Research Article

Pricing Warrant Bonds with Credit Risk under a Jump Diffusion Process

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This article investigates the pricing of the warrant bonds with default risk under a jump diffusion process. We assume that the stock price follows a jump diffusion model while the interest rate and the default intensity have the feature of mean reversion. By the risk neutral pricing theorem, we obtain an explicit pricing formula of the warrant bond. Furthermore, numerical analysis is provided to illustrate the sensitivities of the proposed pricing model.

1. Introduction

In recent years, warrant bond is one of the major investment instruments in financial market. The warrant bonds are made to keep the features of both convertible bonds and warrants. The holder may convert the bond into a predetermined number of stock or continue to hold the bond to maturity depending on the market. Differently from the convertible bond, the essential characteristic of the warrant bond is that the bond and the option are separable. That is to say, when the bond is converted into stock, the value of the bond still exists.

The seminal work of Brennan and Schwartz [1] and Ingersoll [2] popularized the studies on pricing convertible bond. Liao and Huang [3] considered the pricing of convertible bond with credit risk under the geometric Brownian motion model. Zhou and Wang [4] assumed that the interest rate follows the geometric Brownian motion and obtained the valuation of convertible bond by the method of measure transformation. Laura and Ioannis [5] defined the firm’s optimal call policy and proposed the pricing framework for convertible bond based on a structural default model. There has been a considerable interest in investigating the valuation of warrant bond since the study of Payne et al. [6]. Zhu [7] extended Payne et al. [6] to a stochastic interest rate frame and considered the pricing of warrant bond. It is well known that traditional asset price models fail to handle discrete movements (such as random environment, market trends, interest rates, business cycles, etc.). To reflect the reality, Wang and Zhao [8] used a regime switching model to describe the price dynamics of asset and investigated the pricing of warrant bond. Chen [9] assumed that the stochastic interest rate and the underlying stock follow fractional Brownian motion, respectively, and deduced the pricing formula of warrant bond. Hu et al. [10] built a structure model under portfolio constraints, discussed the pricing of warrant bond and investment portfolios under prohibition of short-selling and borrowing, and obtained an arbitrage-free price interval.

The aforementioned papers have made significant contributions to the study of pricing convertible bonds and warrant bonds. Since the 2008 financial crisis, the credit risk has been one of the most important sources of risks that should be taken into account. Bond holders also face credit risk as bonds issuer may default before the bond is delivered. Among a vast amount of literature on credit risk, two main approaches are used to model credit risk: structural model and reduced form model. The structural model is originated by Black and Scholes [11]. Furthermore, Merton [12] assumed that the default is specified as the firm’s asset value reaches...
a specific threshold boundary. Major investigations about the
specific model are to characterize the evolution of the firm’s
value and capital structure. Related papers include Merton
[12], Johnson and Stulz [13], Klein and Inglis [14], Ammann
[15], and Wang and Wang [16]. In contrast, the reduced form
model which considers that the default is controlled by an
exoogenous intensity process is more flexible and tractable in
the real market. Since the pioneering work by Jarrow and
Turnbull [17], more advanced settings and methods have been
proposed on the reduced form model, such as Jarrow and Yu
[18], Su and Wang [19], Liang et al. [20], and Wang et al. [21].

This article investigates the pricing of warrant bonds with
credit risk. From the characteristic of the warrant bond, we
find that its value can be divided into the value of a bond and
the value of a call option. In order to price the warrant bond,
we should utilize the theory of option pricing. It is known
that certain vital features of financial time series cannot be
depicted by the classical Black-Scholes models. Therefore,
Merton [12] and many scholars introduced the jump diffusion
process to describe the pricing dynamics of assets and improved
the pricing model of Black-Scholes. Comparing with these
studies about warrant bonds, the differences between theirs
and ours are evident. First, based on Merton [12], we assume
that the stock price follows a jump diffusion model in order
to capture its large or sudden changes. Second, we use a
reduced form model to describe the default risk. Finally, we
provide numerical experiments to illustrate the effect of some
parameters on the price of the warrant bond.

The rest of the paper is organized as follows. In Section 2,
we give some basic assumptions of the model. In Section 3,
we derive the pricing of the warrant bonds. In Section 4, we
present some numerical analysis of the result obtained.

