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Research Article

The Metric Dimension of Some Generalized Petersen Graphs

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The distance $d(u,v)$ between two distinct vertices u and v in a graph G is the length of a shortest (u,v)-path in G. For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in G, the code of v with respect to W is the ordered k-tuple $c_W(v) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$. The set W is a resolving set for G if every two vertices of G have distinct codes. The *metric dimension* of G is the minimum cardinality of a resolving set of G. In this paper, we first extend the results of the metric dimension of $P(n,3)$ and $P(n,4)$ and study bounds on the metric dimension of the families of the generalized Petersen graphs $P(2k,k)$ and $P(3k,k)$. The obtained results mean that these families of graphs have constant metric dimension.

1. Introduction

Let G be a connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The distance between two distinct vertices u and v in G, denoted by $d(u,v)$, is the length of a shortest (u,v)-path. For positive integer k and a vertex $v \in V(G)$, the k-neighborhood of v is the set $N_k(v) = \{u \mid d(u,v) = k\}$. For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in G, the code of v with respect to W is the ordered k-tuple $c_W(v) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$. The set W is a resolving set [1] (or locating set [2]) for G if every two vertices of G have distinct codes. The *metric dimension* of G, denoted by $\dim(G)$, is the minimum cardinality of a resolving set of G. A resolving set containing a minimum number of vertices is called a basis for G [3].

Graph theory is a powerful tool to model the real world applications such as physical-chemical property testing [4, 5]. Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension of a graph was introduced by Slater in [6], where the metric generators were called locating sets. The concept of metric dimension of a graph was also introduced by Harary and Melter in [1], where metric generators were called resolving sets. Applications of this invariant to the navigation of robots in networks are discussed in [7] and applications to chemistry in [8, 9]. This graph parameter was studied further in a number of other papers including, for instance, [10–19]. Several variations of metric generators including resolving dominating sets [20], independent resolving sets [21], local metric sets [22], strong resolving sets [23], mixed metric dimension [24], and k-metric dimension [25] have since been introduced and studied.

We observe from definition that the property of a given set W of vertices of a graph G to be a resolving set of G can be tested by investigating the vertices of $V(G) \setminus W$ because every vertex $w \in W$ is the unique vertex of G whose distance from w is 0. If $d(x,t) \neq d(y,t)$, we say that vertex t distinguishes vertices x and y.

For natural numbers n and k, where $n > 2k$, a generalized Petersen graph $P(n,k)$ is a graph with vertex set $U \cup V$, where $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$, and edge set $E_1 \cup E_2 \cup E_3$, where $E_1 = \{u_i u_{i+1} \mid 1 \le i \le n$, $E_2 = \{u_i v_i \mid 1 \le i \le n\}$, and $E_3 = \{v_i v_{i+k} \mid 1 \le i \le n\}$, where subscripts are taken modulo n (see [2, 26]). We observe that, for each $1 \le i \le n$,

$$N_2(u_i) = \{u_{i-2}, u_{i+2}, v_{i-1}, v_{i+1}, v_{i-k}, v_{i+k}\}, \quad (1)$$

$$N_2(v_i) = \{u_{i-1}, u_{i+1}, v_{i-2k}, v_{i+2k}, u_{i-k}, u_{i+k}\}, \quad (2)$$

Table 1: The code $c_W(v)$ of v with respect to $W = \{u_1, u_7, v_5\}$ in $P(9,3)$.

v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$
u_1	$(0,3,3)$	u_2	$(1,4,2)$	u_3	$(2,4,3)$	u_4	$(3,3,2)$	u_5	$(4,2,1)$
u_6	$(4,1,2)$	u_7	$(3,0,3)$	u_8	$(2,1,2)$	u_9	$(1,2,3)$	v_1	$(1,2,4)$
v_2	$(2,3,1)$	v_3	$(3,3,4)$	v_4	$(2,2,3)$	v_5	$(3,3,0)$	v_6	$(3,2,3)$
v_7	$(2,1,4)$	v_8	$(3,2,1)$	v_9	$(2,3,4)$				

$$N_3(u_i) = \{u_{i-3}, u_{i+3}, v_{i-2}, v_{i+2}, v_{i+1-k}, v_{i+k+1}, v_{i-1-k}, v_{i+k-1}, v_{i-2k}, v_{i+2k}, u_{i-k}, u_{i+k}\}, \tag{3}$$

$$N_3(v_i) = \{u_{i-2}, u_{i+2}, v_{i+1}, v_{i-1}, u_{i+1+k}, u_{i+k-1}, u_{i-1-k}, u_{i-k+1}, u_{i-2k}, u_{i+2k}, v_{i-3k}, v_{i+3k}\}. \tag{4}$$

If $k = 3$ and $n \geq 16$, then clearly $|N_2(x)| \geq 6$ and $|N_3(x)| \geq 8$ for each $x \in V(P(n,k))$.

Javaid et al. [27] proved that $\dim(P(n,2)) = 3$ for $n \geq 5$ and posed the following problem.

Problem 1. Is the generalized Petersen graphs $P(n,k)$, for $n \geq 7$ and $3 \leq k \leq \lfloor (n-1)/2 \rfloor$, a family of graphs with constant metric dimension?

Some partial answers are given to aforementioned problem as follows.

Theorem 2 (see [28]). *For $n \geq 3$, $\dim(P(n,1)) = 2$ when $n \equiv 1 \pmod 2$ and $\dim(P(n,1)) = 3$ if $n \equiv 0 \pmod 2$.*

Theorem 3 (see [29]).

$$\dim(P(n,3)) \begin{cases} = 4, & \text{if } n \equiv 0 \pmod 6 \text{ and } n \geq 24; \\ = 3, & \text{if } n \equiv 1 \pmod 6 \text{ and } n \geq 25; \\ \leq 5, & \text{if } n \equiv 2 \pmod 6 \text{ and } n \geq 8; \\ \leq 4, & \text{if } n \equiv 3,4,5 \pmod 6 \text{ and } n \geq 17; \end{cases} \tag{5}$$

In [30, 31], it was showed that

Theorem 4. *(i) $\dim(P(2k+1,k)) = 3$ if $k \geq 2$.*

(ii)

$$\dim(P(2k,k-1)) = \begin{cases} 3, & \text{if } k = 2 \text{ or } k \geq 3 \text{ is odd}; \\ 4, & \text{if } k \geq 4 \text{ is odd}. \end{cases} \tag{6}$$

In [32], it was proved that

Theorem 5. *For $n \geq 4$, $\dim(P(n,4)) = 3$ if $n \equiv 0 \pmod 4$, $\dim(P(n,4)) = 4$ when $n = 4k+3$ for even k, and $\dim(P(n,4)) \leq 4$ otherwise.*

In this paper, we first extend the results of Theorems 3 and 5.

