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Research Article

A Family of Integrable Different-Difference Equations, Its Hamiltonian Structure, and Darboux-Bäcklund Transformation

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An integrable family of the different-difference equations is derived from a discrete matrix spectral problem by the discrete zero curvature representation. Hamiltonian structure of obtained integrable family is established. Liouville integrability for the obtained family of discrete Hamiltonian systems is proved. Based on the gauge transformation between the Lax pair, a Darboux-Bäcklund transformation of the first nonlinear different-difference equation in obtained family is deduced. Using this Darboux-Bäcklund transformation, an exact solution is presented.

1. Introduction

During the past decades, study of the integrable different-difference equations (or integrable lattice equations) has received considerable attention. They play the important roles in mathematical physics, lattice soliton theory, cellular automata, and so on. Many nonlinear integrable different-difference equations have been proposed and discussed, such as Ablowitz-Ladik lattice [1], Toda lattice [2], relativistic Toda lattices in polynomial form and rational form [3, 4], modified Toda lattice [5, 6], Volterra (or Langmuir) lattice [7], deformed reduced semi-discrete Kaup-Newell lattice [8], Merola-Ragnisco-Tu lattice [9], and so forth [10–18]. As we know, searching for novel integrable nonlinear different-difference equations is still an important and very difficult research topic. In the lattice soliton theory, the discrete zero curvature representation is a significant way to derive the integrable different-difference equations [18].

Starting from a suitable matrix spectral problem

$$E\varphi_n = U_n(y_n, \lambda)\varphi_n \tag{1}$$

and a series of auxiliary spectral problems

$$\varphi_{t_m} = V_n^{(m)}\varphi_n, \quad m \geq 1, \tag{2}$$

where for a lattice function $f_n = f(n,t)$, the shift operator E and the inverse of E are defined by

$$\begin{aligned} E(f_n) &= f_{n+1} = f(n+1,t), \\ E^{-1}(f_n) &= f_{n-1} = f(n-1,t), \\ & \qquad n \in Z, \end{aligned} \tag{3}$$

and $(\varphi_n^1, \varphi_n^2)$ is eigenfunction vector, and $y_n = (r_n, s_n)^T$ is a potential vector.

A family of evolution different-difference equations

$$y_{n_{t_m}} = F_m(y_n), \quad m \geq 1, \tag{4}$$

is called to be integrable in Lax sense (or lattice soliton equations), if it can be written as a compatibility condition of equations (1) and (2):

$$\begin{aligned} &U_n(y_n,\lambda)_{t_m} - \left(EV_n^{(m)}\right)U(y_n,\lambda) + U_n(y_n,\lambda)V_n^{(m)} \\ &\quad = 0. \end{aligned} \tag{5}$$

Moreover, in the lattice soliton theory, if we can seek out a Hamiltonian operator J and a series of conserved functionals

$\widetilde{H}_n^{(m)}(m \geq 1)$ so that (4) may be represented as the following forms

$$y_{n,t_m} = J\frac{\delta \widetilde{H}_n^{(m)}}{\delta y_n}, \tag{6}$$

where the Hamiltonian functionals $\widetilde{H}_n^{(m)} = \sum_{n\in Z} H_n^{(m)}, \delta\widetilde{H}_n/\delta y_n = \sum_{n\in Z} E^{-m}(\partial H_n^{(m)}/\partial y_{n+m}),\ m \geq 1$, then Hamiltonian structure of family (4) is established. In lattice soliton theory, (6) is called discrete Hamiltonian systems. Further, if we can deduce infinitely many involutive conserved functionals of family (6), then Liouville integrability of the integrable lattice family (4) is proved.

In addition, it is well known that Darboux-Bäcklund transformation is an important and very effective method for solving integrable different-difference equations. This transformation is a formula between the new solution and the old solution of the different-difference equation. From a known solution, through this transformation, we can obtain another solution. According to [19], Bäcklund transformation are usually divided into three types: the Wahlquist-Estabrook (WE) type [20, 21], the Darboux-Bäcklund type [5, 6, 22, 23], and the Hirota type [24].

This paper is organized as follows. In Section 2, we introduce a novel discrete spectral problem

$$\begin{gathered} E\varphi_n = U_n(y_n,\lambda)\varphi_n, \\ U_n(y_n,\lambda) = \begin{pmatrix} 0 & r_n\lambda \\ s_n\lambda & r_n s_n \lambda^2 \end{pmatrix}, \end{gathered} \tag{7}$$

where λ is the spectral parameter and $\lambda_t = 0$. It is easy to see that the matrix spectral problem (7) is equivalent to the following eigenvalue problem:

$$s_n E^{-1}(r_n\xi_n) + r_n s_n \xi_n = \frac{1}{\lambda^2} E(\xi_n), \tag{8}$$

the eigenfunction $\xi_n = \varphi_n^2$. Based on this discrete matrix spectral problem, we derive a family of integrable different-difference equations through the discrete zero curvature representation. Then, we establish the Hamiltonian structure of the obtained family by means of the discrete trace identity [18]. Afterwards, infinitely many common commuting conserved functionals of the obtained family are worked out. This guarantees Liouville integrability of the obtained family. In Section 3, we would like to derive a Bäcklund transformation of Darboux type (or Darboux-Bäcklund transformation) of the first nonlinear integrable different-difference equation in the obtained family; this transformation is constructed by means of the gauge transformation of Lax pair of the spectral problem, as application of Darboux-Bäcklund transformation, an exact solution is given. Some conclusions and remarks are given in the final section.

2. An Integrable Different-Difference Family and Its Hamiltonian Structure

We first solve the stationary discrete zero curvature equation

$$(E\Gamma_n)U_n - U_n\Gamma_n = \Gamma_{n+1}U_n - U_n\Gamma_n = 0, \tag{9}$$

with

$$\Gamma_n = \begin{pmatrix} a_n & b_n \\ c_n & -a_n \end{pmatrix}. \tag{10}$$

Equation (9) implies

$$\begin{aligned} r_n s_n (a_{n+1} - a_n)\lambda &= r_n c_{n+1} - s_n b_n, \\ r_n s_n b_{n+1}\lambda &= -r_n(a_n + a_{n+1}), \\ r_n s_n c_n \lambda &= -s_n(a_n + a_{n+1}), \\ s_n b_{n+1} - r_n c_n &= 0. \end{aligned} \tag{11}$$

Substituting Laurent series expansions

$$\begin{aligned} a_n &= \sum_{m=0}^{\infty} a_n^{(m)}\lambda^{-2m}, \\ b_n &= \sum_{m=0}^{\infty} b_n^{(m)}\lambda^{-2m+1}, \\ c_n &= \sum_{m=0}^{\infty} c_n^{(m)}\lambda^{-2m+1} \end{aligned} \tag{12}$$

into (11), we obtain the initial conditions:

$$\begin{aligned} r_n s_n \left(a_{n+1}^{(0)} - a_n^{(0)}\right) &= r_n c_{n+1}^{(0)} - s_n b_n^{(0)}, \\ b_{n+1}^{(0)} &= 0, \\ c_n^{(0)} &= 0 \end{aligned} \tag{13}$$

and the recursion relations:

$$\begin{aligned} r_n s_n \left(a_{n+1}^{(m+1)} - a_n^{(m+1)}\right) &= r_n c_{n+1}^{(m+1)} - s_n b_n^{(m)}, \\ r_n s_n b_{n+1}^{(m+1)} &= -r_n\left(a_n^{(m)} + a_{n+1}^{(m)}\right), \\ r_n s_n c_n^{(m+1)} &= -s_n\left(a_n^{(m)} + a_{n+1}^{(m)}\right). \end{aligned} \tag{14}$$