2. Modeling Framework

2.1. The Underlying Market. Let \( T > 0 \) be a finite time
horizon and \((\Omega, F , (F_t )_{t \geq 0}, Q)\) be a filtered probability
space satisfying the usual conditions. Let \( Q \) represent an equivalent
martingale measure under which the discounted asset price
\( \mathcal{Q} \)

\[
\begin{align*}
&dS_t = r_t S_t dt + \sigma_t S_t dW_t^S - \nu B_t dt + \delta \sum_{i=1}^N X_i,
\end{align*}
\]

where \( r_t \) is the instantaneous interest rate and \( \sigma_t > 0 \) is
the volatility of \( S_t \). If the jump happens, the jump size is
controlled by independent identical distributed random vari-
able \( X_i(X_i > -1, i = 1, 2, \ldots) \). Here, \( X_i > -1 \) is to make
sure that the stock price is nonnegative. Furthermore, we
denote \( f(y) \) as the probability density of \( \ln(1 + X_i) \) and \( \beta =
E_Q[X_i] \), where \( E_Q[\cdot] \) denotes the mathematical expectation
under the probability measure \( Q \). Throughout this paper,
we suppose that \( \{N_t \}_{t \in [0,T]} \) and \( \{W_t \}_{t \in [0,T]} \) are mutually
independent.

In addition, the money market account \( B = (B_t) \) and the
market interest rate \( r = (r_t) \) are governed by

\[
\begin{align*}
dB_t &= r_t B_t dt, \\
B_0 &= 1,
\end{align*}
\]

where \( k > 0, \theta > 0, \) and \( \sigma_\gamma > 0 \) represent the speed of
reversion, the long term mean level, and the volatility of \( r_t \),
respectively.

In this article, we use the reduced form model proposed in
Jarrow and Turnbull [17] to model the default risk. Let \( \tau \) denote
the default time of the warrant bonds issuer with
default intensity process \( \lambda_t \). We model the default intensity
\( \lambda_t \) having the feature of mean reversion

\[
d\lambda_t = a(b - \lambda_t) dt + \sigma_\gamma dW_t^\gamma,
\]

where \( a > 0, b > 0, \) and \( \sigma_\gamma > 0 \) represent the speed of
reversion, the long term mean level, and the volatility of \( \lambda_t \),
respectively.

Furthermore, the filtration \( \mathcal{F}_t \) is generated by
\( \mathcal{F}_t = \mathcal{F}^S_t \lor \mathcal{F}_t^\gamma \lor \mathcal{H}_t \), where \( \mathcal{F}^S_t = \sigma(S_s, s \leq t), \mathcal{F}^\gamma_t = \sigma(r_s, s \leq t), \mathcal{F}_t^\gamma = \sigma(\lambda_s, s \leq t), \) and \( \mathcal{H}_t = \sigma(\theta, s \leq t) \). Define a new
filtration \( \mathcal{G}_t = \mathcal{F}^S_t \lor \mathcal{F}^\gamma_t \lor \mathcal{H}_t \), and \( \mathcal{G}_0 = \mathcal{F}^S_0 \lor \mathcal{F}^\gamma_0 \lor \mathcal{H}_0 \).

We adopt the assumption of Jarrow and Yu [18]; the
conditional and unconditional distributions of \( \tau \) are given by

\[
Q(\tau > t | \mathcal{G}_0) = \exp\left( -\int_0^t \lambda_s ds \right),
\]

\[
Q(\tau > t) = E_Q\left[ \exp\left( -\int_0^t \lambda_s ds \right) \right], \quad t \in [0, T].
\]

2.2. Warrant Bonds. A warrant bond (see Payne et al. [6])
offers the investor the option to convert it into a predetermined
amount of stock or continue to hold the bond to
maturity. When the bond is converted into stock, the value
of the bond still exists. We assume that the holder chooses
to convert the bond into stock only at expiration time \( T \).
Thus, the value of the warrant bond can be divided into two
parts, the value of a bond and the value of a European call
option. The assumption about the conversion time \( T^* \) may be
more realistic if we assume that \( T^* \in [0, T] \). Wang and Bian
[22], Yang et al. [23], and Laura and Ioannis [5] considered
the pricing of convertible bonds when the holder converts
the bond into stocks before maturity. The major differences
between their papers and this one are the following: first,
Wang and Bian [22] assumed that the stock price is driven
by a Poisson process and the interest rate is constant. Second, in Yang et al. [23] the interest rate and default intensity were assumed to be constants. Finally, Laura and Ioannis [5] described the default risk based on a structural default model. As mentioned above, we make assumptions about the stock price, the interest rate, and the default intensity as described by (2), (3), and (4). In fact, the result may not have explicit solution for the price of the warrant bond if the conversion time is chosen at any time before $T$ under our pricing frame and we shall explore such extension in future works. Then, the cash flows of the warrant bond at $T$ can be expressed as follows:

$$V(T) = I_{(t<T)} \Psi_T + I_{(t\geq T)} \omega \Psi_T.$$  \hspace{1cm} (7)

