{\{\tau^B > t, \tau^C > t\}} dt \mid H_t \right] &= P(\tau^B > t, \tau^C > t) \\ &= e^{-(b_0\bar{b}_1 + c_0\bar{c}_1)t}, \end{aligned}$$

$$\begin{aligned} E^Q \left[1_{\{\tau^C > T\}} \exp \left(- \int_0^T (\bar{r}_\mu + \bar{\lambda}_\mu^B) d\mu \right) \mid H_t \right] &= E^Q \left[1_{\{\tau^C > T\}} \right. \\ &\cdot \exp \left(- \int_0^T (\bar{r}_\mu + b_0\bar{b}_1 + \bar{b}_2 1_{\{\tau^C \leq \mu\}}) d\mu \right) \mid H_t \left. \right] \\ &= E^Q \left[1_{\{\tau^C > T\}} \right. \\ &\cdot \exp \left(-b_0\bar{b}_1 T - \int_0^T \bar{r}_\mu d\mu - \bar{b}_2 (T - \tau^C) 1_{\{\tau^C \leq T\}} \right) \mid \end{aligned}$$

$$\begin{aligned} H_t &= E^Q \left[1_{\{\tau^C > T\}} e^{-b_0\bar{b}_1 T - \int_0^T \bar{r}_\mu d\mu} \right] \\ &= e^{-b_0\bar{b}_1 T} E^Q \left[1_{\{\tau^C > T\}} e^{-\int_0^T \bar{r}_\mu d\mu} \right] = e^{-b_0\bar{b}_1 T} \bar{V}^C(0, T). \end{aligned} \quad (31)$$

Therefore, $\bar{s} = (\bar{V}^B(0, T) - e^{-b_0\bar{b}_1 T} \bar{V}^C(0, T)) / \int_0^T \bar{p}(0, \mu) e^{-(b_0\bar{b}_1 + c_0\bar{c}_1)\mu} d\mu$,

Meanwhile, based on the monotony of the functions, we can derive the γ -cut of \bar{s}_γ as $\bar{s}_\gamma = [\bar{s}_\gamma^-, \bar{s}_\gamma^+]$, where

$$\begin{aligned} \bar{s}_\gamma &= \left[\frac{\bar{V}^B(0, T) - e^{-b_0\bar{b}_1 T} \bar{V}^C(0, T)}{\int_0^T \bar{p}(0, \mu) e^{-(b_0\bar{b}_1 + c_0\bar{c}_1)\mu} d\mu} \right]_\gamma^- \\ &= \frac{[\bar{V}^B(0, T)]_\gamma^- - [e^{-b_0\bar{b}_1 T}]_\gamma^+ [\bar{V}^C(0, T)]_\gamma^+}{\int_0^T [\bar{p}(0, \mu)]_\gamma^+ [e^{-(b_0\bar{b}_1 + c_0\bar{c}_1)\mu}]_\gamma^+ d\mu} \\ &= \frac{\bar{V}_\gamma^{B,-}(0, T) - e^{-b_0(1-(1-\gamma)a)b_1 T} \bar{V}_\gamma^{C,+}(0, T)}{\int_0^T \bar{p}_\gamma^+(0, \mu) e^{-(b_0(1-(1-\gamma)a)b_1 + c_0(1-(1-\gamma)a)c_1)\mu} d\mu}, \end{aligned} \quad (32)$$

a similar conclusion \bar{s}_γ^+ can be proved. At the same time, we have

$$\begin{aligned} \bar{V}_\gamma^{B,-}(0, T) &= E^Q \left[\exp \left(- \int_t^T \bar{r}_\mu d\mu \right) 1_{\{\tau^B > T\}} \right]_\gamma^- = [\bar{P}(0, T)]_\gamma^- E^Q [1_{\{\tau^B > T\}}]_\gamma^- = \bar{P}_\gamma^-(0, T) [P(\tau^B > T)]_\gamma^- \\ &= \begin{cases} \bar{P}_\gamma^-(0, T) \left\{ \frac{c_0 c_1}{(b_2 - c_0 c_1)} e^{-(b_0 b_1 + b_2)t} [e^{(b_2 - c_0 c_1)t} - 1] + e^{-(c_0 c_1 + b_0 b_1)t} \right\}_\gamma^-, & b_2 \neq c_0 c_1 \\ \bar{P}_\gamma^-(0, T) \left\{ c_0 c_1 e^{-(b_0 b_1 + b_2)t} + e^{-(c_0 c_1 + b_0 b_1)t} \right\}_\gamma^-, & b_2 = c_0 c_1 \end{cases} \\ &= \begin{cases} \bar{P}_\gamma^-(0, T) \left\{ \frac{c_0 (1 - (1 - \gamma) a) c_1}{((1 + (1 - \gamma) b) b_2 - c_0 (1 - (1 - \gamma) a) c_1)} e^{-(b_0(1+(1-\gamma)a)b_1+(1+(1-\gamma)b)b_2)T} \times \right. \\ \left. [e^{((1-(1-\gamma)b)b_2-c_0(1+(1-\gamma)a)c_1)T} - 1] + e^{-(c_0(1+(1-\gamma)a)c_1+b_0(1+(1-\gamma)a)b_1)T} \right\}, & \bar{b}_2 \neq c_0 \bar{c}_1 \\ \bar{P}_\gamma^-(0, T) \left\{ c_0 (1 - (1 - \gamma) a) c_1 e^{-(b_0(1+(1-\gamma)a)b_1+(1+(1-\gamma)b)b_2)T} + e^{-(c_0(1+(1-\gamma)a)c_1+b_0(1+(1-\gamma)a)b_1)T} \right\}, & \bar{b}_2 = c_0 \bar{c}_1 \end{cases} \end{aligned} \tag{33}$$