We make use of the following result in this paper.

Theorem 6 (see [7]). *If G is a graph of order n, diameter d and metric dimension k, then $n < d^k + k$.*

2. Main Results

Next result extends Theorem 3.

Lemma 7. *Let G be a connected graph and let $|N_2(v)| \geq 6$ or $|N_3(v)| \geq 8$ for each $v \in V(G)$. Then $\dim(G) \geq 3$.*

Proof. Clearly, for any $w, v \in V(G)$ and for any $z \in N_k(w)$ we have

$$d(v,w) - k \leq d(v,z) \leq d(v,w) + k. \tag{7}$$

Suppose, to the contrary, that $S = \{w_1, w_2\}$ is a resolving set of G. Since $|N_2(w_1)| \geq 6$ or $|N_3(w_1)| \geq 8$, we deduce from (7) and the Pigeonhole principle that there exist two vertices $x_1, x_2 \in N_2(w_1)$ such that $d(x_1, w_2) = d(x_2, w_2)$, a contradiction. □

Theorem 8. *For $n \geq 7$,*

(i) If $n \in \{9, 10, 11, 15\}$ or $n \equiv 1 \pmod 6$, then $\dim(P(n,3)) = 3$.

(ii) If $n = 20$, then $\dim(P(n,3)) = 5$.

Proof. If $n = 9$, then let $W = \{u_1, u_7, v_5\}$. The code of v with respect to W in $P(9,3)$ is presented in Table 1 yielding $\dim(P(9,3)) \leq 3$.

Now, we show that $\dim(P(9,3)) \geq 3$. Suppose, to the contrary, there exists a resolving set $W = \{x, y\}$ of $P(9,3)$. First let $W \cap U \neq \emptyset$. We may assume w.l.o.g. that $x \in W \cap U$. By (1), we have $|N_2(x)| = 6$. For each $u \in N_2(x)$, we have $d(y,x) - 2 \leq d(y,u) \leq d(y,x) + 2$. By the Pigeonhole principle, we have $d(y,u) = d(y,v)$ for some $u, v \in N_2(x)$ and this leads to a contradiction. Now let $W \cap U = \emptyset$. Assume without loss of generality that $x = v_1$ and $y = v_i$ for some $i \in \{2,3,4,5\}$. If $i \in \{2,3\}$, then $(d(x,u_4), d(y,u_4)) = (d(x,u_9), d(y,u_9))$, and if $i \in \{4,5\}$, then $(d(x,u_{7-i}), d(y,u_{7-i})) = (d(x,u_{9-i}), (y,u_{9-i}))$, a contradiction. Thus, $\dim(P(9,3)) \geq 3$ and so $\dim(P(9,3)) = 3$.

If $n = 10$, then let $W = \{u_1, v_9, v_{10}\}$. The code of v with respect to W in $P(10,3)$ is presented in Table 2 showing that $\dim(P(10,3)) \leq 3$.

Next, we show that $\dim(P(10,3)) \geq 3$. Suppose, to the contrary, there exists a resolving set $W = \{x, y\}$ of $P(10,3)$. As above, we may assume that $W \cap U = \emptyset$. We may assume w.l.o.g. that $x = v_1$ and $y = v_i$ for some $i \in \{2,3,\ldots,6\}$. If $i \in \{2,4,6\}$, then $(d(x,u_3), d(y,u_3)) = (d(x,u_5), d(y,u_5))$, and if $i \in \{3,5\}$, then we have $(d(x,u_2), d(y,u_2)) = (d(x,u_4), d(y,u_4))$, a contradiction.

If $n = 11$, then let $W = \{u_1, u_5, v_2\}$. The code of v with respect to W in $P(11,3)$ is presented in Table 3 yielding $\dim(P(11,3)) \leq 3$.

Table 2: The code $c_W(v)$ of v with respect to $W = \{u_1, v_9, v_{10}\}$ in $P(10, 3)$.

v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$
u_1	$(0, 3, 2)$	u_2	$(1, 2, 3)$	u_3	$(2, 3, 2)$	u_4	$(3, 4, 3)$	u_5	$(4, 3, 4)$
u_6	$(5, 2, 3)$	u_7	$(4, 3, 2)$	u_8	$(3, 2, 3)$	u_9	$(2, 1, 2)$	u_{10}	$(1, 2, 1)$
v_1	$(1, 4, 3)$	v_2	$(2, 1, 4)$	v_3	$(3, 2, 1)$	v_4	$(2, 5, 2)$	v_5	$(3, 2, 5)$
v_6	$(4, 1, 2)$	v_7	$(3, 4, 1)$	v_8	$(2, 3, 4)$	v_9	$(3, 0, 3)$	v_{10}	$(2, 3, 0)$

Table 3: The code $c_W(v)$ of v with respect to $W = \{u_1, u_5, v_2\}$ in $P(11, 3)$.

v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$
u_1	$(0, 4, 2)$	u_2	$(1, 3, 1)$	u_3	$(2, 2, 2)$	u_4	$(3, 1, 3)$	u_5	$(4, 0, 2)$
u_6	$(4, 1, 3)$	u_7	$(4, 2, 3)$	u_8	$(4, 3, 3)$	u_9	$(3, 4, 3)$	u_{10}	$(2, 4, 2)$
u_{11}	$(1, 4, 3)$	v_1	$(1, 3, 3)$	v_2	$(2, 2, 0)$	v_3	$(3, 3, 3)$	v_4	$(2, 2, 3)$
v_5	$(3, 1, 1)$	v_6	$(3, 2, 4)$	v_7	$(3, 3, 2)$	v_8	$(3, 2, 2)$	v_9	$(2, 3, 4)$
v_{10}	$(3, 3, 1)$	v_{11}	$(2, 3, 3)$						

Table 4: The code $c_W(v)$ of v with respect to $W = \{u_1, u_3, v_{11}\}$ in $P(15, 3)$.