We choose the initial values satisfying the above initial conditions,

$$\begin{aligned} a_n^{(0)} &= -\frac{1}{2}, \\ b_n^{(0)} &= 0, \end{aligned} \tag{15}$$

and require selecting zero constant for the inverse operation of the difference operator $(E-1)$ in computing $a_n^{(m)}(\ m \geq 1)$, then the recursion relation (14) uniquely determines $a_n^{(m)},\ b_n^{(m)},\ c_n^{(m)}(\ m \geq 1)$. In addition, we have the following assertion.

Proposition 1. *$\{a_n^{(m)}\}_{m\geq 1}$ may be deduced through an algebraic method rather than by solving the difference equation.*

Proof. From (9), we know that

$$(E-1)\operatorname{tr}\left(\Gamma_n^2\right) = (E-1)\left(a_n^2 + b_n c_n\right) = 0.$$

$$a_n^{(m+1)} = \sum_{j=1}^{m} a_n^{(j)} a_n^{(m+1-j)} + \sum_{j=1}^{m} b_n^{(j)} c_n^{(m+1-j)} + \gamma(t). \tag{16}$$

Here $\gamma(t)$ is an arbitrary function of time variable t only. Further, we choose $\gamma(t) = 0$. Then, we get a recursion relation for $a_n^{(m)}(m \geq 1)$. Thus, $a_n^{(m)}(m \geq 1)$ are all local, and they are just the rational function in two dependent variables r_n, s_n.

The proof is completed. □

The first few terms are given by

$$\begin{aligned}
a_n^{(1)} &= \frac{1}{r_n s_{n-1}}, \\
b_n^{(1)} &= \frac{1}{s_{n-1}}, \\
c_n^{(1)} &= \frac{1}{r_n}, \\
a_n^{(2)} &= -\frac{1}{r_n^2 s_{n-1}^2} - \frac{1}{r_{n+1} r_n s_n s_{n-1}} - \frac{1}{r_{n-1} r_n s_{n-1} s_{n-2}}, \\
b_n^{(2)} &= -\frac{1}{r_{n-1} s_{n-1} s_{n-2}} - \frac{1}{r_n s_{n-1}^2}, \\
c_n^{(2)} &= -\frac{1}{r_n^2 s_{n-1}} - \frac{1}{r_n r_{n+1} s_n}, \cdots
\end{aligned} \tag{17}$$

Let us denote

$$V_n^{(m)} = \begin{pmatrix} \sum_{i=0}^{m} a_n^{(i)} \lambda^{2m-2i} & \sum_{i=0}^{m} b_n^{(i)} \lambda^{2m-2i+1} \\ \sum_{i=0}^{m} c_n^{(i)} \lambda^{2m-2i+1} & -\sum_{i=0}^{m} a_n^{(i)} \lambda^{2m-2i} \end{pmatrix}, \quad m \geq 0. \tag{18}$$

On the basis of the recursion relations (14), we have

$$\left(E V_n^{(m)}\right) U_n - U_n V_n^{(m)} = \begin{pmatrix} 0 & -r_n s_n b_{n+1}^{(m+1)} \lambda \\ r_n s_n c_n^{(m+1)} \lambda & r_n s_n \left(a_{n+1}^{(m)} - a_n^{(m)}\right) \end{pmatrix}. \tag{19}$$

It is obvious that (19) is not compatible with $(U_n)_{t_m}$. Therefore, we choose the following modification term:

$$\Delta_n^{(m)} = \begin{pmatrix} 0 & 0 \\ 0 & a_n^{(m)} \end{pmatrix}. \tag{20}$$

Then we introduce auxiliary matrix spectral problem

$$\Omega_n^{(m)} = V_n^{(m)} + \Delta_n^{(m)}, \quad m \geq 1. \tag{21}$$

Let the time evolution of the eigenfunction of the spectral problem (7) obey the differential equation

$$\varphi_{n,t_m} = \Omega_n^{(m)} \varphi_n, \quad m \geq 0. \tag{22}$$

Then the compatibility conditions of (7) and (22) are

$$(U_n)_{t_m} = \left(E\Omega_n^{(m)}\right) U_n - U_n \Omega_n^{(m)}, \quad m \geq 0 \tag{23}$$

It implies the family of integrable (in Lax sense) different-difference equations.

$$\begin{aligned}
r_{n,t_m} &= -r_n s_n b_{n+1}^{(m+1)} - r_n a_n^{(m)}, \\
s_{n,t_m} &= r_n s_n c_n^{(m+1)} + s_n a_{n+1}^{(m)}.
\end{aligned} \qquad m \geq 0. \tag{24}$$

When $m = 0$, (24) becomes a trivial linear system

$$\begin{aligned}
r_{n,t_0} &= -\frac{r_n}{2}, \\
v_{n,t_0} &= \frac{s_n}{2}.
\end{aligned} \tag{25}$$

When $m = 1$ in (24), we obtain the first nontrivial integrable lattice equation

$$\begin{aligned}
r_{n,t_1} &= \frac{r_n}{r_{n+1} s_n}, \\
s_{n,t_1} &= -\frac{s_n}{r_n s_{n-1}}.
\end{aligned} \tag{26}$$

In Section 3, we are going to construct its Darboux-Bäcklund transformation.

Now let us introduce some concepts for further discussion. The Gateaux derivative, the variational derivative, and the inner product are defined, respectively, by

$$\begin{aligned}
J'(y_n)[v_n] &= \frac{\partial}{\partial \varepsilon} J(y_n + \varepsilon v_n)\big|_{\varepsilon=0}, \\
\frac{\delta H_n}{\delta y_n} &= \sum_{m \in Z} E^{(-m)} \left(\frac{\partial H_n}{\partial y_{n+m}}\right), \\
\langle f_n, g_n \rangle &= \sum_{n \in Z} (f_n, g_n)_{R^2}.
\end{aligned} \tag{27}$$

f_n, g_n are required to be rapidly vanished at the infinity, and $(f_n, g_n)_{R^2}$ denotes the standard inner product of f_n and g_n in the Euclidean space R^2. Operator J^* is defined by $\langle f_n, J^* g_n \rangle = \langle J f_n, g_n \rangle$; it is called the adjoint operator of J. If an operator J has the property $J^* = -J$, then J is called to be skew-symmetric. An operator J is called a Hamiltonian operator, if J is a skew-symmetric operator satisfying the Jacobi identity, i.e.,

$$\begin{aligned}
&\langle f_n, J g_n \rangle = -\langle J f_n, g_n \rangle, \\
&\left\langle J'(u_n)[J f_n] g_n, h_n \right\rangle + Cycle(f_n, g_n, h_n) = 0.
\end{aligned} \tag{28}$$

