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Research Article

Backwards Asymptotically Autonomous Dynamics for 2D MHD Equations

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We consider the backwards topological property of pullback attractors for the nonautonomous MHD equations. Under some backwards assumptions of the nonautonomous force, it is shown that the theoretical existence result for such an attractor is derived from an increasing, bounded pullback absorbing and the backwards pullback flattening property. Meanwhile, some abstract results on the convergence of nonautonomous pullback attractors in asymptotically autonomous problems are established and applied to MHD equations.

1. Introduction

In this paper, we consider the existence and backwards compactness of pullback attractors for the nonautonomous MHD equations on a bounded domain $\mathcal{O} \subset \mathbb{R}^2$:

$$\begin{aligned} & u_t + (u \cdot \nabla) u - \nu_1 \Delta u - \nu_3 (B \cdot \nabla) B + \nabla \left(p + \frac{\nu_3 |B|^2}{2} \right) \\ & \quad = f(x,t), \\ & B_t + (u \cdot \nabla) B - (B \cdot \nabla) u - \nu_2 \Delta B = 0, \\ & \operatorname{div} u = 0, \\ & \operatorname{div} B = 0. \end{aligned} \tag{1}$$

The unknown $u = (u_1, u_2)$ is the velocity vector, $B = (B_1, B_2)$ is the magnetic field, and p is the pressure. The positive constants $\nu_1 = 1/Re$, $\nu_2 = 1/Rm$, and $\nu_3 = M^2 \nu_1 \nu_2$, where Re, Rm, and M are called the Reynolds number, magnetic Reynolds number, and Hartman number, respectively (see [1]). The external force $f \in L^2_{loc}(\mathbb{R}, \mathbb{L}^2(\mathcal{O}))$.

The system of equations describes a magnetized plasma as a one-component fluid and the magnetic field polarizes the conductive fluid, which changes the magnetic field reciprocally. Because of the important physical applications and the mathematical properties, MHD equations have been widely investigated in the literatures (see [2–8]).

When the body force f is time-independent, i.e., the MHD equation is autonomous, both well-posedness and ergodicity of the stochastic MHD equation were discussed in some papers (see [9, 10]) and the reference therein, while the existence of attractors was proved by many authors (see [5, 11]).

Since the force is time-dependent, the dynamics is nonautonomous which is described by an important concept of pullback attractors. It is well-known that a pullback attractor is a time-dependent family of compact, invariant, and pullback attracting sets with the minimality, which was studied by many authors (see [12–16]).

In this paper, we focus on a relatively new subject about *backwards compactness* of a pullback attractor, which means that the union of a pullback attractor over the past time is precompact; i.e., $\bigcup_{s \leq t} \mathcal{A}(s)$ is precompact for all $t \in \mathbb{R}$. To the best of our knowledge, there has been very little information on nonautonomous pullback attractors for evolution problems involving the backwards compactness (see [17–19]). To establish the theoretical results of a backwards compact attractor, we will introduce the flattening property presented by Kloeden [20] and promote this nature as a *backwards pullback flattening property*. We will prove that a nonautonomous system has a backwards compact attractor if it has an increasing, bounded, and pullback absorbing set and this system is *backwards pullback flattening*. Similarly, we can introduce other relative concepts of *backwards pullback asymptotic compactness*. In fact, the two concepts mentioned above are equivalent in a uniform convex Banach space.

As the application of theoretical results, we obtain that 2D MHD equations have a backwards compact attractor in H and V, respectively. In this case, we need only to assume that the nonautonomous external force f is *backwards tempered* and *backwards limiting*. The spectrum decomposition technique is used to give required backwards uniform estimates in V.

Finally, we consider the asymptotically autonomous dynamics of PDE. Let S be an evolution process with a pullback attractor $\mathscr{A} = \{\mathscr{A}(t) : t \in \mathbb{R}\}$ and T a semigroup with a global attractor $\mathscr{A}_\infty$ on a Banach space X. We say that S is *asymptotically autonomous* to T if

$$\lim_{\tau \longrightarrow -\infty} \left\| S(\tau + t, \tau) x_\tau - T(t) x_0 \right\|_X = 0, \quad \text{for each } t \geqslant 0, \tag{2}$$

whenever $\|x_\tau - x_0\|_X \longrightarrow 0$ as $\tau \longrightarrow -\infty$, while S is *uniformly asymptotically autonomous* to T if the convergence in (2) is uniform in $t \geqslant 0$; i.e.,

$$\lim_{\tau \longrightarrow -\infty} \sup_{t \geqslant 0} \left\| S(\tau + t, \tau) x_\tau - T(t) x_0 \right\|_X = 0. \tag{3}$$

There is not much research on this kind of problem. The representative literature is published by Kloeden [21] which proved that if S is *uniformly asymptotically autonomous* to T and the pullback attractor $\mathscr{A}$ is *uniformly compact* (i.e., $\bigcup_{s \in \mathbb{R}} \mathscr{A}(s)$ is precompact), then the pullback attractor converges to the global attractor in the Hausdorff semidistance sense:

$$\lim_{\tau \longrightarrow +\infty} \operatorname{dist}_X \left(\mathscr{A}(\tau), \mathscr{A}_\infty \right) = 0. \tag{4}$$

where $\tau \longrightarrow +\infty$ which is different in this paper. Other forms of results can be found in [22–24] but all known results involved uniform convergence and uniform compactness.

However, the uniformness condition is hard to verify in realistic models. Motivated by this dilemma, we establish an abstract result to reduce the uniformness condition (only $\bigcup_{s \leqslant t} \mathscr{A}(s)$ is precompact) and find that $\mathscr{A}$ is backwards compact if and only if the upper semicontinuity holds; i.e.,

$$\lim_{\tau \longrightarrow -\infty} \operatorname{dist}_X \left(\mathscr{A}(\tau), \mathscr{A}_\infty \right) = 0, \tag{5}$$

if S is weakly asymptotically autonomous ($\tau \longrightarrow -\infty$) to T, in this paper.

2. Preliminaries and Abstract Results

First, we review some basic concepts related to pullback attractors for nonautonomous dynamical system (see [12, 13, 15, 16]) and introduce the concept of a backwards compact attractor and then investigate its existence.

Let $(X, \|\cdot\|_X)$ be a Banach space and $\mathfrak{D}$ is the collection of all bounded nonempty subsets of X. A set-valued mapping $\mathscr{D} : \mathbb{R} \longrightarrow 2^X \setminus \emptyset$ is called a nonautonomous set in X, and it is said to have a topological property (such as boundedness, compactness, or closedness) if $\mathscr{D}(t)$ has this property for each $t \in \mathbb{R}$. We also say that a nonautonomous set $\mathscr{D}(t)$ is increasing if $\mathscr{D}(s) \subset \mathscr{D}(t)$ for $s \leqslant t$.

Definition 1. A nonautonomous set $\mathscr{D} \subset X$ is called backwards compact (resp., backwards bounded) if $\bigcup_{s \leqslant t} \mathscr{D}(s)$ is precompact (resp., bounded) in X with each $t \in \mathbb{R}$.