v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$
u_1	$(0, 2, 4)$	u_2	$(1, 1, 3)$	u_3	$(2, 0, 4)$	u_4	$(3, 1, 4)$	u_5	$(4, 2, 3)$
u_6	$(5, 3, 4)$	u_7	$(4, 4, 3)$	u_8	$(5, 5, 2)$	u_9	$(5, 4, 3)$	u_{10}	$(4, 5, 2)$
u_{11}	$(5, 5, 1)$	u_{12}	$(4, 4, 2)$	u_{13}	$(3, 5, 3)$	u_{14}	$(2, 4, 2)$	u_{15}	$(1, 3, 3)$
v_1	$(1, 3, 5)$	v_2	$(2, 2, 2)$	v_3	$(3, 1, 5)$	v_4	$(2, 2, 5)$	v_5	$(3, 3, 2)$
v_6	$(4, 2, 5)$	v_7	$(3, 3, 4)$	v_8	$(4, 4, 1)$	v_9	$(4, 3, 4)$	v_{10}	$(3, 4, 3)$
v_{11}	$(4, 4, 0)$	v_{12}	$(3, 3, 3)$	v_{13}	$(2, 4, 4)$	v_{14}	$(3, 3, 1)$	v_{15}	$(2, 2, 4)$

Table 5: The code $c_W(v)$ of v with respect to $W = \{u_1, u_3, u_{11}, v_8, v_{10}\}$ in $P(20, 3)$.

v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$
u_1	$(0, 2, 6, 4, 4)$	u_2	$(1, 1, 5, 3, 5)$	u_3	$(2, 0, 6, 4, 4)$	u_4	$(3, 1, 5, 3, 3)$
u_5	$(4, 2, 4, 2, 4)$	u_6	$(5, 3, 5, 3, 3)$	u_7	$(4, 4, 4, 2, 2)$	u_8	$(5, 5, 3, 1, 3)$
u_9	$(6, 4, 2, 2, 2)$	u_{10}	$(5, 5, 1, 3, 1)$	u_{11}	$(6, 6, 0, 2, 2)$	u_{12}	$(5, 5, 1, 3, 3)$
u_{13}	$(6, 6, 2, 4, 2)$	u_{14}	$(5, 5, 3, 3, 3)$	u_{15}	$(4, 6, 4, 4, 4)$	u_{16}	$(5, 5, 5, 5, 3)$
u_{17}	$(4, 4, 4, 4, 4)$	u_{18}	$(3, 5, 5, 5, 5)$	u_{19}	$(2, 4, 6, 4, 4)$	u_{20}	$(1, 3, 5, 5, 5)$
v_1	$(1, 3, 5, 5, 3)$	v_2	$(2, 2, 4, 2, 4)$	v_3	$(3, 1, 5, 5, 5)$	v_4	$(2, 2, 4, 4, 2)$
v_5	$(3, 3, 3, 1, 5)$	v_6	$(4, 2, 4, 4, 4)$	v_7	$(3, 3, 3, 3, 1)$	v_8	$(4, 4, 2, 0, 4)$
v_9	$(5, 3, 3, 3, 3)$	v_{10}	$(4, 4, 2, 4, 0)$	v_{11}	$(5, 5, 1, 1, 3)$	v_{12}	$(4, 4, 2, 4, 4)$
v_{13}	$(5, 5, 3, 5, 1)$	v_{14}	$(4, 4, 2, 2, 4)$	v_{15}	$(3, 5, 3, 5, 5)$	v_{16}	$(4, 4, 4, 4, 2)$
v_{17}	$(3, 3, 3, 3, 5)$	v_{18}	$(2, 4, 4, 6, 4)$	v_{19}	$(3, 3, 5, 3, 3)$	v_{20}	$(2, 2, 4, 4, 6)$

Since $\mathrm{diam}(P(11, 3)) = 4$, we deduce from Theorem 6 that $\dim(P(11, 3)) \geq 3$. Thus, $\dim(P(11, 3)) = 3$.

If $n = 15$, then let $W = \{u_1, u_3, v_{11}\}$. The code of v with respect to W in $P(15, 3)$ is presented in Table 4 implying that $\dim(P(15, 3)) \leq 3$.

Since $\mathrm{diam}(P(15, 3)) = 5$, it follows from Theorem 6 that $\dim(P(15, 3)) \geq 3$. Hence, $\dim(P(15, 3)) = 3$.

If $n = 20$, then let $W = \{u_1, u_3, u_{11}, v_8, v_{10}\}$. The code of v with respect to W in $P(20, 3)$ is presented in Table 5. This implies that $\dim(P(20, 3)) \leq 5$.

Analogous to the proof of the case $n = 9$, we can obtain the desired lower bound with a more complicated analysis. Also it can be verified by computer search.

If $n \leq 15$, we can verify the results by computer. If $n \geq 16$ and $n \equiv 1 \pmod 6$, we have $|N_2(v)| = 6$ and $|N_3(v)| \geq 8$ for any $v \in P(n, 3)$. Now by Lemma 7, we have $\dim(P(n, 3)) \geq 3$. Now, the proof is complete. □

The following theorem extends the result of Theorem 5.

Theorem 9. *Let G be the graph $G = P(n, 4)$ with $n \geq 9$; then if $n \in \{9, 10, 11, 13, 17, 21\}$ or $n \equiv 0 \pmod 4$, then $\dim(G) = 3$.*

Proof. If $n = 9$, let $W = \{u_1, v_8, v_9\}$. The code of v with respect to W in $P(9, 4)$ is presented in Table 6 showing that $\dim(P(9, 4)) \leq 3$.

If $n = 10$, let $W = \{u_1, u_5, v_3\}$. The code of v with respect to W in $P(10, 4)$ is presented in Table 7 showing that $\dim(P(10, 4)) \leq 3$.

Note that the diameter of $P(10, 4)$ is 4; by Theorem 6, we have $\dim(P(10, 4)) \geq 3$.