Based on a given Hamiltonian operator J, we can define the corresponding Poisson bracket [10, 11, 18]

$$\{f_n, g_n\}_J = \left\langle \frac{\delta f_n}{\delta y_n}, J\frac{\delta g_n}{\delta y_n} \right\rangle = \sum_{n\in Z} \left\langle \frac{\delta f_n}{\delta y_n}, J\frac{\delta g_n}{\delta y_n} \right\rangle_{R^2}. \quad (29)$$

Following [18], we set

$$R_n = \Gamma_n (U_n)^{-1} = \begin{pmatrix} -a_n + \dfrac{b_n}{r_n\lambda} & \dfrac{a_n}{s_n\lambda} \\ -c_n - \dfrac{a_n}{r_n\lambda} & \dfrac{c_n}{s_n\lambda} \end{pmatrix} \quad (30)$$

and $\langle M, N\rangle = \operatorname{tr}(MN)$, where M and N are some order square matrices. It is easy to calculate that

$$\begin{aligned} \frac{\partial U_n}{\partial \lambda} &= \begin{pmatrix} 0 & r_n \\ s_n & 2r_n s_n \lambda \end{pmatrix}, \\ \frac{\partial U_n}{\partial r_n} &= \begin{pmatrix} 0 & \lambda \\ 0 & s_n\lambda^2 \end{pmatrix}, \\ \frac{\partial U_n}{\partial s_n} &= \begin{pmatrix} 0 & 0 \\ \lambda & r_n\lambda^2 \end{pmatrix}. \end{aligned} \quad (31)$$

Hence

$$\begin{aligned} \left\langle R_n, \frac{\partial U_n}{\partial \lambda} \right\rangle &= r_n c_n, \\ \left\langle R_n, \frac{\partial U_n}{\partial r_n} \right\rangle &= -\frac{a_n}{r_n}, \\ \left\langle R_n, \frac{\partial U_n}{\partial s_n} \right\rangle &= -\frac{a_{n+1}}{s_n}. \end{aligned} \quad (32)$$

By virtue of the discrete trace identity [18]

$$\begin{aligned} \frac{\delta}{\delta r_n}\sum_{n\in Z}\left\langle R_n, \frac{\partial U_n}{\partial \lambda}\right\rangle &= \lambda^{-\varepsilon}\frac{\partial}{\partial\lambda}\lambda^{\varepsilon}\left\langle R_n, \frac{\partial U_n}{\partial r_n}\right\rangle, \\ \frac{\delta}{\delta s_n}\sum_{n\in Z}\left\langle R_n, \frac{\partial U_n}{\partial \lambda}\right\rangle &= \lambda^{-\varepsilon}\frac{\partial}{\partial\lambda}\lambda^{\varepsilon}\left\langle R_n, \frac{\partial U_n}{\partial s_n}\right\rangle. \end{aligned} \quad (33)$$

Substituting expansions $a_n = \sum_{m=0}^{\infty} a_n^{(m)}\lambda^{-2m}, b_n = \sum_{m=0}^{\infty} b_n^{(m)}\lambda^{-2m+1}, c_n = \sum_{m=0}^{\infty} c_n^{(m)}\lambda^{-2m+11}$ into (33) and comparing the coefficients of λ^{-2m-1} in (33), we obtain

$$\begin{pmatrix} \dfrac{\delta}{\delta r_n} \\ \dfrac{\delta}{\delta s_n} \end{pmatrix} \sum_{n\in Z}\left(r_n c_n^{(m+1)}\right) = (\varepsilon - 2m)\begin{pmatrix} -\dfrac{a_n^{(m)}}{r_n} \\ \dfrac{-a_{n+1}^{(m)}}{s_n} \end{pmatrix}. \quad (34)$$

When $m = 0$ in (34), through a direct calculation, we find that $\varepsilon = 0$. Therefore, (34) can be written as

$$\begin{pmatrix} \dfrac{\delta}{\delta r_n} \\ \dfrac{\delta}{\delta s_n} \end{pmatrix} \sum_{n\in Z}\left(-\frac{r_n c_n^{(m+1)}}{2m}\right) = \begin{pmatrix} -\dfrac{a_n^{(m)}}{r_n} \\ -\dfrac{a_{n+1}^{(m)}}{s_n} \end{pmatrix} \quad m \geq 1. \quad (35)$$

Set

$$\widetilde{H}_n^{(m)} = \sum_{n\in Z}\left(-\frac{r_n c_n^{(m+1)}}{2m}\right). \quad (36)$$

We have

$$\begin{pmatrix} \dfrac{\delta}{\delta r_n} \\ \dfrac{\delta}{\delta s_n} \end{pmatrix} \sum_{n\in Z}\left(\widetilde{H}_n^{(m)}\right) = \begin{pmatrix} -\dfrac{a_n^{(m)}}{r_n} \\ -\dfrac{a_{n+1}^{(m)}}{s_n} \end{pmatrix} \quad m \geq 1. \quad (37)$$

Set

$$\begin{pmatrix} -r_n s_n b_{n+1}^{(m+1)} - r_n a_n^{(m)} \\ r_n s_n c_n^{(m+1)} + s_n a_{n+1}^{(m)} \end{pmatrix} = J\begin{pmatrix} -\dfrac{a_n^{(m)}}{r_n} \\ -\dfrac{a_{n+1}^{(m)}}{s_n} \end{pmatrix} \quad m \geq 1. \quad (38)$$

We can obtain

$$J = \begin{pmatrix} 0 & -r_n s_n \\ r_n s_n & 0 \end{pmatrix}. \quad (39)$$

Evidently, the operator J is a skew-symmetric operator, i.e., $J = -J^*$. In addition, through a direct calculation, we can prove that the operator J satisfies the Jacobi identity (28). Thus, we obtain the following assertion.