Definition 2. An evolution process S in X is a family of mappings $S(t, \tau) : X \longrightarrow X$ with $t \geqslant \tau$, which satisfies

$$\begin{aligned} &S(\tau, \tau) = id_X, \\ &S(t, \tau) = S(t, s) S(s, \tau), \\ &\qquad \text{for all } t \geqslant s \geqslant \tau \text{ with } t \in \mathbb{R}, \end{aligned} \tag{6}$$

$$\begin{aligned} \text{and } &(t, \tau, x) \longrightarrow S(t, \tau) x \\ &\text{is continuous for } t \geqslant \tau \text{ and } x \in X. \end{aligned}$$

Definition 3. A nonautonomous set $\mathscr{A}(\cdot)$ in X is called a backwards compact attractors for a process $S(\cdot, \cdot)$ if

(1) $\mathscr{A}(\cdot)$ is backwards compact;

(2) $\mathscr{A}(\cdot)$ is invariant, i.e., $S(t, s)\mathscr{A}(s) = \mathscr{A}(t)$ for all $t \geqslant s$;

(3) $\mathscr{A}(\cdot)$ is pullback attracting set, which means that it pullback attracts every bounded subset $D \in \mathfrak{D}$, i.e.,

$$\lim_{\tau \longrightarrow \infty} \operatorname{dist}_X \left(S(t, t - \tau) D, \mathscr{A}(t) \right) = 0, \quad \forall t \in \mathbb{R}, \tag{7}$$

where and throughout this paper $\operatorname{dist}(\cdot, \cdot)$ is Hausdorff semidistance, i.e.,

$$\operatorname{dist}_X (A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_X, \quad \forall A, B \subseteq X. \tag{8}$$

Remark 4. Through the above definitions, a backwards compact attractor must be the minimal family of closed sets with property (3). This term can be interpreted as if there is another family $\mathscr{A}_1(\cdot)$ of closed sets that pullback attracts bounded subsets of X, then $\mathscr{A}(t) \subset \mathscr{A}_1(t)$. Meanwhile, in general this is required to guarantee the uniqueness of the backwards compact attractor and by the minimality, it is shown that a backwards compact attractor must be a pullback attractor in the sense of [14, p.12]. If a pullback attractor is backwards compact, then it is a backwards compact attractor.

Definition 5. A nonautonomous set $\mathscr{K}$ in X is a pullback absorbing set at time $t \in \mathbb{R}$ for an evolution process S if, for each bounded subset D in X, there is $\tau_0 := \tau_0(t, D) > 0$ such that

$$S(t, t - \tau) D \subset \mathscr{K}(t), \quad \text{for all } \tau \geqslant \tau_0. \tag{9}$$

Definition 6. An evolution process S in X is said to possess the backwards pullback flattening condition if given a bounded set $D \subset X, t \in \mathbb{R}$ and $\varepsilon > 0$; there exist $\tau_0 := \tau_0(\varepsilon, t, D) > 0$ and a finite dimensional subspace X_1 of X such that, for a bounded projector $P : X \longrightarrow X_1$,

$$P\left(\bigcup_{\tau \geqslant \tau_0} \bigcup_{s \leqslant t} S(s, s-\tau) D\right) \text{ is bounded in } X, \quad (10)$$

and

$$\left\| (I-P)\left(\bigcup_{\tau \geqslant \tau_0} \bigcup_{s \leqslant t} S(s, s-\tau) D\right) \right\|_X < \varepsilon \quad (11)$$

Theorem 7 (see [18]). *Let S be an evolution process in a Banach space X; assume that*

(i) *S has an increasing and bounded absorbing set $\mathscr{K}(\cdot)$,*

(ii) *S is backwards pullback flattening.*

Then S has a backwards compact attractor $\mathscr{A}$ given by

$$\mathscr{A}(t) = \omega(\mathscr{K}(t), t) := \bigcap_{\tau_0 > 0} \overline{\bigcup_{\tau \geqslant \tau_0} S(t, t-\tau) \mathscr{K}(t)}^X, \quad \text{for each } t \in \mathbb{R}. \quad (12)$$

Let an evolution process S have a pullback attractor $\mathscr{A}$ and a semigroup T with a global attractor $\mathscr{A}_\infty$.

Definition 8. An evolution process S is said to be weakly asymptotically autonomous to T if for each $t \geqslant 0$,

$$\lim_{\tau \longrightarrow -\infty} \operatorname{dist}_X (S(\tau + t, \tau) x_\tau, T(t) x_0) = 0, \quad (13)$$

whenever $x_\tau \in \mathscr{A}(\tau)$, $x_0 \in \mathscr{A}_\infty$ and $x_\tau \longrightarrow x_0$.

Theorem 9. *Let S be weakly asymptotically autonomous to T. Then the upper semicontinuity holds; i.e.,*

$$\lim_{\tau \longrightarrow -\infty} \operatorname{dist}_X (\mathscr{A}(\tau), \mathscr{A}_\infty) = 0 \quad (14)$$

if and only if $\mathscr{A}$ is backwards compact.

Proof.

Sufficiency. We argue by contradiction. Since $\mathscr{A}$ is backwards compact, then $C := \overline{\bigcup_{s \leqslant 0} \mathscr{A}(s)}$ is compact. Suppose that the semicontinuity (14) is not true, then there are $\delta > 0$ and $0 > \tau_n \downarrow -\infty$ such that $\operatorname{dist}_X(\mathscr{A}(\tau_n), \mathscr{A}_\infty) \geqslant 4\delta$ for all $n \in \mathbb{N}$. We choose $x_n \in \mathscr{A}(\tau_n)$ such that

$$d(x_n, \mathscr{A}_\infty) \geqslant \operatorname{dist}_X (\mathscr{A}(\tau_n), \mathscr{A}_\infty) - \delta \geqslant 3\delta. \quad (15)$$

By the attraction of $\mathscr{A}_\infty$ under the semigroup, there is a $n_0 \in \mathbb{N}$ such that

$$\operatorname{dist}_X \left(T\left(\left|\tau_{n_0}\right|\right) C, \mathscr{A}_\infty\right) < \delta. \quad (16)$$

By the invariance of the pullback attractor $\mathscr{A}$, we see that, for any $x_n \in \mathscr{A}(\tau_n)$, there exists $y_n \in \mathscr{A}(\tau_n - |\tau_{n_0}|)$ such that

$$x_n = S\left(\tau_n - \left|\tau_{n_0}\right| + \left|\tau_{n_0}\right|, \tau_n - \left|\tau_{n_0}\right|\right) y_n. \quad (17)$$

Since $\{y_n\}$ is included into the compact set C, it follows that there exist a subsequence $\{y_{n_k}\}$ and $y \in C$ such that $y_{n_k} \longrightarrow y$ in X as $k \longrightarrow \infty$.

Applying the (13) in the case that $t = |\tau_{n_0}|$ and $\tau = \tau_{n_k} - |\tau_{n_0}| \longrightarrow -\infty$ as $k \longrightarrow \infty$, we find

$$\begin{aligned} d\left(x_{n_k}, T\left(\left|\tau_{n_0}\right|\right) y\right) &= d\left(S\left(\left(\tau_{n_k} - \left|\tau_{n_0}\right|\right) + \left|\tau_{n_0}\right|, \tau_{n_k} - \left|\tau_{n_0}\right|\right)\right. \\ &\quad \left. \cdot\, y_{n_k}, T\left(\left|\tau_{n_0}\right|\right) y\right) < \delta, \end{aligned} \quad (18)$$

if k is large enough. From (16) and (18), we obtain that

$$\begin{aligned} d\left(x_{n_k}, \mathscr{A}_\infty\right) &\leqslant d\left(x_{n_k}, T\left(\left|\tau_{n_0}\right|\right) y\right) \\ &\quad + \operatorname{dist}_X \left(T\left(\left|\tau_{n_0}\right|\right) C, \mathscr{A}_\infty\right) < 2\delta, \end{aligned} \quad (19)$$

which contradicts with (15). Therefore the semicontinuity (14) holds true.

Necessity. Suppose the semicontinuity (14) holds true. We need to prove the precompactness of $\bigcup_{s \leqslant t} \mathscr{A}(s)$ for each fixed $t \in \mathbb{R}$. Taking a sequence $\{x_n\}$ from this set, we then choose $s_n \leqslant t$ such that $x_n \in \mathscr{A}(s_n)$. We will prove that the sequence $\{x_n\}$ has a convergent subsequence in two case.