3. Pricing the Warrant Bonds with Credit Risk

In this section we investigate the pricing of the warrant bonds with credit risk. By the risk neutral valuation formula, under the equivalent martingale measure $Q$, the valuation at time $t$ of the warrant bond is given by

$$V(t, T) = \mathbb{E}_Q \left[ e^{-\int_t^T r_s du} \left( I_{(t<T)} \Psi_T + I_{(t\geq T)} \omega \Psi_T \right) \mid \mathcal{F}_t \right].$$  \hspace{1cm} (8)

In terms of the default intensity, we obtain the following expression:

$$V(t, T) = \omega \mathbb{E}_Q \left[ e^{-\int_t^T r_s du} \Psi_T \mid \mathcal{F}_t \right] + (1 - \omega) \mathbb{E}_Q \left[ e^{-\int_t^T r_s du} I_{(t<T)} \Psi_T \mid \mathcal{F}_t \right].$$  \hspace{1cm} (9)

We substitute formula (6) into (9) and obtain

$$V(t, T) = \omega \left\{ \mathbb{E}_Q \left[ e^{-\int_t^T r_s du} P_b \mid \mathcal{F}_t \right] + \mathbb{E}_Q \left[ e^{-\int_t^T r_s du} \alpha S_T I_{(S_T \geq C_v)} \mid \mathcal{F}_t \right] - \mathbb{E}_Q \left[ e^{-\int_t^T r_s du} \gamma C_v I_{(S_T \geq C_v)} \mid \mathcal{F}_t \right] \right\} + (1 - \omega) \cdot I_{(t\geq T)} \left\{ \mathbb{E}_Q \left[ e^{-\int_t^T (r_s + \lambda_b) du} P_b \mid \mathcal{F}_t \right] + \mathbb{E}_Q \left[ e^{-\int_t^T (r_s + \lambda_b) du} \gamma C_v I_{(S_T \geq C_v)} \mid \mathcal{F}_t \right] - \mathbb{E}_Q \left[ e^{-\int_t^T (r_s + \lambda_b) du} \gamma S_T I_{(S_T \geq C_v)} \mid \mathcal{F}_t \right] \right\}.$$  \hspace{1cm} (10)

For simplifying the notations, denote

$$I_1 = \mathbb{E}_Q \left[ e^{-\int_t^T r_s du} P_b \mid \mathcal{F}_t \right];$$
$$I_2 = \mathbb{E}_Q \left[ e^{-\int_t^T r_s du} \gamma S_T I_{(S_T \geq C_v)} \mid \mathcal{F}_t \right];$$
$$I_3 = \mathbb{E}_Q \left[ e^{-\int_t^T r_s du} \gamma C_v I_{(S_T \geq C_v)} \mid \mathcal{F}_t \right];$$
$$I_4 = \mathbb{E}_Q \left[ e^{-\int_t^T (r_s + \lambda_b) du} P_b \mid \mathcal{F}_t \right];$$
$$I_5 = \mathbb{E}_Q \left[ e^{-\int_t^T (r_s + \lambda_b) du} \gamma S_T I_{(S_T \geq C_v)} \mid \mathcal{F}_t \right];$$
$$I_6 = \mathbb{E}_Q \left[ e^{-\int_t^T (r_s + \lambda_b) du} \gamma C_v I_{(S_T \geq C_v)} \mid \mathcal{F}_t \right].$$  \hspace{1cm} (11 to 14)

Then $V(t, T)$ can be rewritten as

$$V(t, T) = \omega (I_1 + I_2 - I_3) + (1 - \omega) I_{(t\geq T)} (I_4 + I_5 - I_6).$$  \hspace{1cm} (15)

3.1. The Useful Lemmas. In the following, we calculate $I_1, I_2, I_3, I_4, I_5, I_6$, respectively. In order to use the method of measure transformation to obtain the price of the warrant bonds, we first present two lemmas to introduce two new measures $Q^T$ and $Q^*$. Let $P(t, T)$ denote the price of the zero coupon bond at time $t$, with maturity $T$. From (11), we have

$$I_1 = R^T P(t, T).$$  \hspace{1cm} (16)

According to Jaimungal and Wang [24], we get the zero coupon with the affine structure as follows:

$$P(t, T) = \exp \left( -r_t \sigma(t, T, k) + A(t, T) \right).$$  \hspace{1cm} (17)
where
\[ \sigma(t,T,k) = \frac{1 - e^{-k(T-t)}}{k}, \]
\[ A(t,T) = \left( \theta - \frac{\sigma^2}{2k^2} \right) \left( \sigma(t,T,k) - (T-t) \right) - \frac{\sigma^2}{4k} \sigma^2(t,T,k). \]

Moreover, \( P(t,T) \) satisfies

\[ dP(t,T) = r_t P(t,T) \, dt - \sigma \sigma(t,T,k) P(t,T) \, dW_t^r. \]  

In the presence of stochastic interest rate, we will define the forward-neutral measure \( Q^T \) equivalent to the risk neutral measure \( Q \) by Lemma 1.