Table 6: The code $c_W(v)$ of v with respect to $W = \{u_1, v_8, v_9\}$ in $P(9, 4)$.

v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$
u_1	$(0,3,2)$	u_2	$(1,3,3)$	u_3	$(2,2,3)$	u_4	$(3,2,2)$	u_5	$(3,3,2)$
u_6	$(3,3,3)$	u_7	$(3,2,3)$	u_8	$(2,1,2)$	u_9	$(1,2,1)$	v_1	$(1,4,2)$
v_2	$(2,3,4)$	v_3	$(3,1,3)$	v_4	$(3,1,1)$	v_5	$(2,3,1)$	v_6	$(2,4,3)$
v_7	$(3,2,4)$	v_8	$(3,0,2)$	v_9	$(2,2,0)$				

Table 7: The code $c_W(v)$ of v with respect to $W = \{u_1, u_5, v_3\}$ in $P(10, 4)$.

v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$
u_1	$(0,3,3)$	u_2	$(1,3,2)$	u_3	$(2,2,1)$	u_4	$(3,1,2)$	u_5	$(3,0,3)$
u_6	$(4,1,3)$	u_7	$(3,2,2)$	u_8	$(3,3,3)$	u_9	$(2,3,2)$	u_{10}	$(1,4,3)$
v_1	$(1,2,2)$	v_2	$(2,3,3)$	v_3	$(3,3,0)$	v_4	$(3,2,3)$	v_5	$(2,1,2)$
v_6	$(3,2,4)$	v_7	$(2,3,1)$	v_8	$(3,3,4)$	v_9	$(3,2,1)$	v_{10}	$(2,3,4)$

Table 8: The code $c_W(v)$ of v with respect to $W = \{u_1, v_8, v_{11}\}$ in $P(11, 4)$.

v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$
u_1	$(0,2,2)$	u_2	$(1,3,3)$	u_3	$(2,3,3)$	u_4	$(3,2,2)$	u_5	$(3,3,3)$
u_6	$(4,3,3)$	u_7	$(4,2,2)$	u_8	$(3,1,3)$	u_9	$(3,2,3)$	u_{10}	$(2,3,2)$
u_{11}	$(1,3,1)$	v_1	$(1,1,3)$	v_2	$(2,4,4)$	v_3	$(3,4,2)$	v_4	$(3,1,1)$
v_5	$(2,2,4)$	v_6	$(3,4,4)$	v_7	$(3,3,1)$	v_8	$(2,0,2)$	v_9	$(3,3,4)$
v_{10}	$(3,4,3)$	v_{11}	$(2,2,0)$						

Table 9: The code $c_W(v)$ of v with respect to $W = \{u_1, v_{14}, v_{17}\}$ in $P(17, 4)$.

v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$
u_1	$(0,2,2)$	u_2	$(1,3,3)$	u_3	$(2,4,3)$	u_4	$(3,4,2)$	u_5	$(3,3,3)$
u_6	$(4,3,4)$	u_7	$(5,4,4)$	u_8	$(5,4,3)$	u_9	$(4,3,3)$	u_{10}	$(4,2,4)$
u_{11}	$(5,3,4)$	u_{12}	$(5,3,3)$	u_{13}	$(4,2,2)$	u_{14}	$(3,1,3)$	u_{15}	$(3,2,3)$
u_{16}	$(2,3,2)$	u_{17}	$(1,3,1)$	v_1	$(1,1,3)$	v_2	$(2,3,4)$	v_3	$(3,5,4)$
v_4	$(3,5,1)$	v_5	$(2,2,3)$	v_6	$(3,2,5)$	v_7	$(4,5,5)$	v_8	$(4,5,2)$
v_9	$(3,3,2)$	v_{10}	$(3,1,5)$	v_{11}	$(4,4,5)$	v_{12}	$(4,4,3)$	v_{13}	$(3,3,1)$
v_{14}	$(2,0,4)$	v_{15}	$(3,3,4)$	v_{16}	$(3,4,3)$	v_{17}	$(2,4,0)$		

If $n = 11$, let $W = \{u_1, v_8, v_{11}\}$. The code of v with respect to W in $P(11, 4)$ is presented in Table 8 showing that $\dim(P(11, 4)) \leq 3$.

Note that the diameter of $P(11, 4)$ is 4; by Theorem 6, we have $\dim(P(11, 4)) \geq 3$.

If $n = 17$, let $W = \{u_1, v_{14}, v_{17}\}$. The code of v with respect to W in $P(17, 4)$ is presented in Table 9 showing that $\dim(P(17, 4)) \leq 3$.

Note that the diameter of $P(17, 4)$ is 5; by Theorem 6, we have $\dim(P(17, 4)) \geq 3$.

If $n = 21$, let $W = \{u_1, u_{16}, v_{14}\}$. The code of v with respect to W in $P(21, 4)$ is presented in Table 10 showing that $\dim(P(21, 4)) \leq 3$.

Note that the diameter of $P(21, 4)$ is 6; by Theorem 6, we have $\dim(P(21, 4)) \geq 3$. □

Theorem 10. *Let G be the graph $G = P(2k, k)$ with $k \geq 3$; then*

$$\dim(G) \begin{cases} = 3, & \text{if } k = 3 \text{ or } k \equiv 0 \pmod 2; \\ \leq 4, & \text{otherwise.} \end{cases} \tag{8}$$

Proof.

Case 1 ($k = 3$ or $k \equiv 0 \pmod 2$). If $k = 3$, let $W = \{u_3, v_1, v_2\}$. The code of v with respect to W in $P(6, 3)$ is presented in Table 11 showing that $\dim(P(6, 3)) \leq 3$.

If $k = 4$, let $W = \{v_4, v_7, v_8\}$. The code of v with respect to W in $P(8, 4)$ is presented in Table 12 showing that $\dim(P(8, 4)) \leq 3$.

If $k \geq 6$, let $W = \{u_1, u_2, u_{k+1}\}$. Then the codes of the outer vertices are $c_W(u_i) = (h_1(i), h_2(i), h_3(i))$ and the codes of the inner vertices are $c_W(v_i) = (g_1(i), g_2(i), g_3(i))$, where

$$h_1(i) = \begin{cases} i - 1, & 3 \leq i \leq \dfrac{k}{2} + 2; \\ k + 4 - i, & \dfrac{k}{2} + 3 \leq i \leq k; \\ i + 2 - k, & k + 1 \leq i \leq \dfrac{3k}{2} - 1; \\ 2k + 1 - i, & \dfrac{3k}{2} \leq i \leq 2k. \end{cases}$$

Table 10: The code $c_W(v)$ of v with respect to $W = \{u_1, u_{16}, v_{14}\}$ in $P(21, 4)$.