Case 1. If $s_0 = \inf_{n \in \mathbb{N}} s_n \neq -\infty$, then for $n \in \mathbb{N}$, $s_n \in [s_0, t]$, and so $\{x_n\} \subset \bigcup_{s_0 \leqslant s \leqslant t} \mathscr{A}(s)$. Define a mapping $\Upsilon : [s_0, +\infty) \times X \longrightarrow X$, $(s, x) \longrightarrow S(s, s_0)x$, then the continuity assumption implies that Υ is a continuous mapping. By the invariance of the pullback attractor $\mathscr{A}$, it is easy to see that

$$\bigcup_{s_0 \leqslant s \leqslant t} \mathscr{A}(s) = \Upsilon([s_0, t] \times \mathscr{A}(s_0)). \quad (20)$$

Then $\bigcup_{s_0 \leqslant s \leqslant t} \mathscr{A}(s)$ is a compact set since the range of a continuous mapping on a compact set is compact. Hence $\{x_n\}$ is precompact as required.

Case 2. $s_0 = \inf_{n \in \mathbb{N}} s_n = -\infty$. In this case, passing to a subsequence, we may assume $s_n \downarrow -\infty$. By the upper semicontinuity assumption (14), we have

$$d(x_n, \mathscr{A}_\infty) \leqslant \operatorname{dist}_X (\mathscr{A}(s_n), A_\infty) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (21)$$

For each $n \in \mathbb{N}$ we choose a $y_n \in A_\infty$ such that $d(x_n, y_n) < d(x_n, A_\infty) + 1/n$. Since the global attractor A_∞ is a compact set, it implies that the sequence $\{y_n\}$ has a convergent subsequence such that $y_{n_k} \longrightarrow y$ as $k \longrightarrow \infty$. Therefor,

$$\begin{aligned} d\left(x_{n_k}, y\right) &\leqslant d\left(x_{n_k}, y_{n_k}\right) + d\left(y_{n_k}, y\right) \\ &\leqslant d\left(y_{n_k}, y\right) + d\left(x_{n_k}, \mathscr{A}_\infty\right) + \frac{1}{n_k}, \end{aligned} \quad (22)$$

which together with (21) implies that $x_{n_k} \longrightarrow y$ as required. $\square$

Remark 10. This proof (sufficiency) is different from Kloeden given in [21, Theorem 3.2]. At this moment, we only need that the convergence from S to T holds true at every single time (e.g., $t = |\tau_{n_0}|$), not uniformly in $t \geqslant 0$. So we reduce the uniformness condition in [21, Theorem 3.2] successfully.

3. Nonautonomous 2D MHD Equations

3.1. Functional Spaces and Operators. Let $\mathcal{O} \subset \mathbb{R}^2$ be a bounded, open, and simply connected subset with regular boundary Γ. We consider the following MHD equations defined on $\mathcal{O} \times [\tau, +\infty)$:

$$u_t + (u \cdot \nabla) u - \nu_1 \Delta u - \nu_3 (B \cdot \nabla) B + \nabla P_0 = f(x,t), \quad (23)$$

$$B_t + (u \cdot \nabla) B - (B \cdot \nabla) u - \nu_2 \Delta B = 0, \quad (24)$$

$$\begin{gathered} \operatorname{div} u = 0, \\ \operatorname{div} B = 0, \end{gathered} \quad (25)$$

where $P_0 = p + \nu_3 |B|^2/2$ is the total pressure and ν_i are positive constants.

We consider the initial problem of (23)-(25) with mixed boundary conditions:

$$\begin{gathered} u(x,\tau) = u_0(x), \\ B(x,\tau) = B_0(x) \\ \text{on } \mathcal{O} \\ u(x,t) = 0, \\ B \cdot \mathbf{n} = 0. \\ \text{on } \Gamma, t \geqslant \tau \end{gathered} \quad (26)$$

where $\mathbf{n}$ is the unit outward normal on Γ. For the mathematical setting of this problem, we introduce some Hilbert spaces. We set $H = H_1 \times H_2$ and $V = V_1 \times V_2$ with

$$\begin{gathered} H_1 = H_2 = \left\{\varphi \in \mathbb{L}^2(\mathcal{O}) : \operatorname{div}\varphi = 0, \varphi \cdot \mathbf{n}|_\Gamma = 0\right\}, \\ V_1 = \left\{\varphi \in \mathbb{H}_0^1(\mathcal{O}) : \operatorname{div}\varphi = 0\right\}, \\ V_1' = \left\{\varphi \in \mathbb{H}^{-1}(\mathcal{O}) : \operatorname{div}\varphi = 0\right\}, \\ V_2 = \left\{\varphi \in \mathbb{H}^1(\mathcal{O}) : \operatorname{div}\varphi = 0, \varphi \cdot \mathbf{n}|_\Gamma = 0\right\}, \end{gathered} \quad (27)$$

where $\mathbb{L}^2(\mathcal{O}) = L^2(\mathcal{O})^2$, $\mathbb{H}^1(\mathcal{O}) = H^1(\mathcal{O})^2$, and so on. We use $(\cdot,\cdot)$ to denote the usual scalar product in $\mathbb{L}^2(\mathcal{O})$ and equip $H = H_1 \times H_2$ with the scalar product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$ by

$$\begin{gathered} (\nu_1, \nu_2)_H = (u_1, u_2) + \nu_3 (B_1, B_2), \\ \|u\|_H = (u,u)_H^{1/2}, \\ \nu_i = (u_i, B_i) \in H_i. \end{gathered} \quad (28)$$

We take the scalar product in V_1 and V_2 with the general forms denoted by $((\cdot,\cdot))$ and since $\mathcal{O} \subset \mathbb{R}^2$ is a bounded smooth domain, we take equivalent norms on V_1 and V_2 to be the same symbol $\|\nabla \cdot\|$; that is,

$$\begin{gathered} ((u,v)) = \sum_{i=1,j=1}^{2} \int_{\mathcal{O}} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx, \\ \|\nabla u\| = ((u,u))^{1/2}, \\ u, v \in V_1. \end{gathered} \quad (29)$$

We equip $V = V_1 \times V_2$ with the scalar product $((\cdot,\cdot))_V$ and the norm $\|\cdot\|_V$ given by

$$\begin{gathered} ((\nu_1, \nu_2))_V = ((u_1,u_2)) + \nu_3 ((B_1, B_2)), \\ \|\nu\|_V = ((\nu,\nu))_V^{1/2}, \quad \nu_i = (u_i, B_i) \in V. \end{gathered} \quad (30)$$

The trilinear form $b(u,v,w)$ and the bilinear operator $\mathbb{B}$ from $\mathbb{H}^1(\mathcal{O}) \times \mathbb{H}^1(\mathcal{O})$ into $\mathbb{H}^{-1}(\mathcal{O})$ are defined by

$$\langle \mathbb{B}(u,v), w\rangle = b(u,v,w) = \sum_{i=1,j=1}^{2} \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j dx. \quad (31)$$

Moreover, we have the following useful relations (see [11, 25]):

$$\begin{gathered} b(u,v,v) = 0, \\ b(u,v,w) = -b(u,w,v), \\ u \in V_2, \ v, w \in V_1, \end{gathered} \quad (32)$$

$$\begin{gathered} |b(u,v,w)| \leqslant c \|u\|^{1/2} \|\Delta u\|^{1/2} \|\nabla v\| \|w\|, \\ u \in \mathbb{H}^2(\mathcal{O}), \ v \in \mathbb{H}^1(\mathcal{O}), \ w \in \mathbb{L}^2(\mathcal{O}), \end{gathered} \quad (33)$$

$$\begin{gathered} |b(u,v,w)| \leqslant c \|\nabla u\| \|\nabla v\|^{1/2} \|\Delta v\|^{1/2} \|w\|, \\ u \in \mathbb{H}^1(\mathcal{O}), \ v \in \mathbb{H}^2(\mathcal{O}), \ w \in \mathbb{L}^2(\mathcal{O}), \end{gathered} \quad (34)$$

where c is an intrinsic positive constant.