**Lemma 1.** Let \( \eta_{IT} \) denote the Radon-Nikodým derivative

\[ \eta_{IT} = \frac{dQ^T}{dQ} = \frac{P(T,T)}{P(0,T)B_T^r} \]

and, then,

\[ \tilde{W}_t^r = W_t^r + \int_0^t \sigma \sigma(u,T,k) \, du, \]
\[ \tilde{W}_t^s = W_t^s + \int_0^t \rho \sigma \sigma(u,T,k) \, du, \]
\[ \tilde{W}_t^\lambda = W_t^\lambda + \int_0^t \rho \sigma \sigma(u,T,k) \, du \]

are the standard Brownian motions under measure \( Q^T \). The covariance matrix of \( \tilde{W}_t^r, \tilde{W}_t^s, \tilde{W}_t^\lambda \) is the same as \( W_t^r, W_t^s, W_t^\lambda \). Moreover, the intensity of \( N_t \) and the distribution of \( X_t \) under \( Q^T \) are the same as those under \( Q \).

**Proof.** From (19) and (20), the Radon-Nikodým derivative \( \eta_{IT} \) is given by

\[ \eta_{IT} = \frac{dQ^T}{dQ} = \exp \left\{ -\int_0^T \sigma \sigma(u,T,k) \, dW_u^r - \frac{1}{2} \int_0^T \sigma \sigma^2(u,T,k) \, du \right\}. \]

By virtue of Girsanov’s theorem, we immediately get the result of Lemma 1.

By Bayes rule, \( I_3 \) can be calculated under \( Q^T \):

\[ I_3 = E_Q \left[ e^{-\int_0^T r_u \, du} \alpha \gamma C \alpha \gamma C \delta \left( t, r \right) \bigg| \mathcal{F}_t \right] \]

\[ = \alpha \gamma C \alpha \gamma C p(t,T) E_{Q^T} \left[ \delta \left( t, r \right) \bigg| \mathcal{F}_t \right]. \]

According to Lemma 1 and the Itô lemma, we can rewrite \( S_T \) under \( Q^T \) as

\[ S_T = S_t \exp \left\{ \theta (T-t) \right\} \left( r_t - \theta \right) + \left( r_t - \theta \right) \sigma(t,T,k) \]

\[ - \frac{\sigma^2}{2} (T-t) - \frac{\sigma^2}{4k} \sigma^2(t,T,k), \]

\[ + \sigma \left( \tilde{W}_t^r \tilde{W}_t^r - \tilde{W}_t^r \tilde{W}_t^r \right) + \int_t^T \sigma \sigma(u,T,k) \, dW_u^r \]

\[ - \left[ \int_t^T \sigma \sigma^2(u,T,k) \, du \right] \]

\[ + \left[ \int_t^T \rho \sigma \sigma(u,T,k) \, du \right]. \]

By the law of iterated conditional expectation, we obtain that

\[ I_3 = \alpha \gamma C \alpha \gamma C P(t,T) E_{Q^T} \left[ I_{[\delta \geq \theta]} \bigg| \mathcal{F}_t \right]. \]

\[ \left\{ \sum_{n=1}^{N_t} e^{-\nu(t-T)} \nu(t-T)^n \right\} \]

\[ - \left[ \int_{-\infty}^{\infty} N(\nu(t,T,y)) f^n(y) \, dy \right] \]

\[ + e^{-\nu(t-T)} N(\nu(t,T,0)) \right\}, \]

where \( N(\cdot) \) denotes the cumulative distribution function for a standard normal random variable, \( f^n(y) \) is the \( n \)-th convolution of the density function \( f(y) \) of \( \ln(1 + X_t) \), and \( d_3(t,T,y) \) is given by formula (39) in Theorem 3. Further,

\[ \Lambda(t,T) = (\theta - \frac{\sigma^2}{2} \nu(T-t)) + (r_t - \theta) \sigma(t,T,k) \]

\[ - \int_t^T \sigma \sigma^2(u,T,k) \, du \]

\[ - \int_t^T \rho \sigma \sigma(u,T,k) \, du. \]

Let

\[ X(t,T) = E_{Q^T} \left[ e^{-\int_0^T (r_u + k) \, du} \bigg| \mathcal{F}_t \right]. \]