where similar conclusions $\bar{V}_\gamma^{B,+}(0, T)$, $\bar{V}_\gamma^{C,-}(0, T)$, $\bar{V}_\gamma^{C,+}(0, T)$ can also be proved, so the proof is completed. \square

3. Simulation Analysis

In this part, we conduct a simulation on Theorem 6 and set the parameter as shown in Table 1.

Other related parameters are considered as follows: in the CIR interest rate model, we suppose that $\alpha = 0.03$, $k = 0.04$, and $\sigma = 0.07$; in the looping default models (4) and (5), we suppose that the initial default intensity parameters $b_0 = 0.6$, $c_0 = 0.7$ and hypothesize that the triangular fuzzy type risk-free interest rate, external shocks, and counterparty risk are

$$\begin{aligned} \bar{r}_{t,\gamma} &= [(1 - (1 - \gamma) 0.3) 0.07, (1 + (1 - \gamma) 0.2) 0.07]; \\ \bar{b}_{1,\gamma} &= [(1 - (1 - \gamma) 0.1) 1.04, (1 + (1 - \gamma) 0.1) 1.04]; \\ \bar{c}_{1,\gamma} &= [(1 - (1 - \gamma) 0.1) 1.07, (1 + (1 - \gamma) 0.1) 1.07]; \\ \bar{b}_{2,\gamma} &= [(1 - (1 - \gamma) 0.4) 0.06, (1 + (1 - \gamma) 0.4) 0.06]; \\ \bar{c}_{2,\gamma} &= [(1 - (1 - \gamma) 0.4) 0.08, (1 + (1 - \gamma) 0.4) 0.08], \end{aligned} \tag{34}$$

where $\gamma \in [0, 1]$, which represents the subjective reliability of investors. According to the parameters set by Table 1 and by means of simulation through the use of MATLAB R2010a, we obtain the simulation results in Figures 2–7.

In Figures 2–4, we see that the price range of simulation will become larger as the fuzzy degree of interest rates, external shocks, and counterparty risk increase, with the fuzzy credit spreads related to outside market shocks much more sensitive. This implies that all kinds of fuzziness in the market will have a significant impact on credit spreads. From Figure 5, we can see that, along with the continuous improvement in personal subjective judgment reliability, the fuzzy credit spreads interval is gradually narrowed, until

subjective reliability reaches 1; at that moment, the price range changes back to a real number; that is, there is no fuzziness in the market. This implies that people can improve their personal subjective judgment reliability to narrow the range of fuzzy credit spreads and select the optimal credit spread range. From Figures 6–7, we can see that external shocks and counterparty risk have a reverse effect on credit spreads (pricing environments with or without fuzziness have the same effect); increasing the external shocks and counterparty risk causes credit spreads to become smaller and smaller. In the fuzzy environment, credit spreads, compared to changes in counterparty risk, are much more sensitive. The above shows that both external shocks and counterparty risk have a significant effect on credit spreads. This is especially true when there is a certain degree of fuzziness in the market, when the spreads are caused by the counterparty risk.

Compared to the literature [14, 17, 22, 23], our new fuzzy form CDS pricing model with counterparty risk has more default influence factors, is much closer to the real complexity of default dynamics, employs the membership function to describe the fuzzy phenomenon, enables the fuzzy phenomenon to be estimated in two kinds of state, the possible degree and impossible degree, and can simultaneously reflect the fuzziness and the randomness in financial markets.

4. Conclusions

Using random analysis and fuzzy numbers, one can think of yet more uncertain sources leading to looping default, sources such as the subjective judgment of financial market investors leading to a broadening of possible credit spreads. The proposed model can thus be used as a new tool for CDS pricing. Of course, there are some deficiencies in this paper. Due to limited market data, we did not check market data parameters for the model, and this is what we will study in the future. Likewise, the stability of the model's results is also the focus of our further research.