On the other hand, through the above terms, (23)-(25) can be rewritten in a weak form as follows:

$$\frac{du}{dt} - \nu_1 \Delta u = \left(-\mathbb{B}(u,u) + \nu_3 \mathbb{B}(B,B)\right) + f(x,t), \quad (35)$$

$$\frac{dB}{dt} - \nu_2 \Delta B = \left(-\mathbb{B}(u,B) + \mathbb{B}(B,u)\right), \quad (36)$$

$$\begin{gathered} \operatorname{div} u = 0, \\ \operatorname{div} B = 0, \end{gathered} \quad (37)$$

with the initial-boundary condition (26).

3.2. Assumptions on the Nonautonomous Force. In order to obtain a backwards compact attractor in H for (35)-(37), a basic assumption for external force is $f \in L^2_{loc}(\mathbb{R}, \mathbb{L}^2(\mathcal{O}))$. Furthermore, one has the following.

Assumption F1. f is backwards tempered; i.e.,

$$\begin{gathered} \sup_{s \leqslant t} \int_{-\infty}^{s} e^{\gamma(r-s)} \|f(r,\cdot)\|^2 dr < \infty, \\ \text{for all } \gamma > 0 \text{ and } t \in \mathbb{R}. \end{gathered} \quad (38)$$

To prove the existence of backwards compact attractor in V for (35)-(37), we assume further the following.

Assumption F2. f is backwards limiting; i.e.,

$$\lim_{\gamma\longrightarrow+\infty}\sup_{s\leqslant t}\int_{-\infty}^{s}e^{\gamma(r-s)}\|f(r,\cdot)\|^2\,dr=0,\quad\text{for all } t\in\mathbb{R}. \tag{39}$$

By employing Galerkin method, we have the following well-possessedness of problem (35)-(37), which is similar to the nonautonomous case as given in [26].

Lemma 11. *Let $f\in L^2_{loc}(\mathbb{R},\mathbb{L}^2(\mathcal{O}))$. Then for each $(u_0,B_0)\in H$ and for each $\tau\in\mathbb{R}$, there exists a unique weak solution*

$$(u,B)\in L^2_{loc}(\tau,\infty;V)\cap C([\tau,\infty);H) \tag{40}$$

satisfying (35)-(37) in distribution sense with $(u,B)|_{t=\tau}=(u_0,B_0)$. Moreover, the mapping $(u_0,B_0)\longmapsto(u,B)$ is continuous in H.

For convenience, we rewrite the solution of (35)-(37) by $\varphi:=(u,B)$ and the initial data by $\varphi_0:=(u_0,B_0)$.

By Lemma 11, we can use the unique weak solution to define an evolution process $S(t,\tau):H\longrightarrow H$ by

$$S(t,\tau)\varphi_0:=\varphi(t,\tau,\varphi_0),\quad\text{for all } t\geqslant\tau \text{ and } \varphi_0\in H. \tag{41}$$

4. Backwards Compact Attractors for 2D MHD Equations

4.1. Backwards Compact Attractors in H. In this subsection, our main work is to prove that the evolution process has an increasing bounded pullback absorbing set in H. From now on, we assume without loss of generality that c will be a positive constant which may alter its values everywhere.

Lemma 12. *Let f be backwards tempered, then for each $t\in\mathbb{R}$ and $R>0$, there exists $\tau_0:=\tau_0(R)\geqslant 2$ such that, for all $\tau\geqslant\tau_0$ and $\|\varphi_0\|_H\leqslant R$,*

$$\sup_{s\leqslant t}\sup_{r\in[s-2,s]}\|\varphi(r,s-\tau,\varphi_0)\|_H^2\leqslant c(1+M(t)), \tag{42}$$

$$\sup_{s\leqslant t}\int_{s-\tau}^{s}e^{\lambda(r-s)}\|\nabla\varphi(r,s-\tau,\varphi_0)\|_H^2\,dr\leqslant c(1+M(t)), \tag{43}$$

where λ is given by (48) and $M(t)$ is a nonnegative increasing function defined by

$$M(t):=\sup_{s\leqslant t}\int_{-\infty}^{s}e^{\lambda(r-s)}\|f(r,\cdot)\|^2\,dr<\infty. \tag{44}$$

Proof. Let $t\in\mathbb{R}$ be fixed. For each $s\leqslant t$, we multiply equation in (35) by u and (36) by $\nu_3 B$ respectively and integrate over $\mathcal{O}$, then the sum of them is

$$\begin{aligned}\frac{d}{ds}\left(\|u\|^2+\nu_3\|B\|^2\right)&+2\nu_1\|\nabla u\|^2+2\nu_2\nu_3\|\nabla B\|^2\\&=2\left((-\mathbb{B}(u,u)+\nu_3\mathbb{B}(B,B),u)\right.\\&\quad\left.+(-\mathbb{B}(u,B)+\mathbb{B}(B,u),\nu_3 B)\right)+2\int_{\mathcal{O}}f\cdot u\,dx.\end{aligned} \tag{45}$$

Notice from (31) and (32) that

$$\begin{aligned}&(\mathbb{B}(u,u),u)=0;\\&(\mathbb{B}(u,B),\nu_3 B)=0;\\&(\nu_3\mathbb{B}(B,B),u)=\nu_3 b(B,B,u);\\&(\mathbb{B}(B,u),\nu_3 B)=-\nu_3 b(B,B,u).\end{aligned} \tag{46}$$

For the nonlinear term, we have

$$\begin{aligned}2\int_{\mathcal{O}}f\cdot u\,dx&\leqslant\lambda\|u\|^2+c\|f(s,\cdot)\|^2\\&\leqslant\lambda\|\varphi\|^2+c\|f(s,\cdot)\|^2,\end{aligned} \tag{47}$$

where we have used the notation $\varphi=(u,B)$, $\|\varphi\|^2=\|u\|^2+\nu_3\|B\|^2$, and $\lambda>0$ is given by

$$\begin{aligned}&2\lambda\|u\|^2\leqslant\nu\|\nabla u\|^2,\\&2\lambda\|B\|^2\leqslant\nu\|\nabla B\|^2,\\&\qquad\forall u\in V_1,\ B\in V_2,\ \nu=\min\{\nu_1,\nu_2\}.\end{aligned} \tag{48}$$

Substituting the above into (45), we have

$$\frac{d}{ds}\|\varphi\|_H^2+\lambda\|\varphi\|_H^2+\nu\|\nabla\varphi\|_H^2\leqslant c\|f(s,\cdot)\|^2. \tag{49}$$

Multiplying (49) by $e^{\lambda s}$ and integrating it over $[s-\tau,s]$, we obtain

$$\begin{aligned}\|\varphi(s)\|_H^2&+\int_{s-\tau}^{s}e^{\lambda(r-s)}\|\nabla\varphi(r)\|_H^2\,dr\\&\leqslant e^{-\lambda\tau}\|\varphi_0\|_H^2+c\int_{s-\tau}^{s}e^{\lambda(r-s)}\|f(r,\cdot)\|^2\,dr\\&\leqslant c\left(1+\int_{-\infty}^{s}e^{\lambda(r-s)}\|f(r,\cdot)\|^2\,dr\right),\end{aligned} \tag{50}$$

for all $\tau\geqslant\tau_0$ with some $\tau_0:=\tau_0(R)\geqslant 2$.

On the other hand, we multiply (49) by $e^{\lambda s}$ and integrating it over $[s-\tau,r]$ with $r\in[s-2,s]$, we obtain

$$\begin{aligned}\|\varphi(r)\|_H^2&\leqslant e^{\lambda(s-r-\tau)}\|\varphi_0\|_H^2\\&\quad+c\int_{s-\tau}^{r}e^{\lambda(r_1-r)}\|f(r_1,\cdot)\|^2\,dr_1\\&\leqslant c\left(1+\int_{-\infty}^{s}e^{\lambda(r_1-s)}\|f(r_1,\cdot)\|^2\,dr_1\right),\end{aligned} \tag{51}$$

for all $\tau\geqslant\tau_0$ with some $\tau_0:=\tau_0(R)\geqslant 2$.