Proposition 2. *J is a discrete Hamiltonian operator. Consequently, (22) have Hamiltonian structures*

$$y_{n_{t_m}} = \begin{pmatrix} r_n \\ s_n \end{pmatrix}_{t_m} = J\frac{\delta \widetilde{H}_n^{(m)}}{\delta y_n} = J\begin{pmatrix} -\dfrac{a_n^{(m)}}{r_n} \\ -\dfrac{a_{n+1}^{(m)}}{s_n} \end{pmatrix}, \quad m \geq 0. \quad (40)$$

$$\begin{aligned} \widetilde{H}_n^{(m)} &= \sum_{n\in Z}\left(H_n^{(m)}\right), \\ H_n^{(m)} &= \left(\frac{r_n c_n^{(m+1)}}{-2m}\right), \quad m \geq 1. \end{aligned} \quad (41)$$

In particular, the different-difference equation (23) possesses the Hamiltonian structure

$$\begin{aligned} y_{n,t} &= \begin{pmatrix} r_n \\ s_n \end{pmatrix}_t = J\frac{\delta \widetilde{H}_n^{(1)}}{\delta u_n}, \\ \widetilde{H}_n^{(1)} &= \sum_{n\in Z}\left(\frac{1}{2r_n s_{n-1}} + \frac{1}{2r_{n+1}s_n}\right). \end{aligned} \quad (42)$$

Following (14) and (35), we have the following recursion relation:

$$\frac{\delta \widetilde{H}_n^{(m+1)}}{\delta u_n} = \Psi\frac{\delta \widetilde{H}_n^{(m)}}{\delta u_n}, \quad m \geq 1, \quad (43)$$

where

$$\Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{12} \end{pmatrix}. \tag{44}$$

Moreover, we have

$$J\Psi = M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{45}$$

with

$$\begin{aligned}
M_{11} &= -r_n\left(1-E^{-1}\right)^{-1}\frac{1}{s_n}E\frac{1}{r_n}(1-E)^{-1}r_n \\
&\quad + r_n\left(1-E^{-1}\right)^{-1}\frac{1}{r_n}E^{-1}\frac{1}{s_n}(1-E)^{-1}r_n, \\
M_{12} &= E^{-1} - \frac{r_n}{s_n}E\frac{s_n}{r_n} - r_n(1-E)^{-1}\frac{1}{r_n}E^{-1} \\
&\quad - E^{-1}\frac{1}{s_n}(1-E)^{-1}s_n + \frac{r_n}{s_n}E\frac{1}{r_n}(1-E)^{-1}s_n \\
&\quad + r_n(1-E)^{-1}\frac{1}{s_n}E\frac{r_n}{s_n} \\
&\quad + r_n(1-E)^{-1}\frac{1}{r_n}E^{-1}\frac{1}{s_n}(1-E)^{-1} \\
&\quad - r_n(1-E)^{-1}\frac{1}{s_n}E\frac{1}{r_n}(1-E)^{-1}s_n, \\
M_{21} &= -E - \frac{s_n}{r_n}E^{-1}\frac{r_n}{s_n} - s_n\left(1-E^{-1}\right)^{-1}\frac{1}{r_n}E^{-1}\frac{r_n}{s_n} \\
&\quad - E\frac{1}{r_n}\left(1-E^{-1}\right)^{-1}r_n + s_n\left(1-E^{-1}\right)^{-1}\frac{1}{s_n}E \\
&\quad - \frac{s_n}{r_n}E^{-1}\frac{1}{s_n}\left(1-E^{-1}\right)^{-1}r_n \\
&\quad - s_n\left(1-E^{-1}\right)^{-1}\frac{1}{r_n}E^{-1}\frac{r_n}{s_n} \\
&\quad - s_n\left(1-E^{-1}\right)^{-1}\frac{1}{s_n}E^{-1}\frac{1}{r_n}\left(1-E^{-1}\right)^{-1}r_n \\
&\quad - \frac{s_n}{r_n}E^{-1}\frac{1}{s_n}\left(1-E^{-1}\right)^{-1}r_n, \\
M_{22} &= s_n(1-E)^{-1}\frac{1}{r_n}E^{-1}\frac{1}{s_n}\left(1-E^{-1}\right)^{-1}s_n \\
&\quad - s_n(1-E)^{-1}\frac{1}{s_n}E\frac{1}{r_n}(1-E)^{-1}r_n.
\end{aligned} \tag{46}$$

It is easy to verify that M is a skew-symmetric operator. Namely,

$$M^* = -M. \tag{47}$$

With the help of the operator Ψ, (40) may be written in the following form:

$$\begin{aligned}
y_{n,t_m} &= \begin{pmatrix} r_n \\ s_n \end{pmatrix}_{t_m} = J\frac{\delta\widetilde{H}_n^{(m)}}{\delta u_n} = M\frac{\delta\widetilde{H}_n^{(m-1)}}{\delta u_n} \\
&= J\Psi^m\frac{\delta\widetilde{H}_n^{(0)}}{\delta u_n}, \quad m \geq 1.
\end{aligned} \tag{48}$$

Furthermore, on the basis of [4, 16], we may obtain a recursion operator

$$\Phi = \Psi^* = \begin{pmatrix} \Psi_{11}^* & \Psi_{21}^* \\ \Psi_{12}^* & \Psi_{22}^* \end{pmatrix}. \tag{49}$$

Next, we prove Liouville integrability of the discrete Hamiltonian systems (40). It is crucial to show the existence of infinite involutive conserved functionals.

Proposition 3. *$\{\widetilde{H}_n^{(m)}\}_{m=0}^{\infty}$ are conserved functionals of the whole family (24) or (40). And they are in involution in pairs with respect to the Poisson bracket (29).*

Proof. Due to (47), we know that M is a skew-symmetric operator, and a direct calculation shows

$$(J\Psi)^* = M^* = -M = -J\Psi, \tag{50}$$

namely,

$$\Psi^* J = J\Psi \tag{51}$$

Therefore

$$\begin{aligned}
\left\{\widetilde{H}_n^{(m)}, \widetilde{H}_n^{(l)}\right\}_J &= \left\langle \frac{\delta\widetilde{H}_n^{(m)}}{\delta u_n}, J\frac{\delta\widetilde{H}_n^{(l)}}{\delta u_n} \right\rangle \\
&= \left\langle \Psi^{m-1}\frac{\delta\widetilde{H}_n^{(1)}}{\delta u_n}, J\Psi^{l-1}\frac{\delta\widetilde{H}_n^{(1)}}{\delta u_n} \right\rangle \\
&= \left\langle \Psi^{m-1}\frac{\delta\widetilde{H}_n^{(1)}}{\delta u_n}, \Psi^* J\Psi^{l-2}\frac{\delta\widetilde{H}_n^{(1)}}{\delta u_n} \right\rangle \\
&= \left\langle \Psi^{m}\frac{\delta\widetilde{H}_n^{(1)}}{\delta u_n}, J\Psi^{l-2}\frac{\delta\widetilde{H}_n^{(1)}}{\delta u_n} \right\rangle \\
&= \left\{\widetilde{H}_n^{(m+1)}, \widetilde{H}_n^{(l-1)}\right\}_J = \cdots \\
&= \left\{\widetilde{H}_n^{(m)}, \widetilde{H}_n^{(l)}\right\}_J = \left\{\widetilde{H}_n^{(m+l-1)}, \widetilde{H}_n^{(1)}\right\}_J.
\end{aligned} \tag{52}$$

Repeating the above argument, we can obtain

$$\left\{\widetilde{H}_n^{(l)}, \widetilde{H}_n^{(m)}\right\}_J = \left\{\widetilde{H}_n^{(m+l-1)}, \widetilde{H}_n^{(1)}\right\}_J. \tag{53}$$

Then combining (52) with (53) leads to

$$\left\{\widetilde{H}_n^{(m)}, \widetilde{H}_n^{(l)}\right\}_J = 0, \quad m, l \geq 1, \tag{54}$$

and

$$\left(\widetilde{H}_n^{(m)}\right)_{t_l} = \left\langle \frac{\delta \widetilde{H}_n^{(m)}}{\delta u_n}, u_{t_l} \right\rangle = \left\langle \frac{\delta \widetilde{H}_n^{(m)}}{\delta u_n}, J\frac{\delta \widetilde{H}_n^{(l)}}{\delta u_n} \right\rangle = \left\{\widetilde{H}_n^{(m)}, \widetilde{H}_n^{(l)}\right\}_J = 0, \quad m, l \geq 1. \tag{55}$$

The proof is completed. □

Remark 4. According to the above proposition, we can get that (24) is not only Lax integrable but also Liouville integrable. Based on (40) and Proposition 2, we can obtain the following theorem.