v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$
u_1	$(0,5,3)$	u_2	$(1,5,4)$	u_3	$(2,4,5)$	u_4	$(3,5,5)$	u_5	$(3,6,4)$
u_6	$(4,6,3)$	u_7	$(5,5,4)$	u_8	$(5,4,4)$	u_9	$(4,5,3)$	u_{10}	$(5,5,2)$
u_{11}	$(6,4,3)$	u_{12}	$(6,3,3)$	u_{13}	$(5,3,2)$	u_{14}	$(4,2,1)$	u_{15}	$(5,1,2)$
u_{16}	$(5,0,3)$	u_{17}	$(4,1,3)$	u_{18}	$(3,2,2)$	u_{19}	$(3,3,3)$	u_{20}	$(2,3,4)$
u_{21}	$(1,4,4)$	v_1	$(1,4,2)$	v_2	$(2,4,3)$	v_3	$(3,3,6)$	v_4	$(3,4,6)$
v_5	$(2,5,3)$	v_6	$(3,5,2)$	v_7	$(4,4,5)$	v_8	$(4,3,5)$	v_9	$(3,4,4)$
v_{10}	$(4,4,1)$	v_{11}	$(5,3,4)$	v_{12}	$(5,2,4)$	v_{13}	$(4,3,3)$	v_{14}	$(3,3,0)$
v_{15}	$(4,2,3)$	v_{16}	$(4,1,4)$	v_{17}	$(3,2,4)$	v_{18}	$(2,3,1)$	v_{19}	$(3,3,4)$
v_{20}	$(3,2,5)$	v_{21}	$(2,3,5)$						

Table 11: The code $c_W(v)$ of v with respect to $W = \{u_3, v_1, v_2\}$ in $P(6, 3)$.

v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$
u_1	$(2,1,2)$	u_2	$(1,2,1)$	u_3	$(0,3,2)$	u_4	$(1,2,3)$	u_5	$(2,3,2)$
u_6	$(3,2,3)$	v_1	$(3,0,3)$	v_2	$(2,3,0)$	v_3	$(1,4,3)$	v_4	$(2,1,4)$
v_5	$(3,4,1)$	v_6	$(2,3,4)$						

Table 12: The code $c_W(v)$ of v with respect to $W = \{v_4, v_7, v_8\}$ in $P(8, 4)$.

v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$
u_1	$(3,3,2)$	u_2	$(3,3,3)$	u_3	$(2,2,3)$	u_4	$(1,3,2)$	u_5	$(2,3,3)$
u_6	$(3,2,3)$	u_7	$(3,1,2)$	u_8	$(2,2,1)$	v_1	$(4,4,3)$	v_2	$(4,4,4)$
v_3	$(3,1,4)$	v_4	$(0,4,1)$	v_5	$(3,4,4)$	v_6	$(4,3,4)$	v_7	$(4,0,3)$
v_8	$(1,3,0)$								

$$h_2(i) = \begin{cases} i-2, & 3 \le i \le \frac{k}{2}+3; \\ k+5-i, & \frac{k}{2}+4 \le i \le k+2; \\ i+1-k, & k+3 \le i \le \frac{3k}{2}; \\ 2k+2-i, & \frac{3k}{2}+1 \le i \le 2k. \end{cases}$$

$$h_3(i) = \begin{cases} i+2, & 3 \le i \le \frac{k}{2}-1; \\ k+1-i, & \frac{k}{2} \le i \le k; \\ i-1-k, & k+1 \le i \le \frac{3k}{2}+2; \\ 2k+4-i, & \frac{3k}{2}+3 \le i \le 2k. \end{cases}$$

$$g_1(i) = \begin{cases} i, & 1 \le i \le \frac{k}{2}+1; \\ k+3-i, & \frac{k}{2}+2 \le i \le k+1; \\ i+1-k, & k+2 \le i \le \frac{3k}{2}; \\ 2k+2-i, & \frac{3k}{2} \le i \le 2k. \end{cases}$$

$$g_2(i) = \begin{cases} i+1 & i = 1; \\ i-1, & 2 \le i \le \frac{k}{2}+2; \\ k+4-i, & \frac{k}{2}+3 \le i \le k+1; \\ i-k, & k+2 \le i \le \frac{3k}{2}+1; \\ 2k+3-i, & \frac{3k}{2}+2 \le i \le 2k. \end{cases}$$

$$g_3(i) = \begin{cases} i+1, & 1 \le i \le \frac{k}{2}; \\ k+2-i, & \frac{k}{2}+1 \le i \le k+1; \\ i+2-k, & k+2 \le i \le \frac{3k}{2}+1; \\ 2k+3-i, & \frac{3k}{2}+2 \le i \le 2k. \end{cases} \tag{9}$$

It can be verified that there are no two vertices on the outer cycle with the same codes, and there are no two vertices in the inner cycle and outer cycle with the same codes. Moreover, no two vertices in the inner cycle have same codes. Hence, $W = \{u_1, u_2, u_{k+1}\}$ is a resolving set of $P(2k, k)$ for even $k \ge 6$. This means that $\dim(P(2k, k)) \le 3$ for even $k \ge 6$.

Table 13: The code $c_W(v)$ of v with respect to $W = \{u_1, v_{10}, v_{11}\}$ in $P(21,7)$.

v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$
u_1	$(0,3,3)$	u_2	$(1,2,3)$	u_3	$(2,3,2)$	u_4	$(3,4,3)$	u_5	$(3,3,4)$
u_6	$(4,2,3)$	u_7	$(5,3,2)$	u_8	$(4,3,3)$	u_9	$(3,2,3)$	u_{10}	$(3,1,2)$
u_{11}	$(2,2,1)$	u_{12}	$(1,3,2)$	v_1	$(1,4,4)$	v_2	$(2,1,4)$	v_3	$(3,4,1)$
v_4	$(3,5,4)$	v_5	$(2,4,5)$	v_6	$(3,1,4)$	v_7	$(4,4,1)$	v_8	$(3,4,4)$
v_9	$(2,3,4)$	v_{10}	$(3,0,3)$	v_{11}	$(3,3,0)$	v_{12}	$(2,4,3)$		