Taking the supremum with respect to the past time $s\leqslant t$ in (51) and (50), we get (42) and (43). By the assumption (38), $M(t)$ is finite and increasing. This completes the proof. □

Lemma 13. *Let f be backwards tempered, then for each $t \in \mathbb{R}$ and $R > 0$, there exists $\tau_0 := \tau_0(R) \geqslant 2$ such that, for all $\tau \geqslant \tau_0$ and $\|\varphi_0\|_H \leqslant R$,*

$$\sup_{s \leqslant t} \sup_{r \in [s-1,s]} \left\|\nabla\varphi(r, s-\tau, \varphi_0)\right\|_H^2 \leqslant ce^{(1+M(t))^2}, \quad (52)$$

where $M(t)$ is given by (44).

Proof. Let $t \in \mathbb{R}$ be fixed. For each $s \leqslant t$, we multiply equation in (35) by $-\Delta u$ and (35) by $-\nu_3 \Delta B$, respectively, then integrate over $\mathcal{O}$, and sum the results to find

$$\begin{aligned}
&\frac{d}{ds}\left(\|\nabla u\|^2 + \nu_3\|\nabla B\|^2\right) + 2\nu_1\|\Delta u\|^2 + 2\nu_2\nu_3\|\Delta B\|^2 \\
&\quad = 2\left(\left(-\mathbb{B}(u,u) + \nu_3\mathbb{B}(B,B), -\Delta u\right)\right. \\
&\quad\quad \left.+ \left(-\mathbb{B}(u,B) + \mathbb{B}(B,u), -\nu_3\Delta B\right)\right) - 2\int_{\mathcal{O}} f \cdot \Delta u dx
\end{aligned} \quad (53)$$

Notice from (33) that

$$\begin{aligned}
(-\mathbb{B}(u,u), -\Delta u) &= b(u,u,\Delta u) \\
&\leqslant c\|u\|^{1/2}\|\nabla u\|\|\Delta u\|^{3/2} \\
&\leqslant c\|u\|^2\|\nabla u\|^4 + \frac{\nu_1}{8}\|\Delta u\|^2,
\end{aligned} \quad (54)$$

$$\left|\mathbb{B}(B,u), -\nu_3\Delta B\right| \leqslant c\|B\|^2\|\nabla u\|^4 + \frac{\nu_2\nu_3}{8}\|\Delta B\|^2. \quad (55)$$

On the other hand, by (33) and the inequality that $abc \leqslant C(\varepsilon_1,\varepsilon_2)a^4 + \varepsilon_1 b^4 + \varepsilon_2 c^2$, we have

$$\begin{aligned}
\left|\nu_3\mathbb{B}(B,B), -\Delta u\right| &= \nu_3\left|b(B,B,\Delta u)\right| \\
&\leqslant c\|B\|^{1/2}\|\nabla B\|\|\Delta B\|^{1/2}\|\Delta u\| \\
&\leqslant c\|B\|^2\|\nabla B\|^4 + \frac{\nu_2\nu_3}{8}\|\Delta B\|^2 \\
&\quad + \frac{\nu_1}{8}\|\Delta u\|^2, \\
\left|\mathbb{B}(u,B), \nu_3\Delta B\right| &\leqslant c\|u\|^2\|\nabla B\|^4 + \frac{\nu_1}{8}\|\Delta u\|^2 \\
&\quad + \frac{\nu_2\nu_3}{8}\|\Delta B\|^2.
\end{aligned} \quad (56)$$

The nonlinear term in (53) is controlled by

$$\begin{aligned}
\int_{\mathcal{O}} \left|f \cdot \Delta u\right| dx &\leqslant \frac{\nu_1}{8}\left(\|\Delta u\|^2 + \nu_2\|\Delta B\|^2\right) \\
&\quad + c\left\|f(s,\cdot)\right\|^2,
\end{aligned} \quad (57)$$

Substituting (54)-(57) into (53), we find

$$\begin{aligned}
&\frac{d}{ds}\|\nabla\varphi(s)\|_H^2 + \nu\|\Delta\varphi\|_H^2 \\
&\quad \leqslant h(s)\|\nabla\varphi\|_H^2 + c\left\|f(s,\cdot)\right\|^2,
\end{aligned} \quad (58)$$

where $h(s) = c\|\varphi(s)\|_H^2\|\nabla\varphi(s)\|_H^2$. Integrate (58) over (ξ, r) with $\xi \in [s-2, s-1]$ and $r \in [s-1, s]$ to obtain

$$\begin{aligned}
&\left\|\nabla\varphi(r, s-\tau, \varphi_0)\right\|_H^2 \\
&\quad \leqslant e^{\int_\xi^r ch(\rho)d\rho}\left\|\nabla\varphi(\xi, s-\tau, \varphi_0)\right\|_H^2 \\
&\quad\quad + c\int_\xi^r e^{\int_\rho^r ch(\delta)d\delta}\left\|f(\rho,\cdot)\right\|^2 d\rho \\
&\quad \leqslant e^{\int_{s-2}^r ch(\rho)d\rho}\left\|\nabla\varphi(\xi, s-\tau, \varphi_0)\right\|_H^2 \\
&\quad\quad + c\int_{s-2}^r e^{\int_\rho^r ch(\delta)d\delta}\left\|f(\rho,\cdot)\right\|^2 d\rho
\end{aligned} \quad (59)$$

We integrate (59) with respect to ξ over $[s-2, s-1]$ with $s \leqslant t$; we have

$$\begin{aligned}
&\left\|\nabla\varphi(r, s-\tau, \varphi_0)\right\|_H^2 \\
&\quad \leqslant ce^{\int_{s-2}^s ch(\rho)d\rho}\left(\int_{s-2}^s \left\|\nabla\varphi(\xi, s-\tau, \varphi_0)\right\|_H^2 d\xi\right. \\
&\quad \left.+ \int_{s-2}^s \left\|f(r,\cdot)\right\|^2 dr\right)
\end{aligned} \quad (60)$$

On the other hand, by Lemma 12, we have

$$\begin{aligned}
\int_{s-2}^s h(\rho)d\rho &= c\int_{s-2}^s \|\varphi(\rho)\|_H^2\|\nabla\varphi(\rho)\|_H^2 d\rho \\
&\leqslant c(1+M(t))\int_{s-2}^s \|\nabla\varphi(\rho)\|_H^2 d\rho \\
&\leqslant c(1+M(t))^2
\end{aligned} \quad (61)$$

Therefor, we insert (61) into (60) to obtain that

$$\sup_{r \in [s-1,s]} \left\|\nabla\varphi(r, s-\tau, \varphi_0)\right\|_H^2 \leqslant ce^{(1+M(t))^2}, \quad (62)$$

for all $\tau \geqslant \tau_0$ with some $\tau_0 := \tau_0(R) \geqslant 2$. Hence, we get (52) by taking the supremum in (62) with respect to $s \leqslant t$. □

We now state our result as follows.

Theorem 14. *Assume f is backwards tempered, then the evolution process S generated by nonautonomous 2D MHD equations possesses a backwards compact attractor $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ in H.*

Proof. Define a nonautonomous set by

$$\mathcal{K}_1(t) := \left\{\varphi \in V; \|\varphi\|_V^2 \leqslant ce^{(1+M(t))^2}\right\}, \quad \forall t \in \mathbb{R}, \quad (63)$$

where $M(t)$ is given by (44). By the compactness of the Sobolev embedding and (52), $\mathcal{K}_1(t)$ is compact and pullback absorbing in H. It is readily to check that the process S is backwards pullback asymptotically compact in H and thus is backwards pullback flattening follows from [17, Theorem 2.7]. Then the conclusion can be proved by Theorem 7. □

4.2. Backwards Compact Attractors in V. In this subsection, we prove the existence of backwards compact attractors in V. To do this, we first give a decomposition of an element in V. To this end, we consider the eigenvalue problem:

$$-\Delta v(x) = \lambda v(x), \quad x \in \mathcal{O}, \quad v|_{\partial\mathcal{O}} = 0, \tag{64}$$

Then it is known that the above problem shows a family of complete orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of $L^2(\mathcal{O})$ consisting of eigenvectors of $-\Delta$ who has countable spectrum λ_j, $j = 1, 2, \ldots$, such that

$$\begin{aligned} &0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots \longrightarrow \infty \text{ and} \\ &-\Delta e_j = \lambda_j e_j. \end{aligned} \tag{65}$$

Let $V_m = \operatorname{span}\{e_1, e_2, \ldots, e_m\} \subset V$ and $P_m : V \longrightarrow V_m$ be the canonical projector and I be the identity. Then for every $v \in V$ there exists a unique decomposition

$$\begin{aligned} v &= v_1 + v_2, \\ v_1 &= P_m v \in V_m, \\ v_2 &= (I - P_m) v \in V_m^{\perp}, \end{aligned} \tag{66}$$

where $V_m^{\perp}$ is the orthogonal complement of V_m.