Theorem 5. *The integrable different-difference equations in family (24) are all Liouville integrable discrete Hamiltonian systems.*

3. Darboux-Bäcklund Transformation and Exact Solution

Next we are going to establish a Darboux-Bäcklund transformation of (26).

When $m = 1$ in (23), we obtain the time part of the Lax pair of (26)

$$\varphi_{n,t} = \Omega_n^{(1)}\varphi_n = \begin{pmatrix} -\frac{\lambda^2}{2} + \frac{1}{r_n s_{n-1}} & \frac{1}{s_{n-1}}\lambda \\ \frac{1}{r_n}\lambda & \frac{\lambda^2}{2} \end{pmatrix}\varphi_n. \tag{56}$$

It is crucial to look for a suitable gauge transformation of a matrix spectral problem; it can transform the matrix spectral problem into another spectral problem of the same form [5, 6, 8, 23]. We introduce the following gauge transformation:

$$\widetilde{\varphi_n} = T_n\varphi_n, \tag{57}$$

which can transform two spectral problems (7) and (56) into

$$E\widetilde{\varphi_n} = \widetilde{\varphi}_{n+1} = \widetilde{U}_n\widetilde{\varphi}_n, \tag{58}$$

$$\widetilde{\varphi}_{n_t} = \widetilde{\Omega}_n^{(1)}\widetilde{\varphi}_n, \tag{59}$$

with

$$\begin{aligned} \widetilde{U}_n &= T_{n+1}U_n\left(T_n\right)^{-1}, \\ \widetilde{\Omega}_n^{(1)} &= \left(\Omega_{n_t}^{(1)} + T_n\Omega_n^{(1)}\right)\left(T_n\right)^{-1}. \end{aligned} \tag{60}$$

We suppose that T_n has the following form:

$$\begin{pmatrix} \lambda + \frac{t_{11}[n]}{\lambda} & t_{12}[n] \\ t_{21}[n] & t_{22}[n]\lambda + \frac{1}{\lambda} \end{pmatrix}. \tag{61}$$

In (61), $t_{11}[n]$, $t_{12}[n]$, $t_{21}[n]$, and $t_{22}[n]$ are undetermined functions of variables n and t. Now we would like to construct T_n such that $\widetilde{U}_n$ and $\widetilde{\Omega}_n^{(1)}$ in (60) have the same form with U_n and $\Omega_n^{(1)}$, respectively.

Let $\phi_n = (\phi_n^1, \phi_n^2)^T$, $\psi_n = (\psi_n^1, \psi_n^2)^T$ be two real linear independent solutions of (7) and (56). Let

$$K_n = \begin{pmatrix} \phi_n^1 & \psi_n^1 \\ \phi_n^2 & \psi_n^2 \end{pmatrix}. \tag{62}$$

From equation $\widetilde{K}_n = T_nK_n$, we obtain that

$$\widetilde{K}_n = \begin{pmatrix} \left(\lambda + \frac{t_{11}[n]}{\lambda}\right)\phi_n^1 + t_{12}[n]\phi_n^2 & \left(\lambda + \frac{t_{11}[n]}{\lambda}\right)\psi_n^1 + t_{12}[n]\psi_n^2 \\ t_{21}[n]\phi_n^1 + \left(t_{22}[n]\lambda + \frac{1}{\lambda}\right)\phi_n^2 & t_{21}[n]\psi_n^1 + \left(t_{22}[n]\lambda + \frac{1}{\lambda}\right)\psi_n^2 \end{pmatrix}. \tag{63}$$

We assume that $\pm\lambda_1$ and $\pm\lambda_2$ are four roots of $\det(T_n)$. When $\lambda = \pm\lambda_i (i = 1, 2)$, two columns of (63) are linear dependent. Without loss of generality, there exist two nonzero constants κ_1 and κ_2, which satisfy

$$\begin{aligned} &\left(\lambda + \frac{t_{11}[n]}{\lambda}\right)\phi_n^1 + t_{12}[n]\phi_n^2 \\ &\quad = \kappa_i\left(\left(\lambda + \frac{t_{11}[n]}{\lambda}\right)\psi_n^1 + t_{12}[n]\psi_n^2\right), \\ &t_{21}[n]\phi_n^1 + \left(t_{22}[n]\lambda + \frac{1}{\lambda}\right)\phi_n^2 \\ &\quad = \kappa_i\left(t_{21}[n]\psi_n^1 + \left(t_{22}[n]\lambda + \frac{1}{\lambda}\right)\psi_n^2\right) \end{aligned} \tag{64}$$

where $\lambda = \pm\lambda_i (i = 1, 2)$.

Set

$$\alpha_i[n] = \frac{\phi_n^2 - \kappa_i\psi_n^2}{\phi_n^1 - \kappa_i\psi_n^1}, \quad i = 1, 2. \tag{65}$$

Solving (64), we have

$$\begin{aligned} t_{11}[n] &= \frac{\lambda_1\lambda_2\left(\lambda_1\alpha_2[n] - \lambda_2\alpha_1[n]\right)}{\lambda_1\alpha_1[n] - \lambda_2\alpha_2[n]}, \\ t_{12}[n] &= \frac{\lambda_2^2 - \lambda_1^2}{\lambda_1\alpha_1[n] - \lambda_2\alpha_2[n]}, \\ t_{21}[n] &= \frac{\left(\lambda_2^2 - \lambda_1^2\right)\alpha_1[n]\alpha_2[n]}{\lambda_1\lambda_2\left(\lambda_1\alpha_1[n] - \lambda_2\alpha_2[n]\right)}, \end{aligned}$$

$$t_{22}[n] = \frac{\lambda_1\alpha_2[n] - \lambda_2\alpha_1[n]}{\lambda_1\lambda_2(\lambda_1\sigma_1[n] - \lambda_2\sigma_2[n])}. \tag{66}$$

In (66), λ_i, $\kappa_i (i = 1, 2)$ are suitably chosen such that all the denominations in (65) and (66) are nonzero. Through a direct but tedious calculation, we can get that $\pm\lambda_i (i = 1, 2)$ are four roots of $\det(T_n) = 0$, and then

$$\det(T_n) = t_{22}[n] \frac{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)}{\lambda^2}. \tag{67}$$

From (65), we get

$$\alpha_i[n+1] = \frac{\xi_i(n)}{\chi_i(n)} \quad (i = 1, 2), \tag{68}$$

where $\xi_i(n) = \lambda_i s_n + (r_n s_n \lambda_i^2)\alpha_i[n]$, $\chi_i(n) = r_n\lambda_i\alpha_i[n] (i = 1, 2)$.