From (12), we get

\[ I_3 = P_2 X(t,T). \]
By (3) and (4), we have
\[
\int_t^T (r_u + \lambda_u) \, du = (\theta + b) (r_t - \theta) \sigma (t,T,k) \\
+ (\lambda_t - b) \sigma (t,T,a) \\
+ \int_t^T \sigma_2 \sigma (u,T,k) \, dW_u^r \\
+ \int_t^T \sigma_2 \sigma (u,T,a) \, dW_u^\lambda.
\]

Direct calculation yields
\[
X(t,T) = \exp \left\{ - (\theta + b) (T-t) - (r_t - \theta) \sigma (t,T,k) \\
- \frac{1}{2} \sigma_1^2 (T-t) - \beta (T-t) \\
- \int_t^T \sigma_2 \sigma (u,T,k) M_1(u) \, du \\
- \int_t^T \rho_{12} \sigma_1 M_1(u) \, du \\
- \int_t^T \rho_{13} \sigma_2 M_2(u) \, du \right\}.
\]

Next, we introduce Lemma 2.

**Lemma 2.** Define a measure $Q^\lambda$ by the Radon-Nikodym derivative
\[
\eta_{2T} = \frac{dQ^\lambda}{dQ} = e^{-\int_t^T (r_u + \lambda_u) \, du} \left[ e^{-\int_t^T (r_u + \lambda_u) \, du} \right],
\]
and then
\[
\bar{W}_t^{r} = W_t^{r} + \int_t^T M_1(u) \, du,
\]
\[
\bar{W}_t^{\lambda} = W_t^{\lambda} + \int_t^T M_2(u) \, du,
\]
\[
\bar{W}_t^{S} = W_t^{S} + \rho_{12} \int_t^T M_1(u) \, du + \rho_{13} \int_t^T M_2(u) \, du.
\]
are standard $Q^\lambda$ Brownian motions, where $M_1(u) = \sigma_2 \sigma (u,T,k) + \rho_{23} \sigma_3 \sigma (u,T,a)$, and $M_2(u) = \sigma_2 \sigma (u,T,a) + \rho_{23} \sigma_3 \sigma (u,T,k)$. The covariance matrix of $(\bar{W}_t^{r}, \bar{W}_t^{\lambda}, \bar{W}_t^{S})$ is the same as $(W_t^{r}, W_t^{r}, W_t^{\lambda})$. Moreover, the intensity of $N_t$ and the distribution of $X_t$ under $Q^\lambda$ are the same as those under $Q$.

**Proof.** Analogously to the proof of Lemma 1, we can get the Radon-Nikodym derivative
\[
\eta_{2T} = \exp \left\{ - \int_0^T \sigma_2 \sigma (u,T,k) \, dW_u^r \\
- \int_0^T \sigma_2 \sigma (u,T,a) \, dW_u^\lambda \\
+ \frac{1}{2} \int_0^T \sigma_2 \sigma (u,T,k) \, dW_u^r \\
+ \frac{1}{2} \int_0^T \sigma_2 \sigma (u,T,a) \, dW_u^\lambda \\
+ \int_0^T \rho_{23} \sigma_3 \sigma (u,T,k) \sigma (u,T,a) \, du \right\}.
\]

By virtue of Girsanov’s theorem, we can complete the proof.

From Lemma 2, Itô lemma, and (2), $S_T$ can be written as
\[
S_T = S_t \exp \left\{ M(t,T) + \sigma_1 \left( \bar{W}_T^S - \bar{W}_t^S \right) \\
+ \int_t^T \sigma_2 \sigma (u,T,k) \, d\bar{W}_u^r + \sum_{i=N_1+1}^{N_T} \ln(1 + X_i) \right\},
\]
where
\[
M(t,T) = \theta (T-t) + (r_t - \theta) \sigma (t,T,k) \\
- \frac{1}{2} \sigma_1^2 (T-t) - \gamma \beta (T-t) \\
- \int_t^T \sigma_2 \sigma (u,T,k) M_1(u) \, du \\
- \int_t^T \rho_{12} \sigma_1 M_1(u) \, du \\
- \int_t^T \rho_{13} \sigma_2 M_2(u) \, du.
\]

Thus, by Bayes rule and the law of iterated conditional expectation, we get
\[
I_6 = \alpha \gamma C_s X(t,T) E_{Q^\lambda} \left[ i \left( S_t \geq C_s \right) \mid \mathcal{F}_t \right]
\]
\[
\vee \sigma \left( \sum_{i=N_1+1}^{N_T} \ln(1 + X_i) \right) \mid \mathcal{F}_t \right] = \alpha \gamma C_s X(t,T)
\]
\[
\left[ \sum_{n=1}^{N_T} e^{-\gamma (T-t)^n} (T-t)^n \right]
\]
\[
\cdot \int_{-\infty}^{\infty} N \left( d_0 (t,T,y) \right) f^n (y) \, dy
\]
\[
\cdot e^{-\gamma (T-t)} \left( d_0 (t,T,0) \right).
\]
where $d_0 (t,T,y)$ can be obtained by formula (41) in Theorem 3.
3.2. Main Results. In the following, we give the main result in Theorem 3.