Lemma 15. *Let f be backwards tempered, then for each $t \in \mathbb{R}$ and $R > 0$, there exists $\tau_0 := \tau_0(R) \geqslant 2$ such that, for all $\tau \geqslant \tau_0$ and $\|\varphi_0\|_H \leqslant R$,*

$$\sup_{s \leqslant t} \int_{s-1}^{s} \|\Delta\varphi(r, s-\tau, \varphi_0)\|_H^2 dr \leqslant c e^{2(1+M(t))^2}, \tag{67}$$

where $M(t)$ is given by (44).

Proof. By (58), we have

$$\begin{aligned} &\frac{d}{ds}\|\nabla\varphi(s)\|_H^2 + \nu\|\Delta\varphi\|_H^2 \\ &\quad \leqslant h(s)\|\nabla\varphi\|_H^2 + c\|f(s,\cdot)\|^2, \end{aligned} \tag{68}$$

Integrating (68) over $[s-1, s]$, we can obtain

$$\begin{aligned} &\nu \int_{s-1}^{s} \|\Delta\varphi(r, s-\tau, \varphi_0)\|_H^2 dr \\ &\quad \leqslant c\int_{s-1}^{s} \|\nabla\varphi(r)\|_H^4 \|\varphi(r)\|_H^2 dr + \|\nabla\varphi(s-1)\|_H^2 \\ &\quad + c\int_{s-1}^{s} \|f(r,\cdot)\|^2 dr. \end{aligned} \tag{69}$$

Thus by Lemmas 12 and 13 we have

$$\int_{s-1}^{s} \|\Delta\varphi(r, s-\tau, \varphi_0)\|_H^2 dr \leqslant c e^{2(1+M(t))^2}. \tag{70}$$

Therefor, we obtain (67) by taking the supremum in (70) over all the past time $s \leqslant t$. □

Lemma 16. *Let f be backwards tempered and backwards limiting, then for each $\varepsilon > 0, t \in \mathbb{R}$, and $R > 0$, there exist $\tau_1 := \tau_1(\varepsilon, R) \geqslant 2$ and $N := N(\varepsilon, R) > 0$ such that, for all $\tau \geqslant \tau_1, m \geqslant N$ and $\|\varphi_0\|_H \leqslant R$,*

$$\sup_{s \leqslant t} \|(I - P_m)\varphi(s, s-\tau, \varphi_0)\|_V^2 < \varepsilon. \tag{71}$$

Proof. Let $t \in \mathbb{R}$ be fixed. For each $s \leqslant t$, we multiply equation in (35) by $-\Delta u_2$ and (36) by $-\nu_3\Delta B_2$, respectively, and then integrate over $\mathcal{O}$ to find that

$$\begin{aligned} &\frac{d}{dt}\|\nabla u_2\|^2 + 2\nu_1\|\Delta u_2\|^2 \\ &\quad = 2\left(\left(-\mathbb{B}(u,u) + \nu_3\mathbb{B}(B,B), -\Delta u_2\right)\right) \\ &\quad - 2\int_{\mathcal{O}} f \cdot \Delta u_2 dx \end{aligned} \tag{72}$$

$$\begin{aligned} &\nu_3\frac{d}{dt}\|\nabla B_2\|^2 + 2\nu_2\nu_3\|\Delta B_2\|^2 \\ &\quad = 2\left(\left(-\mathbb{B}(u,B) + \mathbb{B}(B,u), -\nu_3\Delta B_2\right)\right) \end{aligned} \tag{73}$$

Notice from (34) that we have

$$\begin{aligned} \left|-\mathbb{B}(u,u), -\Delta u_2\right| &= \left|b(u,u,\Delta u_2)\right| \\ &\leqslant c\|\nabla u\|^{3/2}\|\Delta u\|^{1/2}\|\Delta u_2\| \\ &\leqslant c\|\nabla u\|^3\|\Delta u\| + \frac{\nu_1}{4}\|\Delta u_2\|^2, \end{aligned} \tag{74}$$

$$\begin{aligned} \left|\mathbb{B}(B,u), -\nu_3\Delta B_2\right| &= \left|b(B,u,-\nu_3\Delta B_2)\right| \\ &\leqslant c\|\nabla B\|\|\nabla u\|^{1/2}\|\Delta u\|^{1/2}\|\Delta B_2\|^2 \\ &\leqslant c\|\nabla B\|^2\|\nabla u\|\|\Delta u\| \\ &\quad + \frac{\nu_2\nu_3}{4}\|\Delta B_2\|^2. \end{aligned} \tag{75}$$

$$\begin{aligned} \left|\nu_3\mathbb{B}(B,B), -\Delta u_2\right| &= \nu_3\left|b(B,B,\Delta u_2)\right| \\ &\leqslant c\|\nabla B\|^{3/2}\|\Delta B\|^{1/2}\|\Delta u_2\| \\ &\leqslant c\|\nabla B\|^3\|\Delta B\| + \frac{\nu_1}{4}\|\Delta u_2\|^2, \end{aligned} \tag{76}$$

$$\begin{aligned} \left|\mathbb{B}(u,B), \nu_3\Delta B_2\right| &= \left|b(B,u,\nu_3\Delta B_2)\right| \\ &\leqslant c\|\nabla u\|\|\nabla B\|^{1/2}\|\Delta B\|^{1/2}\|\Delta B_2\|^2 \\ &\leqslant c\|\nabla u\|^2\|\nabla B\|\|\Delta B\| \\ &\quad + \frac{\nu_2\nu_3}{4}\|\Delta B_2\|^2. \end{aligned} \tag{77}$$

The nonlinear term in (72) is controlled by

$$\begin{aligned} \int_{\mathcal{O}} \left|f \cdot \Delta u_2\right| dx &\leqslant \frac{\nu_1}{4}\|\Delta u_2\|^2 + \frac{\nu_2\nu_3}{4}\|\Delta B_2\|^2 \\ &\quad + c\|f(t,\cdot)\|^2, \end{aligned} \tag{78}$$

Then from (72) to (78) and using $\|\Delta u_2\|^2 \geqslant \lambda_m \|\nabla u_2\|^2$, we find

$$\frac{d}{dt}\|\nabla\varphi_2\|_H^2 + \frac{\nu}{2}\lambda_m\|\nabla\varphi_2\|_H^2 \leqslant c\|\nabla\varphi\|_H^3\|\Delta\varphi\|_H + c\|f(t,\cdot)\|^2. \tag{79}$$

We multiply (79) by $e^{(\nu/2)\lambda_m s}$ with $s \in [r,t]$, integrating the result in $s \in [r,t]$, and then integrating it once again in $r \in [t-1,t]$, we obtain

$$\begin{aligned}\|\nabla\varphi_2(t)\|_H^2 &\leqslant \int_{t-1}^{t} e^{(\nu/2)\lambda_m(r-t)}\|\nabla\varphi_2(r)\|_H^2\,dr \\ &+ \int_{t-1}^{t} e^{(\nu/2)\lambda_m(r-t)}\|\Delta\varphi(r)\|_H\|\nabla\varphi(r)\|_H^3\,dr \\ &+ c\int_{t-1}^{t} e^{(\nu/2)\lambda_m(r-t)}\|f(r,\cdot)\|^2\,dr\end{aligned} \tag{80}$$