TABLE 1: The parameter configuration.

parameter classification	parameter name	parameter configuration
time parameter	expiration date T	at present, the largest trading volume of CDS contract is 5 years, therefore, hypothesize that T=5.
discount parameter	interest rate r	r employs the period counting rates or yields, suppose that r=7%.
default parameter	recovery rate β^i	for the convenience of analysis, assume that $\beta^i = 0$.

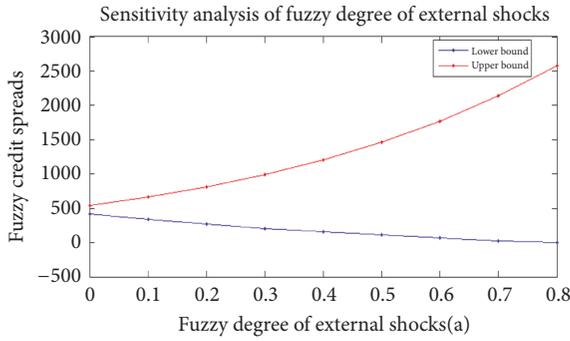


FIGURE 2: The dynamics between fuzzy credit spreads and fuzzy degree of external shocks.

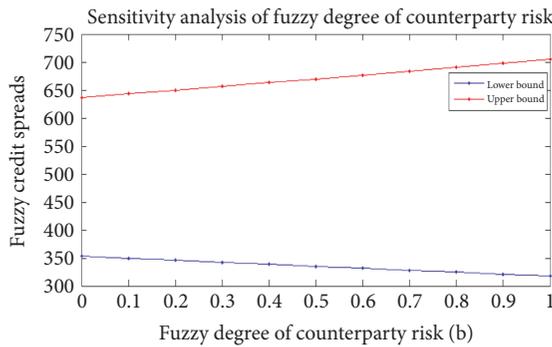


FIGURE 3: The dynamics between fuzzy credit spreads and fuzzy degree of counterparty risk.

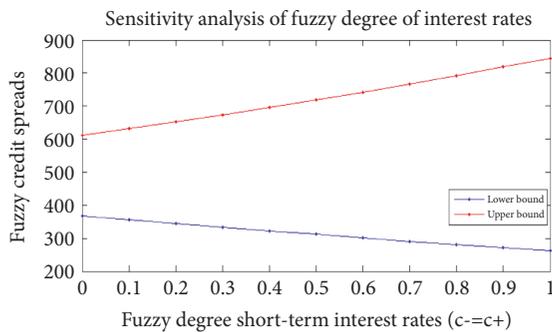


FIGURE 4: The dynamics between fuzzy credit spreads and fuzzy degree of interest rates.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

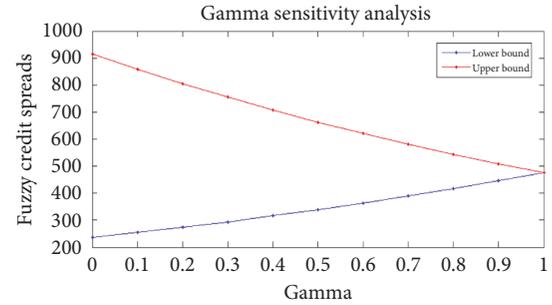


FIGURE 5: The dynamics between fuzzy credit spreads and subjective reliability.

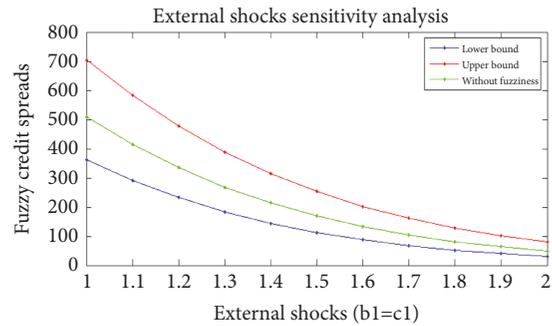


FIGURE 6: The dynamics between fuzzy credit spreads and external shocks.

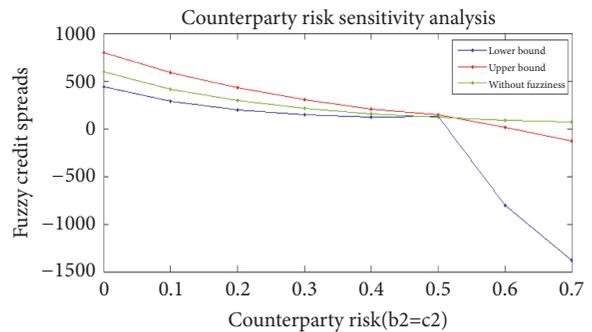


FIGURE 7: The dynamics between fuzzy credit spreads and counterparty risk.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper. The authors confirm that the received funding, grants, or scholarships do not lead

to any conflicts of interest regarding the publication of the manuscript.

Authors' Contributions

Liang Wu is the main writer of the framework and content of this article, Xian-bin Mei carried out numerical simulation, and Jian-guo Sun reviewed the full text.

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