**Theorem 3.** The price of the warrant bond with credit risk under the jump diffusion model at time $t$ is

$$V(t, T) = \omega \left[ P_b P(t, T) - \alpha \gamma C_r P(t, T) \right]$$

$$\cdot \left[ \sum_{n=1}^{\infty} \frac{e^{-\nu(t-t)} \nu^n (T-t)^n}{n!} \right]$$

$$+ \int_{-\infty}^{\infty} N(d_3(t, T, y)) f^m(y) dy$$

$$+ \alpha \gamma S_t \left[ e^{-\nu(T-t)} N(d_2(t, T, 0)) \right]$$

$$+ \sum_{n=1}^{\infty} \frac{e^{-\nu(T-t)} \nu^n (T-t)^n}{n!}$$

$$\cdot \left[ \int_{-\infty}^{\infty} N(d_3(t, T, y)) f^m(y) dy \right] - \alpha \gamma C_r X(t, T)$$

$$+ \sum_{n=1}^{\infty} \frac{e^{-\nu(T-t)} \nu^n (T-t)^n}{n!}$$

$$\cdot \left[ \int_{-\infty}^{\infty} N(d_3(t, T, y)) f^m(y) dy \right]$$

$$+ e^{-\nu(T-t)} N(d_2(t, T, 0)) \right]$$

(37)

where $N(\cdot)$ is the cumulative distribution function of a standard normal distribution, $\nu = \nu^\ast = \nu (\beta + 1)$, and $f^n(y)$, $\tilde{f}^n(y)$, $f^{m}(y)$ denote the $n$-th convolution of $f(y)$, $\tilde{f}(y)$, $f^*(y)$, respectively. The definition of $\Lambda(t, T), X(t, T), M(t, T)$, and $Y(t, T)$ can be referred to in (26), (30), (35), and (44). Further,

$$d_2(t, T, y) = \frac{\ln (S_t / C_r) + \Gamma(t, T) + y}{\sqrt{1/2 \int_{-\infty}^{\infty} \sigma^2 \sigma^2 (u, T, k) du + (1/2) \sigma^2 (T-t) + \int_{-\infty}^{\infty} \rho_{12} \sigma_1 \sigma_2 \sigma (u, T, k) du}},$$

(38)

$$d_3(t, T, y) = \frac{\ln (S_t / C_r) + \Lambda(t, T) + y}{\sqrt{1/2 \int_{-\infty}^{\infty} \sigma^2 \sigma^2 (u, T, k) du + (1/2) \sigma^2 (T-t) + \int_{-\infty}^{\infty} \rho_{12} \sigma_1 \sigma_2 \sigma (u, T, k) du}},$$

(39)

$$d_4(t, T, y) = \frac{\ln (S_t / C_r) + M(t, T) + \int_{-\infty}^{\infty} \sigma_2 \sigma (u, T, k) M_2 (u) du + \int_{-\infty}^{\infty} \sigma_1 \sigma_2 \sigma (u, T, k) du + y}{\sqrt{1/2 \int_{-\infty}^{\infty} \sigma^2 \sigma^2 (u, T, k) du + (1/2) \sigma^2 (T-t) + \int_{-\infty}^{\infty} \rho_{12} \sigma_1 \sigma_2 \sigma (u, T, k) du}},$$

(40)

$$d_6(t, T, y) = \frac{\ln (S_t / C_r) + M(t, T) + y}{\sqrt{1/2 \int_{-\infty}^{\infty} \sigma^2 \sigma^2 (u, T, k) du + (1/2) \sigma^2 (T-t) + \int_{-\infty}^{\infty} \rho_{12} \sigma_1 \sigma_2 \sigma (u, T, k) du}}.$$
and the density function of \(Q\),

\[
\Gamma(t, T) = \left( \theta + \frac{\sigma^2}{2} - \nu \beta \right) (T - t) + \left( \sigma_t - \theta \right) \sigma(t, k) + \int_t^T \rho_{12} \sigma_1 \sigma_2 \sigma(u, T, k) \, du.
\]