We now take into account the supremum of each term in (80) over the past time. From (52) and the increasing property of $M(t)$, we can see that, for all $\tau \geqslant \tau_0$ with some $\tau_0 := \tau_0(R) \geqslant 2$,

$$\begin{aligned}\sup_{s\leqslant t}\int_{s-1}^{s} e^{(\nu/2)\lambda_m(r-s)}\|\nabla\varphi_2(r)\|_H^2\,dr &\leqslant ce^{(1+M(t))^2}\sup_{s\leqslant t}\int_{s-1}^{s} e^{(\nu/2)\lambda_m(r-s)}dr \\ &\leqslant \frac{c}{\nu\lambda_m}e^{(1+M(t))^2}.\end{aligned} \tag{81}$$

Similar, by (52) and (67), we obtain

$$\begin{aligned}\sup_{s\leqslant t}\int_{s-1}^{s} e^{(\nu/2)\lambda_m(r-s)}\|\Delta\varphi(r)\|_H\|\nabla\varphi(r)\|_H^3\,dr &\leqslant ce^{(3/2)(1+M(t))^2}\sup_{s\leqslant t}\int_{s-1}^{s} e^{(\nu/2)\lambda_m(r-s)}\|\Delta\varphi(r)\|_H\,dr \\ &\leqslant ce^{(3/2)(1+M(t))^2}\sup_{s\leqslant t}\left(\left(\int_{s-1}^{s}\|\Delta\varphi(r)\|_H^2\,dr\right)^{1/2}\right. \\ &\left.\cdot\left(\int_{s-1}^{s} e^{\nu\lambda_m(r-s)}dr\right)^{1/2}\right) \leqslant \frac{c}{\sqrt{\nu\lambda_m}}e^{3(1+M(t))^2}.\end{aligned} \tag{82}$$

Finally, f is backwards limiting by assumption (39); thus

$$\lim_{m\longrightarrow\infty}\sup_{s\leqslant t}\int_{-\infty}^{s} e^{(\nu/2)\lambda_m(r-s)}\|f(r,\cdot)\|^2\,dr = 0. \tag{83}$$

Hence, from (80) to (83), for all $\tau \geqslant \tau_1$,

$$\sup_{s\leqslant t}\left\|(I-P_m)\varphi(s,s-\tau,\varphi_0)\right\|_V^2 \longrightarrow 0, \quad \text{as } m\longrightarrow\infty. \tag{84}$$

This completes the proof. □

Theorem 17. *Assume f be backwards tempered and backwards limiting, then the evolution process S generated by nonautonomous 2D MHD equations possesses a backwards compact attractor $\mathscr{A} = \{\mathscr{A}(t)\}_{t\in\mathbb{R}}$ in V.*

Proof. Define a nonautonomous set by

$$\mathscr{K}_2(t) := \left\{\varphi \in V; \|\varphi\|_V^2 \leqslant ce^{(1+M(t))^2}\right\}, \quad \forall t \in \mathbb{R}, \tag{85}$$

where $M(t)$ is given by (44). It is obvious that $\mathscr{K}_2(t)$ is bounded and increasing absorbing set in V. On the other hand, by Lemmas 13 and 16 the process S is backwards pullback flattening in V. Then the all conditions in Theorem 7 are fulfilled. Therefor there exists a backwards compact attractor $\mathscr{A} = \{\mathscr{A}(t)\}_{t\in\mathbb{R}}$ in V. □

5. Asymptotically Autonomous Dynamics

In this section, we will show that the dynamics of the original nonautonomous MHD equations is asymptotically autonomous and its pullback attractor converges upper semicontinuity to the autonomous global attractor $\mathscr{A}_\infty$ of the problem

$$\frac{d\widehat{u}}{dt} - \nu_1\Delta\widehat{u} = \left(-\mathbb{B}(\widehat{u},\widehat{u}) + \nu_3\mathbb{B}\left(\widehat{B},\widehat{B}\right)\right) + f_\infty, \tag{86}$$

$$\frac{d\widehat{B}}{dt} - \nu_2\Delta\widehat{B} = \left(-\mathbb{B}\left(\widehat{u},\widehat{B}\right) + \mathbb{B}\left(\widehat{B},\widehat{u}\right)\right), \tag{87}$$

$$\begin{aligned}&\operatorname{div}\widehat{u} = 0,\\ &\operatorname{div}\widehat{B} = 0,\end{aligned} \tag{88}$$

with initial-boundary values

$$\begin{aligned}&\widehat{u}(x,0) = \widehat{u}_0(x),\\ &\widehat{B}(x,0) = \widehat{B}_0(x)\\ &\qquad \text{on } \mathscr{O}\\ &\widehat{u}(x,t) = 0,\\ &\widehat{B}\cdot\mathbf{n} = 0.\\ &\qquad \text{on } \Gamma, t \geqslant 0\end{aligned} \tag{89}$$

For convenience, we rewrite the solution of (86)-(88) by $\widehat{\varphi} := (\widehat{u},\widehat{B})$ and the initial data by $\widehat{\varphi}_0 := (\widehat{u}_0,\widehat{B}_0)$.

To discuss the asymptotically autonomous problem, we need to give a further assumption about the forcing f. We assume that $f(t,\cdot) \longrightarrow f_\infty$ as S is asymptotically autonomous to T.

Assumption F3. There is a function $f_\infty \in \mathbb{L}^2(\mathscr{O}))$ such that

$$\lim_{\tau\longrightarrow-\infty}\int_0^\infty \|f(\tau+s) - f_\infty\|^2\,ds = 0. \tag{90}$$

Lemma 18. *Suppose assumptions F1 and F3 are satisfied. Then the solution φ of (35)-(37) is asymptotically autonomous to the solution $\widehat{\varphi}$ of (86)-(88). More precisely,*

$$\|\varphi(\tau+t,\tau,\varphi_0) - \widehat{\varphi}(t,\widehat{\varphi}_0)\|_H \longrightarrow 0 \quad \text{as } \tau \longrightarrow -\infty \text{ for each } t \geqslant 0, \tag{91}$$

whenever $\|\varphi_0 - \widehat{\varphi}_0\|_H \longrightarrow 0$ as $\tau \longrightarrow -\infty$.

Proof. Let $Q_1(t) = u(t+\tau,\tau,u_0) - \widehat{u}(t,\widehat{u}_0)$, $Q_2(t) = B(t+\tau,\tau,B_0) - \widehat{B}(t,\widehat{B}_0)$ and $Q = (Q_1,Q_2) = (u-\widehat{u}, B-\widehat{B}) = \varphi - \widehat{\varphi}$. Then subtract (35) from (86) and we obtain

$$\begin{aligned}\frac{dQ_1}{dt} - \nu_1 \Delta Q_1 &= \left(-\mathbb{B}(u,u) + \mathbb{B}(\widehat{u},\widehat{u})\right) \\ &\quad + \nu_3\left(\mathbb{B}(B,B) - \mathbb{B}\left(\widehat{B},\widehat{B}\right)\right) \\ &\quad + f(\tau+t) - f_\infty.\end{aligned} \tag{92}$$

Similarly, subtracting (36) from (87) we find that

$$\begin{aligned}\frac{dQ_2}{dt} - \nu_2 \Delta Q_2 &= \left(-\mathbb{B}(u,B) + \mathbb{B}\left(\widehat{u},\widehat{B}\right)\right) \\ &\quad + \left(\mathbb{B}(B,u) - \mathbb{B}\left(\widehat{B},\widehat{u}\right)\right).\end{aligned} \tag{93}$$