Based on (65) and (66), we can get the expressions of $t_{11}[n+1]$, $t_{12}[n+1]$, $t_{21}[n+1]$, $t_{22}[n+1]$.

$$\begin{aligned}
t_{11}[n+1] &= \frac{\lambda_1\lambda_2(\lambda_1\xi_2(n)\chi_1(n) - \lambda_2\xi_1(n)\chi_2(n))}{\lambda_1\xi(n)\chi_2(n) - \lambda_2\xi_2(n)\chi_1(n)},\\
t_{12}[n+1] &= \frac{(\lambda_2^2 - \lambda_1^2)\chi_1(n)\,chi_2(n)}{\lambda_1\xi_1(n)\chi_2(n) - \lambda_2\xi_2(n)\chi_1(n)},\\
t_{21}[n+1] &= \frac{(\lambda_1^2 - \lambda_2^2)\xi_1(n)\xi_2(n)}{\lambda_1\lambda_2(\lambda_1\xi_1(n)\chi_2(n) - \lambda_2\xi_2(n)\chi_1(n))},\\
t_{22}[n+1] &= \frac{\lambda_1\xi_2(n)\chi_1(n) - \lambda_2\xi_1(n)\chi_2(n)}{\lambda_1\lambda_2(\lambda_1\xi_1(n)\chi_2(n) - \lambda_2\xi_2(n)\chi_1(n))}.
\end{aligned} \tag{69}$$

Proposition 6. *The matrix $\widetilde{U}_n$ defined by (60) has the same form as U_n, that is,*

$$\widetilde{U}_n = \begin{pmatrix} 0 & \widetilde{r}_n\lambda \\ \widetilde{s}_n\lambda & \widetilde{r}_n\widetilde{s}_n\lambda^2 \end{pmatrix}, \tag{70}$$

where the relations between old potentials r_n and s_n and new potentials $\widetilde{r}_n$ and $\widetilde{s}_n$ are given by

$$\begin{aligned}
\widetilde{r}_n &= \frac{r_n}{t_{22}[n] - r_n t_{21}[n]},\\
\widetilde{s}_n &= \frac{s_n}{t_{11}[n]}.
\end{aligned} \tag{71}$$

Proof. One has

$$T_{n+1}U_nT_n^* = \begin{pmatrix} h_{11}(\lambda, n) & h_{12}(\lambda, n) \\ h_{21}(\lambda, n) & h_{22}(\lambda, n) \end{pmatrix}, \tag{72}$$

with

$$\begin{aligned}
h_{11}(\lambda, n) &= s_n t_{12}[n+1] - r_n t_{11}[n+1]\,t_{21}[n]\\
&\quad + (-r_n t_{21}[n] - r_n s_n t_{12}[n+1]\,t_21[n]\\
&\quad + s_n t_{12}[n+1]\,t_{22}[n])\lambda^2,\\
h_{12}(\lambda, n) &= (r_n + r_n s_n t_{12}[n+1])\lambda^3 + (r_n t_{11}[n]\\
&\quad + r_n t_{11}[n+1] + r_n s_n t_{11}[n]\,t_{12}[n+1]\\
&\quad - s_n t_{12}[n]\,t_{12}[n+1])\lambda,\\
&\quad + \frac{r_n t_{11}[n]\,t_{11}[n+1]}{\lambda},\\
h_{21}(\lambda, n) &= (-r_n s_n t_{21}[n]\,t_{22}[n+1]\\
&\quad + s_n t_{22}[n]\,t_{22}[n+1])\lambda^3 + (-r_n s_n t_{21}[n]\\
&\quad - r_n t_{21}[n]\,t_{21}[n+1] + s_n t_{22}[n] + s_n t_{22}[n+1])\lambda\\
&\quad + \frac{s_n}{\lambda},\\
h_{22}(\lambda, n) &= r_n s_n t_{22}[n+1]\lambda^4 + (r_n s_n + r_n t_{21}[n+1]\\
&\quad + r_n s_n t_{11}[n]\,t_{22}[n+1] - s_n t_{12}[n]\,t_{22}[n+1])\lambda^2\\
&\quad + r_n s_n t_{11}[n] - s_n t_{12}[n] + r_n t_{11}[n]\,t_{21}[n].
\end{aligned} \tag{73}$$

According to (66) and (69), we can find that

$$\begin{aligned}
s_n t_{12}[n+1] - r_n t_{11}[n+1]\,t_{21}[n] &= 0,\\
-r_n t_{21}[n] - r_n s_n t_{12}[n+1]\,t_21[n]&\\
+ s_n t_{12}[n+1]\,t_{22}[n] &= 0.
\end{aligned} \tag{74}$$

Thus, we have $h_{11}(\lambda, n) = 0$. Moreover, it is easy to see that $\lambda^2 h_{12}(\lambda, n)$ and $\lambda^2 h_{21}(\lambda, n)$ are five-order polynomial in λ, and $\lambda^2 h_{22}(\lambda, n)$ is a six-order polynomial in λ. Through a tedious but direct computation or by computer algebra system (for instance, Mathematica, Maple), we can obtain that

$$\begin{aligned}
&h_{12}(\pm\lambda_i),\\
&h_{21}(\pm\lambda_i),\\
&h_{22}(\pm\lambda_i)\\
&\qquad (i = 1, 2)
\end{aligned} \tag{75}$$

are all equal zero. Therefore, we have

$$T_{n+1}U_nT_n^* = \det(T_n)\,\tau_n, \tag{76}$$

with

$$\tau_n = \begin{pmatrix} 0 & \tau_{12}^{(1)}\lambda \\ \tau_{21}^{(1)}\lambda & \tau_{22}^{(2)}\lambda^2 + \tau_{22}^{(0)} \end{pmatrix} \tag{77}$$

where $\tau_{jl}^{(i)}(i = 0, 1, 2,\ \ j, l = 1, 2)$ are all independent of λ. We know that $T_n^{-1} = T_n^*/\det(T_n)$. Hence we obtain

$$T_{n+1}U_n = \tau_n T_n. \tag{78}$$

Equating the coefficients of λ^i $(i = 0, 1, 2)$ in (78), we have

$$\begin{aligned}
\tau_{12}^{(1)} &= \frac{r_n}{t_{22}[n] - r_n t_{21}[n]},\\
\tau_{21}^{(1)} &= \frac{s_n}{t_{11}[n]},\\
\tau_{22}^{(2)} &= \frac{r_n}{t_{22}[n] - r_n t_{21}[n]}\left(\frac{s_n}{t_{11}[n]}\right),\\
\tau_{22}^{(0)} &= 0.
\end{aligned} \tag{79}$$