In addition, by Lemma 2 and Bayes rules, we get

\[
I_5 = \alpha \gamma X(t, T) E_{Q^1}\left[ S_T | \mathcal{F}_t \right].
\]

For the calculation of \(I_5\), we perform a measure change to \(Q^1\) by the Radon-Nikodym derivative

\[
\frac{dQ^1}{dQ} = \frac{S_T}{E_{Q^1}[S_T]}.
\]

Then, a direct application of Girsanov’s theorem implies that \(\tilde{W}_t = \tilde{W}_t' - \int_0^t M_2(u) \, du, \tilde{W}_t = \tilde{W}_t' - \int_0^t M_4(u) \, du,\) and \(\tilde{W}_t' = \tilde{W}_t' - \int_0^t M_3(u) \, du - \int_0^t \rho_{13} M_4(u) \, du\) are standard \(Q^1\) Brownian motions, where \(M_2(u) = \sigma_2 \sigma(u, T, k) + \rho_{12} \sigma_1,\) and \(M_4(u) = \sigma_1 + \rho_{12} \sigma_2 (u, T, k).\) The intensity of \(N\) is \(\nu = \nu(\beta + 1),\) and the density function of \(\ln(1 + X_t)\) is \(f^* (y) = e^y f(y) / (1 + \beta).\) Here, we can calculate \(I_5\) in the following way:

\[
I_5 = \alpha \gamma X(t, T) E_{Q^1}\left[ S_T | \mathcal{F}_t \right] E_{Q^1}\left[ I_{(S_T \geq C)} | \mathcal{F}_t \right]
= \alpha \gamma X(t, T) \tilde{Y}(t, T) E_{Q^1}\left[ I_{(S_T \geq C)} | \mathcal{F}_t \right]
= \alpha \gamma X(t, T) \tilde{Y}(t, T) \sum_{n=1}^{\infty} e^{-\nu(t-T)} \left( \nu^* \right)^n (T - t)^n \frac{n!}{n!}
\cdot \int_{-\infty}^{\infty} N \left( d_2 (t, T, y) \right) f^* (y) \, dy + e^{-\nu} (T - t)
\cdot N \left( d_2 (t, T, 0) \right),
\]

where

\[
\tilde{Y}(t, T) = E_{Q^1}\left[ S_T | \mathcal{F}_t \right] = S_t \exp \left\{ \int_t^T \sigma_2 \sigma_2 (u, T, k) \, du \right\}
+ \nu \beta (T - t) + \frac{1}{2} \int_t^T \sigma_2^2 \sigma_2 (u, T, k) \, du
\]

\[
I_5 = \alpha \gamma X(t, T) \tilde{Y}(t, T) \sum_{n=1}^{\infty} e^{-\nu(t-T)} \left( \nu^* \right)^n (T - t)^n \frac{n!}{n!}
\cdot \int_{-\infty}^{\infty} N \left( d_2 (t, T, y) \right) f^* (y) \, dy + e^{-\nu} (T - t)
\cdot N \left( d_2 (t, T, 0) \right).
\]

Combining (15), (16), (25), (28), (36), (43), and (47), we can obtain the result of the theorem.

In the following, we present a few remarks below to discuss some special results.

**Remark 4.** When \(\omega = 1\), (37) reduces to the formula for the price of the warrant bond under a jump diffusion without credit risk. This result is consistent with Wang and Zhao [8]. In this case, (37) is simplified to

\[
V(t, T) = P_2 (t, T) - \alpha \gamma C_2 P(t, T)
\cdot \sum_{n=1}^{\infty} e^{-\nu(t-T)} \left( \nu^* \right)^n (T - t)^n \frac{n!}{n!}
\cdot \int_{-\infty}^{\infty} N \left( d_3 (t, T, y) \right) f^* (y) \, dy
+ e^{-\nu} (T - t) \cdot N \left( d_3 (t, T, 0) \right)
+ \alpha \gamma S_1 \cdot \int_{-\infty}^{\infty} N \left( d_2 (t, T, y) \right) f^* (y) \, dy.
\]