Case 2 ($k \equiv 1 \pmod 2$ and $k \geq 5$). Let $W = \{u_1, u_2, u_{k+1}, u_{k+2}\}$. Then the codes of the outer vertices are $c_W(u_i) = (h_1(i), h_2(i), h_3(i), h_4(i))$, where

$$h_1(i) = \begin{cases} i-1, & 1 \leq i \leq \frac{k+1}{2}+2; \\ k+4-i, & \frac{k+1}{2}+3 \leq i \leq k+1; \\ i+2-k, & k+2 \leq i \leq \frac{3k-1}{2}; \\ 2k+1-i, & \frac{3k+1}{2} \leq i \leq 2k. \end{cases}$$

$$h_2(i) = \begin{cases} i & i = 1; \\ i-2, & 1 \leq i \leq \frac{k+1}{2}+3; \\ k+5-i, & \frac{k+1}{2}+4 \leq i \leq k+2; \\ i+1-k, & k+3 \leq i \leq \frac{3k-1}{2}; \\ 2k+2-i, & \frac{3k+1}{2}+1 \leq i \leq 2k. \end{cases}$$

$$h_3(i) = \begin{cases} i+2, & 1 \leq i \leq \frac{k-1}{2}; \\ k+1-i, & \frac{k+1}{2} \leq i \leq k; \\ i-1-k, & k+1 \leq i \leq \frac{3k+1}{2}+2; \\ 2k+4-i, & \frac{3k+1}{2}+3 \leq i \leq 2k. \end{cases} \tag{10}$$

$$h_4(i) = \begin{cases} i+3 & i = 1; \\ i+1, & 2 \leq i \leq \frac{k+1}{2}; \\ k+2-i, & \frac{k+3}{2} \leq i \leq k+1; \\ i-2-k, & k+2 \leq i \leq \frac{3k+1}{2}+3; \\ 2k+5-i, & \frac{3k+1}{2}+4 \leq i \leq 2k. \end{cases}$$

Then the codes of the inner vertices are $c_W(v_i) = (g_1(i), g_2(i), g_3(i), g_4(i))$, where

$$g_1(i) = \begin{cases} i, & 1 \leq i \leq \frac{k+1}{2}+1; \\ k+3-i, & \frac{k+3}{2} \leq i \leq k; \\ i+1-k, & k+1 \leq i \leq \frac{3k+1}{2}; \\ 2k+2-i, & \frac{3k+3}{2} \leq i \leq 2k. \end{cases}$$

$$g_2(i) = \begin{cases} i+1 & i = 1; \\ i-1, & 2 \leq i \leq \frac{k+3}{2}; \\ k+4-i, & \frac{k+5}{2} \leq i \leq k+1; \\ i-k, & k+2 \leq i \leq \frac{3k+1}{2}; \\ 2k+3-i, & \frac{3k+3}{2} \leq i \leq 2k. \end{cases}$$

$$g_3(i) = \begin{cases} i+1, & 1 \leq i \leq \frac{k+1}{2}; \\ k+2-i, & \frac{k+3}{2} \leq i \leq k; \\ i-k, & k+1 \leq i \leq \frac{3k+1}{2}; \\ 2k+3-i, & \frac{3k+3}{2} \leq i \leq 2k. \end{cases}$$

$$g_4(i) = \begin{cases} i+2 & i = 1; \\ i, & 2 \leq i \leq \frac{k+1}{2}; \\ k+3-i, & \frac{k+3}{2} \leq i \leq k+1; \\ i-1-k, & k+2 \leq i \leq \frac{3k+3}{2}; \\ 2k+4-i, & \frac{3k+5}{2} \leq i \leq 2k. \end{cases} \tag{11}$$

It can be verified that there are no two vertices on the outer cycle with the same codes, and there are no two vertices in the inner cycle and outer cycle with the same codes. Moreover, no two vertices in the inner cycle have same codes. Hence, $W = \{u_1, u_2, u_{k+1}, u_{k+2}\}$ is a resolving set of $P(2k,k)$ for odd $k \geq 5$. This means that $\dim(P(2k,k)) \leq 4$ for odd $k \geq 5$. □

Theorem 11. *Let G be the graph $G = P(3k,k)$ with $k \geq 3$; then*

$$\dim(G) = \begin{cases} 4, & \text{if } k = 7; \\ 3, & \text{otherwise}. \end{cases} \tag{12}$$

Proof.

Case 1 ($k \equiv 0 \pmod 2$). If $k = 4$, let $W = \{u_1, v_{10}, v_{11}\}$. The code of v with respect to W in $P(12,4)$ is presented in Table 13 showing that $\dim(P(12,4)) \leq 3$.

If $k \geq 6$, let $W = \{u_1, v_{k/2+2}, v_{k+1}\}$. Then the codes of the outer vertices are $c_W(u_i) = (h_1(i), h_2(i), h_3(i))$ and the

codes of the inner vertices are $c_W(v_i) = (g_1(i), g_2(i), g_3(i))$, where

$$h_1(i) = \begin{cases} i-1, & 1 \le i \le \frac{k}{2}+2; \\ k+4-i, & \frac{k}{2}+3 \le i \le k+1; \\ i+2-k, & k+2 \le i \le \frac{3k}{2}+1; \\ 2k+4-i, & \frac{3k}{2}+2 \le i \le 2k+1; \\ i-2k+2, & 2k+2 \le i \le \frac{5k}{2}-1; \\ 3k-i+1, & \frac{5k}{2} \le i \le 3k. \end{cases}$$

$$g_1(i) = \begin{cases} i, & 1 \le i \le \frac{k}{2}+1; \\ k+3-i, & \frac{k}{2}+2 \le i \le k+1; \\ i+1-k, & k+2 \le i \le \frac{3k}{2}+1; \\ 2k+3-i, & \frac{3k}{2}+2 \le i \le 2k+1; \\ i-2k+1, & 2k+2 \le i \le \frac{5k}{2}; \\ 3k-i+2, & \frac{5k}{2}+1 \le i \le 3k. \end{cases}$$

$$h_2(i) = \begin{cases} \frac{k}{2}+1, & i = 1; \\ k-i-2, & 2 \le i \le \frac{k}{2}+2; \\ i-k+4, & \frac{k}{2}+3 \le i \le k+2; \\ 2k-i-1, & k+3 \le i \le \frac{3k}{2}+2; \\ i-2k+5, & \frac{3k}{2}+3 \le i \le 2k+2; \\ 3k-i-1, & 2k+3 \le i \le \frac{5k}{2}+2; \\ i-3k+5, & \frac{5k}{2}+3 \le i \le 3k. \end{cases}$$