The proof is completed. □

Proposition 7. *The matrix $\widetilde{\Omega}_n^{(1)}$ defined by (60) has the same form as $\Omega_n^{(1)}$ in (56) under the transformation (71), i.e.,*

$$\widetilde{\Omega}_n^{(1)} = \begin{pmatrix} -\dfrac{\lambda^2}{2} + \dfrac{1}{\widetilde{r}_n \widetilde{s}_{n-1}} & \dfrac{\lambda}{\widetilde{s}_{n-1}} \\ \dfrac{\lambda}{\widetilde{r}_n} & \dfrac{\lambda^2}{2} \end{pmatrix}. \tag{80}$$

Proof. Let

$$\left(T_{n,t} + T_n \Omega_n^{(1)}\right) T_n^* = \begin{pmatrix} \eta_{11}(\lambda, n) & \eta_{12}(\lambda, n) \\ \eta_{21}(\lambda, n) & \eta_{22}(\lambda, n) \end{pmatrix}. \tag{81}$$

Here

$$\begin{aligned}
\eta_{11}(\lambda, n) &= -\frac{1}{2}\lambda^4 t_{22}[n] + \left(-\frac{1}{2} - \frac{t_{21}[n]}{s_{n-1}} - \frac{1}{2}\left(t_{12}[n]\, t_{21}[n] + t_{11}[n]\, t_{22}[n]\right) + \frac{t_{12}[n]\, t_{22}[n]}{r_n}\right)\lambda^2 + \frac{1 + t_{11}[n]\, t_{22}[n]}{r_n s_{n-1}}\\
&\quad - \frac{t_{11}[n]}{2} + \frac{t_{12}[n]}{r_n} - \frac{t_{12}[n]\, t21[n]}{s_{n-1}} + \left(t_{11}[n]\right)_t t_{22}[n] - t_{21}[n]\left(t_{12}[n]\right)_t + \frac{t_{11}[n] + r_n s_{n-1}\left(t_{11}[n]\right)_t}{r_n s_{n-1}\lambda^2},\\
\eta_{12}(\lambda, n) &= \left(\frac{1}{s_{n-1}} + t_{12}[n]\right)\lambda^3 + \left(\frac{2t_{11}[n]}{s_{n-1}} - \frac{t_{12}[n]}{r_n s_{n-1}} - \frac{\left(t_{12}[n]\right)^2}{r_n} + t_{11}[n]\, t_{12}[n] + \left(t_{12}[n]\right)_t\right)\lambda\\
&\quad + \frac{r_n\left(t_{11}[n]\right)^2 - t_{11}[n]\, t_{12}[n] - r_n s_{n-1}\left(t_{12}[n]\left(t_{11}[n]\right)_t - t_{11}[n]\left(t_{12}[n]\right)_t\right)}{r_n s_{n-1}\lambda},\\
\eta_{21}(\lambda, n) &= \left(-t_{21}[n]\, t_{22}[n] + \frac{\left(t_{22}[n]\right)^2}{r_n}\right)\lambda^3\\
&\quad + \left(-t_{21}[n] - \frac{\left(t_{21}[n]\right)^2}{s_{n-1}} + \frac{2t_{22}[n]}{r_n} + \frac{t_{21}[n]\, t_{22}[n]}{r_n s_{n-1}} + t_{22}[n]\left(t_{21}[n]\right)_t - t_{21}[n]\left(t_{22}[n]\right)_t\right)\lambda\\
&\quad + \frac{s_{n-1} + t_{21}[n] + r_n s_{n-1}\left(t_{22}[n]\right)_t}{r_n s_{n-1}\lambda},\\
\eta_{22}(\lambda, n) &= \frac{1}{2}t_{22}[n]\,\lambda^4 + \left(\frac{1}{2} + \frac{t_{21}[n]}{s_{n-1}} + \frac{t_{12}[n]\, t_{21} + t_{11}[n]\, t_{22}[n]}{2} - \frac{t_{12}[n]\, t_{22}[n]}{r_n} + \left(t_{22}[n]\right)_t\right)\lambda^2 + \frac{t_{11}[n]}{2} - \frac{t_{12}[n]}{r_n}\\
&\quad + \frac{t_{11}[n]\, t_{21}[n]}{s_{n-1}} - \frac{t_{12}[n]\, t_{21}[n]}{r_n s_{n-1}} - t_{12}[n]\left(t_{21}[n]\right)_t + t_{11}[n]\left(t_{22}[n]\right)_t.
\end{aligned} \tag{82}$$

It is easy to get that $\lambda^2\eta_{11}(\lambda, n)$ and $\lambda^2\eta_{22}(\lambda, n)$ are six-order polynomials in λ; $\lambda^2\eta_{12}(\lambda, n)$ and $\lambda^2\eta_{21}(\lambda, n)$ are five-order polynomials in λ. According to (56) and (65), we have

$$\alpha_{i,t}[n] = \frac{\lambda_i}{r_n} + \left(\lambda_i^2 - \frac{1}{r_n s_{n-1}}\right)\alpha_i[n] - \frac{\lambda_i}{s_{n-1}}\alpha_i[n]^2, \quad i = 1, 2. \tag{83}$$

Through lengthy but direct calculation, we can verify that

$$\eta_{jk}(\pm\lambda_i, n) = 0, \quad i, j, k = 1, 2. \tag{84}$$

Therefore, the following equation is derived:

$$\left(T_{n_t} + T_n \Omega_n^{(1)}\right) T_n^* = \det(T_n)\,\mu_n, \tag{85}$$

with

$$\mu_n = \begin{pmatrix} \mu_{11}^{(2)}\lambda^2 + \mu_{11}^{(0)} & \mu_{12}^{(1)}\lambda \\ \mu_{21}^{(1)}\lambda & \mu_{22}^{(2)}\lambda^2 + \mu_{22}^{(0)} \end{pmatrix}, \tag{86}$$

where $\mu_{11}^{(2)}$ $\mu_{11}^{(0)}$, $\mu_{12}^{(1)}$, $\mu_{21}^{(1)}$, $\mu_{22}^{(2)}$, $\mu_{22}^{(0)}$ are all independent of λ. From (85), we get

$$\left(T_{n_t} + T_n \Omega_n^{(1)}\right) = \mu_n T_n. \tag{87}$$

Comparing the coefficients of λ in (87), we have

$$\begin{aligned}
\mu_{11}^{(2)} &= -\frac{1}{2}, \\
\mu_{11}^{(0)} &= \frac{1}{\tilde{r}_n \tilde{s}_{n-1}}, \\
\mu_{12}^{(1)} &= \frac{1}{\tilde{s}_{n-1}}, \\
\mu_{21}^{(1)} &= \frac{1}{\tilde{r}_n}, \\
\mu_{22}^{(2)} &= \frac{1}{2}, \\
\mu_{22}^{(0)} &= 0.
\end{aligned} \tag{88}$$

The proof is completed. □

Theorem 8. *Equation (71) is a Darboux-Bäcklund transformation of (26). That is, each solution $(r_n, s_n)^T$ of the integrable lattice system (26) is mapped into its new solution $(\tilde{r}_n, \tilde{s}_n)^T$ under transformation (71).*

Next, using obtained Darboux-*Bäcklund* transformation (71), we derive an exact solution of (26).