Taking the inner product of (92) with Q_1 in H, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|Q_1\|^2 + \nu_1\|\nabla Q_1\|^2 \\ &\quad = \left(-\mathbb{B}(u,u) + \mathbb{B}(\widehat{u},\widehat{u}), Q_1\right) \\ &\quad\quad + \nu_3\left(\mathbb{B}(B,B) - \mathbb{B}\left(\widehat{B},\widehat{B}\right), Q_1\right) \\ &\quad\quad + \left(f(\tau+t) - f_\infty, Q_1\right).\end{aligned} \tag{94}$$

Using the trilinearity of b and relation (31) and (32), we have

$$\begin{aligned}&\left(-\mathbb{B}(u,u) + \mathbb{B}(\widehat{u},\widehat{u}), Q_1\right) \\ &\quad = -b(u,u,Q_1) + b(\widehat{u},\widehat{u}), Q_1) \\ &\quad = -b(u,u,Q_1) + b(u,\widehat{u},Q_1) - b(u,\widehat{u},Q_1) \\ &\quad\quad + b(\widehat{u},\widehat{u}), Q_1) \\ &\quad = -b(u,Q_1,Q_1) - b(u,\widehat{u},Q_1) + b(\widehat{u},\widehat{u}), Q_1) \\ &\quad = -b(Q_1,\widehat{u},Q_1).\end{aligned} \tag{95}$$

Analogously to (95), for the second term on the right hand said of (94), we obtain

$$\begin{aligned}&\nu_3\left(\mathbb{B}(B,B) - \mathbb{B}\left(\widehat{B},\widehat{B}\right), Q_1\right) \\ &\quad = \nu_3 b(B,Q_2,Q_1) + \nu_3 b\left(Q_2,\widehat{B},Q_1\right)\end{aligned} \tag{96}$$

From (94) to (96), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|Q_1\|^2 + \nu_1\|\nabla Q_1\|^2 \\ &\quad = -b(Q_1,\widehat{u},Q_1) + \nu_3 b(B,Q_2,Q_1) \\ &\quad\quad + \nu_3 b\left(Q_2,\widehat{B},Q_1\right) + \left(f(\tau+t) - f_\infty, Q_1\right).\end{aligned} \tag{97}$$

Similarly, we take the inner product of (93) with $\nu_3 Q_2$ in H to get

$$\begin{aligned}&\frac{\nu_3}{2}\frac{d}{dt}\|Q_2\|^2 + \nu_2\nu_3\|\nabla Q_2\|^2 \\ &\quad = -\nu_3 b\left(Q_1,\widehat{B},Q_2\right) + \nu_3 b(B,Q_1,Q_2) \\ &\quad\quad + \nu_3 b(Q_2,\widehat{u},Q_2).\end{aligned} \tag{98}$$

Now, both (97) and (98) imply that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|Q_1\|^2 + \nu_3\|Q_2\|^2\right) + \nu_1\|\nabla Q_1\|^2 + \nu_2\nu_3\|\nabla Q_2\|^2 \\ &\quad = -b(Q_1,\widehat{u},Q_1) + \nu_3 b\left(Q_2,\widehat{B},Q_1\right) \\ &\quad\quad - \nu_3 b\left(Q_1,\widehat{B},Q_2\right) + \nu_3 b(Q_2,\widehat{u},Q_2) \\ &\quad\quad + \left(f(\tau+t) - f_\infty, Q_1\right)\end{aligned} \tag{99}$$

and then that

$$\begin{aligned}&\frac{d}{dt}\|Q\|_H^2 + 2\nu\|\nabla Q\|_H^2 \leqslant 2\left(-b(Q_1,\widehat{u},Q_1)\right. \\ &\quad + \nu_3 b\left(Q_2,\widehat{B},Q_1\right) - \nu_3 b\left(Q_1,\widehat{B},Q_2\right) \\ &\quad \left. + \nu_3 b(Q_2,\widehat{u},Q_2)\right) + 2\left(f(\tau+t) - f_\infty, Q_1\right).\end{aligned} \tag{100}$$

Let χ be a continuous and trilinear operator on $V \times V \times V$ given by

$$\begin{aligned}\chi(v_1,v_2,v_3) &= b(u_1,u_2,u_3) - \nu_3 b(B_1,B_2,u_3) \\ &\quad + \nu_3 b(u_1,B_2,B_3) - \nu_3 b(B_1,u_2,B_3),\end{aligned} \tag{101}$$

for $v_i = (u_i, B_i) \in V$, $i = 1,2,3$. Thanks to the discrete Hölder inequality we have

$$\begin{aligned}&|\chi(v_1,v_2,v_3)| \\ &\quad \leqslant c\|v_1\|_H^{1/2}\|\nabla v_1\|_H^{1/2}\|v_2\|_H\|v_3\|_H^{1/2}\|\nabla v_3\|_H^{1/2}, \\ &\quad\quad v_i \in V.\end{aligned} \tag{102}$$

Therefor (100) can be rewritten as follows:

$$\begin{aligned}&\frac{d}{dt}\|Q\|_H^2 + 2\nu\|\nabla Q\|_H^2 \\ &\quad \leqslant 2\chi(Q,\widehat{\varphi},Q) + 2\left(f(\tau+t) - f_\infty, Q_1\right) \\ &\quad \leqslant c\|\nabla\widehat{\varphi}\|_H^2\|Q\|_H^2 + 2\nu\|\nabla Q\|_H^2 + \|Q\|_H^2 \\ &\quad\quad + \|f(\tau+t) - f_\infty\|_H^2 \\ &\quad \leqslant c\left(\|\nabla\widehat{\varphi}\|_H^2 + 1\right)\|Q\|_H^2 + 2\nu\|\nabla Q\|_H^2 \\ &\quad\quad + \|f(\tau+t) - f_\infty\|_H^2\end{aligned} \tag{103}$$

Then applying the Gronwall inequality to (103), we have

$$\begin{aligned}&\|Q(t)\|_H^2 \\ &\quad \leqslant e^{c\int_0^t (\|\nabla\widehat{\varphi}(r)\|_H^2+1)dr}\|Q(0)\|_H^2 \\ &\quad\quad + ce^{c\int_0^t (\|\nabla\widehat{\varphi}(r)\|_H^2+1)dr}\int_0^t\|f(\tau+s) - f_\infty\|_H^2 ds \\ &\quad \leqslant e^{c\int_0^t (\|\nabla\widehat{\varphi}(r)\|_H^2+1)dr}\|Q(0)\|_H^2 \\ &\quad\quad + ce^{c\int_0^t (\|\nabla\widehat{\varphi}(r)\|_H^2+1)dr}\int_0^\infty\|f(\tau+s) - f_\infty\|_H^2 ds\end{aligned} \tag{104}$$

By Lemma 13, analogous results also hold for $\|\nabla\widehat{\varphi}(r)\|_H^2$. Hence, we have

$$\int_0^t \left(\|\nabla\widehat{\varphi}(r)\|_H^2 + 1\right) dr < \infty \tag{105}$$

Since $\|Q(0)\|_H = \|\varphi_0 - \widehat{\varphi}_0\|_H \longrightarrow 0$, it follows from Assumption F3 and (104) that

$$\|\varphi(\tau + t, \tau, \varphi_0) - \widehat{\varphi}(t, \widehat{\varphi}_0)\|_H \longrightarrow 0 \quad \text{as } \tau \longrightarrow -\infty \text{ for each } t \geqslant 0, \tag{106}$$

This completes the proof. □

Finally, by using the existence of a backwards compact attractor given in Theorem 14 and the asymptotic convergence given in Lemma 18, the following result was established following from Theorem 9 immediately.

Theorem 19. *Suppose assumptions F1 and F3 are satisfied. Then the nonautonomous MHD equations have a backwards compact pullback attractor $\mathscr{A} = \{\mathscr{A}(t) : t \in \mathbb{R}\}$, which converges to the global attractor $\mathscr{A}_\infty$ in H; that is,*

$$\lim_{t\longrightarrow-\infty} \operatorname{dist}_H(\mathscr{A}(t), \mathscr{A}_\infty) = 0. \tag{107}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Each of the authors contributed to each part of this study equally. All authors read and proved the final vision of the manuscript.

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