$$g_2(i) = \begin{cases} \frac{k}{2}+2, & i = 1; \\ 0, & i = \frac{k}{2}+2; \\ 1, & i \in \left\{\frac{3k}{2}+2, \frac{5k}{2}+2\right\}; \\ k-i-1, & 2 \le i \le \frac{k}{2}+1; \\ i-k+5, & \frac{k}{2}+3 \le i \le k+2; \\ 2k-i, & k+3 \le i \le \frac{3k}{2}+1; \\ i-2k+6, & \frac{3k}{2}+3 \le i \le 2k+2; \\ 3k-i, & 2k+3 \le i \le \frac{5k}{2}+1; \\ i-3k+6, & \frac{5k}{2}+3 \le i \le 3k. \end{cases}$$

$$h_3(i) = \begin{cases} i+1, & 1 \le i \le \frac{k}{2}; \\ k+2-i, & \frac{k}{2}+1 \le i \le k+1; \\ i-k, & k+2 \le i \le \frac{3k}{2}+1; \\ 2k+3-i, & \frac{3k}{2}+2 \le i \le 2k+1; \\ i-2k+1, & 2k+2 \le i \le \frac{5k}{2}+1; \\ 3k-i+3, & \frac{5k}{2}+2 \le i \le 3k. \end{cases}$$

$$g_3(i) = \begin{cases} 0, & i = k+1; \\ 1, & i \in \{1, 2k+1\}; \\ i+2, & 2 \le i \le \frac{k}{2}; \\ k-i+3, & \frac{k}{2}+1 \le i \le k; \\ i-k+1, & k+2 \le i \le \frac{3k}{2}+1; \\ 2k-i+4, & \frac{3k}{2}+2 \le i \le 2k; \\ i-2k+2, & 2k+2 \le i \le \frac{5k}{2}+1; \\ 3k-i+4, & \frac{5k}{2}+2 \le i \le 3k. \end{cases} \tag{13}$$

It can be verified that there are no two vertices on the outer cycle with the same codes, and there are no two vertices in the inner cycle and outer cycle with the same codes. Moreover, no two vertices in the inner cycle have same codes. Hence, $W = \{u_1, v_{k/2+2}, v_{k+1}\}$ is a resolving set of $P(3k, k)$ for even $k \ge 6$. This means that $\dim(P(3k, k)) \le 3$ for even $k \ge 6$.

Case 2 ($k \equiv 1 \pmod 2$ and $k \ge 5$). If $k = 5$, let $W = \{v_4, v_{10}, v_{13}\}$. The code of v with respect to W in $P(15, 5)$ is presented in Table 14 showing that $\dim(P(15, 5)) \le 3$.

Note that the diameter of $P(15, 5)$ is 5; by Theorem 6, we have $\dim(P(15, 5)) \ge 3$.

If $k = 7$, we can confirm that $\dim(P(21, 7)) \ge 4$ by an exhaustive search. Let $W = \{v_{14}, v_{17}, v_{20}, v_{21}\}$. The code of v with respect to W in $P(21, 7)$ is presented in Table 15 showing that $\dim(P(21, 7)) \le 4$.

If $k \ge 9$, let $W = \{u_1, v_k, v_{(3k-3)/2}\}$. Then the codes of the outer vertices are $c_W(u_i) = (h_1(i), h_2(i), h_3(i))$ and the codes of the inner vertices are $c_W(v_i) = (g_1(i), g_2(i), g_3(i))$, where

$$h_1(i) = \begin{cases} i-1, & 1 \le i \le \frac{k+5}{2}; \\ k+4-i, & \frac{k+7}{2} \le i \le k+1; \\ i+2-k, & k+2 \le i \le \frac{3k+1}{2}; \\ 2k+2-i, & \frac{3k+3}{2} \le i \le 2k+1; \\ i-2k+2, & 2k+2 \le i \le \frac{5k-1}{2}; \\ 3k-i+1, & \frac{5k+1}{2} \le i \le 3k. \end{cases}$$

Table 14: The code $c_W(v)$ of v with respect to $W = \{v_4, v_{10}, v_{13}\}$ in $P(15,5)$.

v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$
u_1	$(4,3,4)$	u_2	$(3,4,3)$	u_3	$(2,4,2)$	u_4	$(1,3,3)$	u_5	$(2,2,4)$
u_6	$(3,3,4)$	u_7	$(4,4,3)$	u_8	$(3,3,2)$	u_9	$(2,2,3)$	u_{10}	$(3,1,4)$
u_{11}	$(4,2,3)$	u_{12}	$(4,3,2)$	u_{13}	$(3,4,1)$	u_{14}	$(2,3,2)$	u_{15}	$(3,2,3)$
v_1	$(5,4,5)$	v_2	$(4,5,4)$	v_3	$(3,5,1)$	v_4	$(0,4,4)$	v_5	$(3,1,5)$
v_6	$(4,4,5)$	v_7	$(5,5,4)$	v_8	$(4,4,1)$	v_9	$(1,3,4)$	v_{10}	$(4,0,5)$
v_{11}	$(5,3,4)$	v_{12}	$(5,4,3)$	v_{13}	$(4,5,0)$	v_{14}	$(1,4,3)$	v_{15}	$(4,1,4)$

$$g_1(i) = \begin{cases} i, & 1 \le i \le \frac{k+3}{2}; \\ k+3-i, & \frac{k+5}{2} \le i \le k+1; \\ i+1-k, & k+2 \le i \le \frac{3k+1}{2}; \\ 2k+3-i, & \frac{3k+3}{2} \le i \le 2k+1; \\ i-2k+1, & 2k+2 \le i \le \frac{5k+1}{2}; \\ 3k-i+2, & \frac{5k+3}{2} \le i \le 3k. \end{cases}$$

$$h_2(i) = \begin{cases} i+2, & 1 \le i \le \frac{k-1}{2}; \\ k+1-i, & \frac{k+1}{2} \le i \le k; \\ i+1-k, & k+1 \le i \le \frac{3k+1}{2}; \\ 2k+2-i, & \frac{3k+3}{2} \le i \le 2k; \\ i-2k+2, & 2k+1 \le i \le \frac{5k-1}{2}; \\ 3k-i+2, & \frac{5k+1}{2} \le i \le 3k. \end{cases}$$