**Remark 5.** If the stock price is modeled without compound Poisson jump, the result of (37) is given by the following formula which is similar to that of Zhu [7]:

\[
V(t, T) = P_2 (t, T) - \alpha \gamma C_2 P(t, T) N \left( d_3 (t, T) \right)
+ \alpha \gamma S_1 \cdot N \left( d_2 (t, T) \right),
\]

where

\[
d_2 (t, T) = \frac{\ln \left( S_t / C_0 \right) + \Gamma(t, T)}{\sqrt{1/2} \int_t^T \sigma_2^2 \sigma_2 (u, T, k) \, du + (1/2) \sigma_1^2 (T - t) + \int_t^T \rho_{12} \sigma_1 \sigma_2 \sigma(u, T, k) \, du}.
\]
\[ d_3(t,T) = \frac{\ln \left( \frac{S_t}{C_v} \right) + \Lambda(t,T)}{\sqrt{\frac{1}{2} \int_t^T \sigma^2(u,T) \, du + \frac{1}{2} \int_t^T \rho_{12} \sigma_1 \sigma_2 \sigma(u,T,k) \, du}} \]

\[ \Gamma(t,T) = \left( \theta + \frac{\sigma_1^2}{2} - \nu \beta \right)(T-t) + (r_t - \theta) \sigma(t,T,k) + \int_t^T \rho_{12} \sigma_1 \sigma_2 \sigma(u,T,k) \, du, \]

\[ \Lambda(t,T) = \left( \theta - \frac{\sigma_1^2}{2} - \nu \beta \right)(T-t) + (r_t - \theta) \sigma(t,T,k) - \int_t^T \int_t^T \rho_{12} \sigma_1 \sigma_2 \sigma(u,T,k) \, du - \int_t^T \rho_{13} \sigma_1 \sigma_3 \sigma(u,T,k) \, du. \]

(51)

4. Numerical Experiments

In this section, we shall perform the numerical analysis of the results obtained in Theorem 3. We assume that the parameters are as follows if there is no special instruction: \( a = 0.25 \), \( b = 0.1 \), \( k = 0.1 \), \( \theta = 0.05 \), \( \sigma_1 = 0.2 \), \( \sigma_2 = 0.2 \), \( \sigma_3 = 0.25 \), \( \nu = 2 \), \( \alpha = 2 \), \( \omega = 0.8 \), \( i = 0.05 \), \( M = 100 \), \( r_0 = 0.03 \), \( S_0 = 100 \), \( \gamma = 1 \), \( \rho_{12} = 0.2 \), \( \rho_{13} = 0.1 \), \( \rho_{23} = 0.3 \), \( t = 0 \), and \( T = 2 \). Furthermore, we assume that \( \ln(1 + X_i) \) satisfies the standard normal distribution for obtaining the numerical results of the price of the warrant bond.

In Figure 1, for each \( \omega = 0.4, 0.6, 0.8 \), we consider the impact of conversion price \( C_v \) on the warrant bond price. As mentioned above, the value of the warrant bond includes the value of a European call option, and the conversion price amounts to the exercised price of the option. So, the warrant bond price decreases as \( C_v \) increases. It is also found in Figure 1 that the price of the warrant bond increases with the value of the recovery rate, i.e., \( \omega \). In fact, the greater \( \omega \) means that the holder of the warrant bond will obtain more payoff once a credit event occurs. Hence, it is not surprising that the value of the warrant bond increases as the recovery rate \( \omega \) increases.

Figure 2 indicates that the initial stock price \( S_0 \) has a significant effect on the price of the warrant bond. As the values of \( S_0 \) increase, the values of the warrant bond increase as well. In fact, the greater the stock price is, the more likely the convertible bond will be converted. Hence the holder of the warrant bond can get more benefit from the higher stock price.

Figure 3 provides the impact of the exercise proportion \( y \) on the warrant bond. As we can see, the price of the warrant bond increases as \( y \) increases. In fact, the larger the exercise proportion \( y \), the more the stocks that can be converted into and the more the profit the holder may get which leads to the higher price of the warrant bonds.

As assumed in (4), the default intensity \( \lambda \) has the property of mean reversion with the long term mean level \( b \). Larger \( b \) leads to the more chances of default which implies that the valuation of the warrant bond may be lower. It is shown in Figure 4 that the price of the warrant bond decreases as \( b \) increases.

Finally, in Figure 5 we compare the warrant bond price with different \( \theta \) which is the long term mean level of the interest rate. The value of a warrant bond includes the value of a bond and the value of an option. The higher interest rate makes the value of the bond lower but makes the value of the
option price higher. Combining the two facets, the higher the interest rate, the lower the price of the warrant bond.

### 5. Conclusion

The primary purpose of this paper is to value the warrant bond with credit risk under the jump diffusion model. We assume that the stock price follows a jump diffusion model while the market interest rate and the default intensity are described by mean reversion models. The technique of measure transformation is applied to provide an efficient way to evaluate the warrant bond prices. Finally, from the numerical analysis, we obtain the effects of the recovery rate $\omega$, the agreed conversion price $C_v$, the initial price of stock, the exercise proportion $\gamma$, and the long term mean level of interest rate and default intensity $b$ and $\theta$ on the warrant bond price.

### Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

### Conflicts of Interest

There are no conflicts of interest related to this paper.

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