First, it is easy to find that $r_n = e^{-t}$, $s_n = -e^t$ constitute a trivial solution of (71). Substituting this solution into the corresponding Lax pair, we get

$$\begin{aligned}
E\varphi_n &= \begin{pmatrix} 0 & e^{-t}\lambda \\ -e^t\lambda & -\lambda^2 \end{pmatrix} \varphi_n, \\
\varphi_{n_t} &= \begin{pmatrix} -\dfrac{\lambda^2}{2} - 1 & -e^{-t}\lambda \\ e^t\lambda & \dfrac{\lambda^2}{2} \end{pmatrix} \varphi_n.
\end{aligned} \tag{89}$$

Solving the above two equations, we get two real linear independent solutions:

$$\begin{aligned}
\begin{pmatrix} \phi_n^1 \\ \phi_n^2 \end{pmatrix} &= C_1(\lambda, n) \begin{pmatrix} 2\sqrt{2+\sqrt{5}}\, exp\left(-\dfrac{t}{2}\right) \\ -\left(3+\sqrt{5}\right) exp\left(\dfrac{t}{2}\right) \end{pmatrix}, \\
\begin{pmatrix} \psi_n^1 \\ \psi_n^2 \end{pmatrix} &= C_2(\lambda, n) \begin{pmatrix} 2\sqrt{2+\sqrt{5}}\, exp\left(-\dfrac{t}{2}\right) \\ \left(3+\sqrt{5}\right) exp\left(\dfrac{t}{2}\right) \end{pmatrix},
\end{aligned} \tag{90}$$

where

$$\begin{aligned}
C_1(\lambda, n) &= \left(-7\sqrt{2+\sqrt{5}} - 3\sqrt{5\left(2+\sqrt{5}\right)}\right) \\
&\quad \cdot \left(\lambda\left(-\lambda + \sqrt{\lambda^2 - 4}\right)\right)^n - \left(\left(-\lambda\left(\lambda + \sqrt{\lambda^2-4}\right)\right)^n\right) \\
&\quad \cdot \left(29\left(13+\sqrt{5}\right)\sqrt{\lambda^2-4}\right)\left(\lambda\left(-\lambda+\sqrt{\lambda^2-4}\right)\right)^n \\
&\quad + \left(\left(-\lambda\left(\lambda+\sqrt{\lambda^2-4}\right)\right)^n\right), \\
C_2(\lambda, n) &= \left(\lambda\left(-\lambda+\sqrt{\lambda^2-4}\right)\right)^n \\
&\quad - \left(\left(-\lambda\left(\lambda+\sqrt{\lambda^2-4}\right)\right)^n\right).
\end{aligned} \tag{91}$$

From (65), we have

$$\begin{aligned}
\alpha_i[n] &= \frac{\phi_n^2 - \kappa_i \psi_n^2}{\phi_n^1 - \kappa_i \psi_n^1} \\
&= \frac{-\left(3+\sqrt{5}\right) exp(t/2)\left(C_1(\lambda_i, n) - \kappa_i C_2(\lambda_i, n)\right)}{2\sqrt{2+\sqrt{5}\, exp(-t/2)}\left(C_1(\lambda_i, n) + \kappa_i C_2(\lambda_i, n)\right)}, \\
& \qquad i = 1, 2.
\end{aligned} \tag{92}$$

In (92), $\lambda_1 = (2+\sqrt{5})$, $\lambda_2 = -(2+\sqrt{5})$.

By means of the Darboux-Bäcklund transformation (71), we deduce a new exact solution of (26).

$$\begin{aligned}
\tilde{r}_n &= \frac{\lambda_1\lambda_2\left(\lambda_1\sigma_1[n] - \lambda_2\sigma_2[n]\right) exp(-t)}{\left(\lambda_1\alpha_2[n] - \lambda_2\alpha_1[n]\right) - \left(\left(\lambda_2^2 - \lambda_1^2\right)\alpha_1[n]\alpha_2[n]\right) exp(-t)}, \\
\tilde{s}_n &= -\frac{\left(\lambda_1\alpha_1[n] - \lambda_2\alpha_2[n]\right) exp(t)}{\lambda_1\lambda_2\left(\lambda_1\alpha_2[n] - \lambda_2\alpha_1[n]\right)}.
\end{aligned} \tag{93}$$

This transformation can be done continually. Thus, we can obtain a series of exact solutions of (26).

4. Conclusions and Remarks

In this work, based on a Lax pair, we have deduced a novel family of integrable different-difference equations using the discrete zero curvature equation and established the Hamiltonian structure of the obtained integrable family of different-difference equations in virtue of the discrete trace identity. And then the Liouville integrability of the obtained family is demonstrated. With the help of a gauge transformation of the Lax pair, a Darboux-Bäcklund transformation for the first nonlinear different-difference equation in the obtained family is presented, and using the obtained Darboux-Bäcklund transformation, an exact solution is derived. It is worth noting that in (61) we can also study the generalized form of T_n

$$T_n^{(2N+1)} = \begin{pmatrix} T_n^{11} & T_n^{12} \\ T_n^{21} & T_n^{22} \end{pmatrix}, \tag{94}$$

with

$$\begin{aligned}
T_n^{11} &= \lambda^{2N+1} + t_{11}^{(2N-1)}[n]\lambda^{2N-1} + t_{11}^{(2N-3)}[n]\lambda^{2N-3} \\
&\quad + \cdots + \frac{t_{11}^{-2N+1}[n]}{\lambda^{-2N+1}} + \frac{t_{11}^{-2N-1}[n]}{\lambda^{2N+1}}, \quad N \geq 1, \\
T_n^{12} &= t_{12}^{(2N)}[n]\lambda^{2N} + t_{12}^{(2N-2)}[n]\lambda^{2N-2} + \cdots \\
&\quad + \frac{t_{12}^{-2N}[n]}{\lambda^{2N}}, \quad N \geq 1, \\
T_n^{21} &= t_{21}^{(2N)}[n]\lambda^{2N} + t_{21}^{(2N-2)}[n]\lambda^{2N-2} + \cdots \\
&\quad + \frac{t_{21}^{-2N}[n]}{\lambda^{2N}}, \quad N \geq 1, \\
T_n^{22} &= \lambda^{2N+1} t_{22}^{(2N+1)}[n] + t_{22}^{(2N-1)}[n]\lambda^{2N-3} \\
&\quad + t_{22}^{(2N-5)}[n]\lambda^{2N-5} + \cdots + \frac{t_{11}^{-2N+1}[n]}{\lambda^{2N-1}} \\
&\quad + \frac{1}{\lambda^{2N+1}}, \quad N \geq 1.
\end{aligned} \tag{95}$$

In addition, many interesting problems deserve further investigation for obtained integrable family, for example, symmetry constraint [7], symmetries and master symmetries [13], and integrable coupling systems [25, 26].

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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