$$g_2(i) = \begin{cases} 0, & i = k; \\ 1, & i \in \{2k, 3k\}; \\ i+3, & 1 \le i \le \frac{k-1}{2}; \\ k+2-i, & \frac{k+1}{2} \le i \le k-1; \\ i+2-k, & k+1 \le i \le \frac{3k+1}{2}; \\ 2k+3-i, & \frac{3k+3}{2} \le i \le 2k-1; \\ i-2k+3, & 2k+1 \le i \le \frac{5k-1}{2}; \\ 3k-i+3, & \frac{5k+1}{2} \le i \le 3k-1. \end{cases}$$

$$h_3(i) = \begin{cases} \frac{k+1}{2} - i, & 1 \le i \le \frac{k-3}{2}; \\ i-1, & \frac{k-1}{2} \le i \le k-2; \\ k-i+4, & k-1 \le i \le \frac{3k-3}{2}; \\ i-k-2, & \frac{3k-1}{2} \le i \le 2k-1; \\ 2k-i+5, & 2k \le i \le \frac{5k-3}{2}; \\ i-2k-1, & \frac{5k-1}{2} \le i \le 3k-2; \\ 3k-i+5, & 3k-1 \le i \le 3k. \end{cases}$$

$$g_3(i) = \begin{cases} 0, & i = \frac{3k-3}{2}; \\ 1, & i \in \left\{\frac{k-3}{2}, \frac{5k-3}{2}\right\}; \\ \frac{k+3}{2} - i, & 1 \le i \le \frac{k-5}{2}; \\ i, & \frac{k-1}{2} \le i \le k-2; \\ k-i+5, & k-1 \le i \le \frac{3k-5}{2}; \\ i-k-1, & \frac{3k-1}{2} \le i \le 2k-1; \\ 2k-i+6, & 2k \le i \le \frac{5k-5}{2}; \\ i-2k, & \frac{5k-1}{2} \le i \le 3k-2; \\ 3k-i+6, & 3k-1 \le i \le 3k. \end{cases} \tag{14}$$

It can be verified that there are no two vertices on the outer cycle with the same codes, and there are no two vertices in the inner cycle and outer cycle with the same codes. Moreover, no two vertices in the inner cycle have same codes. Hence, $W = \{u_1, v_k, v_{(3k-3)/2}\}$ is a resolving set of $P(3k,k)$ for odd $k \ge 9$. This means that $\dim(P(3k,k)) \le 3$ for odd $k \ge 9$.

Now, we will show that $\dim(P(3k,k)) \ge 3$. Note that for any $w \in U$, say $w = u_i$, we have $|N_2(w)| = 6$ (see Figure 1) and for any $w \in V$, say $w = v_i$, we have $|N_3(w)| = 8$ (see Figure 2). By Lemma 7, we have $\dim(P(3k,k)) \ge 3$. □

Table 15: The code $c_W(v)$ of v with respect to $W = \{v_{14}, v_{17}, v_{20}, v_{21}\}$ in $P(21, 7)$.

v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$	v	$c_W(v)$
u_1	$(3,4,3,2)$	u_2	$(4,3,4,3)$	u_3	$(5,2,5,4)$	u_4	$(5,3,4,5)$	u_5	$(4,4,3,4)$
u_6	$(3,5,2,3)$	u_7	$(2,5,3,2)$	u_8	$(3,4,4,3)$	u_9	$(4,3,5,4)$	u_{10}	$(5,2,5,5)$
u_{11}	$(4,3,4,5)$	u_{12}	$(3,4,3,4)$	u_{13}	$(2,5,2,3)$	u_{14}	$(1,4,3,2)$	u_{15}	$(2,3,4,3)$
u_{16}	$(3,2,5,4)$	u_{17}	$(4,1,4,5)$	u_{18}	$(5,2,3,4)$	u_{19}	$(4,3,2,3)$	u_{20}	$(3,4,1,2)$
u_{21}	$(2,5,2,1)$	v_1	$(4,5,4,3)$	v_2	$(5,4,5,4)$	v_3	$(6,1,6,5)$	v_4	$(6,4,5,6)$
v_5	$(5,5,4,5)$	v_6	$(4,6,1,4)$	v_7	$(1,6,4,1)$	v_8	$(4,5,5,4)$	v_9	$(5,4,6,5)$
v_{10}	$(6,1,6,6)$	v_{11}	$(5,4,5,6)$	v_{12}	$(4,5,4,5)$	v_{13}	$(3,6,1,4)$	v_{14}	$(0,5,4,1)$
v_{15}	$(3,4,5,4)$	v_{16}	$(4,3,6,5)$	v_{17}	$(5,0,5,6)$	v_{18}	$(6,3,4,5)$	v_{19}	$(5,4,3,4)$
v_{20}	$(4,5,0,3)$	v_{21}	$(1,6,3,0)$						

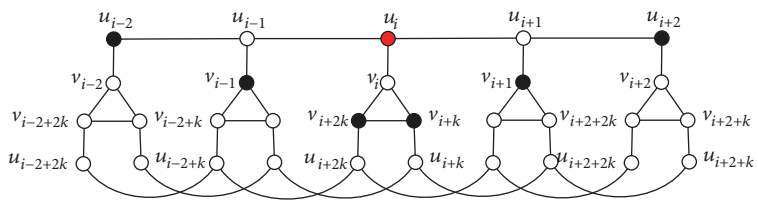

Figure 1: The set of black vertices is $N_2(w)$ for $w = u_i$.

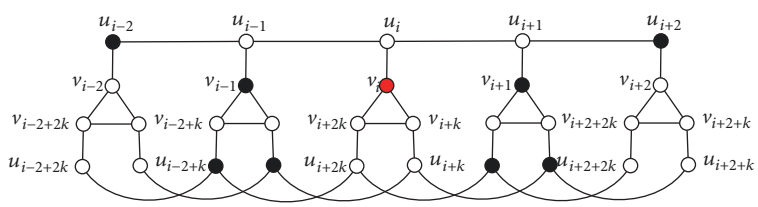

Figure 2: The set of black vertices is $N_3(w)$ for $w = v_i$.

Data Availability

The resolving sets of some generalized Petersen graphs can also be found in https://www.researchgate.net/publication/324182258_Resolving_sets_of_some_generalized_Petersen_graphs_providing_the_corresponding_upper_bounds_for_metric_dimension?.

Disclosure

The authors confirm that the paper has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. The authors further confirm that the order of authors listed in the paper has been approved by all of them.

Conflicts of Interest

The authors wish to confirm that there are no known conflicts of interest associated with this paper and there has been no significant financial support for this work that could have influenced its outcome.

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