Research Article

Existence Results of Initial Value Problems for Hybrid Fractional Sum-Difference Equations

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We consider a hybrid fractional sum-difference initial value problem and a hybrid fractional sequential sum-difference initial value problem. The existence results of these two problems are proved by using the hybrid fixed point theorem for three operators in a Banach algebra and the generalized Krasnoselskii's fixed point theorem, respectively.

1. Introduction

As recognized that fractional difference calculus is a powerful tool used to describe many real world phenomena problems such as physics, chemistry, mechanics, control systems, flow in porous media, and electrical networks [1, 2], this is an impact on a researcher's motivation to develop the research works in this area. Basic definitions and properties of fractional difference calculus were proposed by Goodrich and Peterson [3]. The developments of the theory related to discrete fractional boundary value problems were studied by many authors (see [4–48]). In particular Sitthiwirattam [41, 42] studied three-point Caputo fractional difference-fractional sum boundary value problem for sequential Caputo fractional difference equation of the form

$$\begin{aligned} \Delta_C^\alpha [\phi_p(\Delta_C^\beta x)](t) &= f(t + \alpha + \beta - 1, x(t + \alpha + \beta - 1)), \\ \Delta_C^\beta x(\alpha - 1) &= 0, \\ x(\alpha + \beta + T) &= \rho \Delta^{-\gamma} x(\eta + \gamma), \end{aligned} \quad (1)$$

and three-point fractional sum boundary value problem for sequential Riemann-Liouville fractional difference equation of the form

$$\begin{aligned} \Delta_\alpha^\alpha (\Delta_{\alpha+\beta-1}^\beta + \lambda E_\beta) x(t) &= f(t + \alpha + \beta - 1, x(t + \alpha + \beta - 1)), \\ x(\alpha + \beta - 2) &= 0, \\ x(\alpha + \beta + T) &= \rho \Delta_{\alpha+\beta-1}^{-\gamma} x(\eta + \gamma), \end{aligned} \quad (2)$$

where $t \in \mathbb{N}_{0,T}$, $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, $0 < \gamma \leq 1$, $\eta \in \mathbb{N}_{\alpha+\beta-1, \alpha+\beta+T-1}$, ρ is a constant, $f: \mathbb{N}_{\alpha+\beta-2, \alpha+\beta+T} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $E_\beta x(t) = x(t + \beta - 1)$, and ϕ_p is the p -Laplacian operator.

Calculus which deals with derivatives and integrals of arbitrary orders is known as hybrid differential equations (i.e., quadratic perturbation of a nonlinear differential equation). Hybrid fractional differential equations are initialized to be used to model successfully several physical phenomena (see [49–51]). Apparently, this issue has found numerous miscellaneous applications connected with real world problems as they appear in many fields of engineering and science,

including biology, chemistry, diffusion, control theory, electromagnetic theory, fluid flow, signal and image processing, fractals theory, fitting of experimental data, potential theory, and viscoelasticity. For some recent developments on the topic, see [52–57]. Recently, there are several research works related to boundary value problems for hybrid differential equations (see [58–65]). For example, Sun et al. [60] studied the existence of solutions for the boundary value problem of fractional hybrid differential equations

$$\begin{aligned} D^\nu \left[\frac{x(t)}{f(t, x(t))} \right] + g(t, x(t)) &= 0, \quad 0 < t < 1, \\ x(0) &= x(1) = 0, \end{aligned} \quad (3)$$

where D^ν denotes the Riemann-Liouville fractional derivative of order ν , $1 < \nu \leq 2$.

Sitho et al. [64] studied existence results for the following initial value problem for hybrid fractional integrodifferential equations,

$$\begin{aligned} D^\alpha \left[\frac{x(t) - \sum_{i=1}^m I^{\beta_i} h_i(t, x(t))}{f(t, x(t))} \right] &= F(t, x(t)), \\ t &\in J : [0, T], \\ x(0) &= 0, \end{aligned} \quad (4)$$

and the initial value problem for hybrid fractional sequential integrodifferential equations,

$$\begin{aligned} D^\alpha \left[\frac{D^\omega x(t) - \sum_{i=1}^m I^{\beta_i} h_i(t, x(t))}{f(t, x(t))} \right] &= G(t, x(t), I^\gamma x(t)), \quad t \in J, \\ x(0) &= D^\omega x(0) = 0, \end{aligned} \quad (5)$$

where D^α denotes the Riemann-Liouville fractional derivative of order α , I^ϕ is the Riemann-Liouville fractional integral of order $\phi > 0$, $\phi \in \{\beta_1, \beta_2, \dots, \beta_m\}$, $0 < \alpha, \omega \leq 1$, $1 < \alpha + \omega \leq 2$, and functions are $F \in C(J \times \mathbb{R}, \mathbb{R})$, $G \in C(J \times \mathbb{R}^2, \mathbb{R})$, $f \in C(J \times \mathbb{R}, \mathbb{R} - \{0\})$, and $h_i \in C(J \times \mathbb{R}, \mathbb{R})$ with $h_i(0, 0) = 0$, $i = 1, 2, \dots, m$.

While the boundary value problem for hybrid fractional difference equations has not been studied, to fill this gap, we study a hybrid fractional difference initial value problem of the form,

$$\begin{aligned} \Delta^\alpha \left[\frac{u(t) - \Delta^{-\gamma} p(t + \gamma, u(t + \gamma))}{f(t, u(t))} \right] &= F[t + \alpha, u(t + \alpha)], \\ u(\alpha - 1) &= 0, \end{aligned} \quad (6)$$

and a hybrid fractional sequential sum-difference initial value problem of the form

$$\begin{aligned} \Delta^\alpha \left[\frac{\Delta^\beta u(t) - \Delta^{-\gamma} g(t + \beta + \gamma - 1, u(t + \beta + \gamma - 1))}{h(t, u(t))} \right] &= H[t + \alpha + \beta \\ &\quad - 1, u(t + \alpha + \beta - 1), \Delta^{-\omega} u(t + \alpha + \beta + \omega - 1)], \\ u(\alpha + \beta - 2) &= \Delta^\beta u(\alpha - 1) = 0, \end{aligned} \quad (7)$$

where $t \in \mathbb{N}_{0,T} := \{0, 1, \dots, T\}$, $\alpha, \beta, \gamma, \omega \in (0, 1]$, $1 < \alpha + \beta \leq 2$ are given constants, $f \in C(\mathbb{N}_{\alpha-1, T+\alpha} \times \mathbb{R}, \mathbb{R} - \{0\})$, $h \in C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta} \times \mathbb{R}, \mathbb{R} - \{0\})$, $p \in C(\mathbb{N}_{\alpha-1, T+\alpha} \times \mathbb{R}, \mathbb{R})$ with $p(\alpha - 1, 0) = 0$, $g \in C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta} \times \mathbb{R}, \mathbb{R})$ with $g(\alpha + \beta - 2, 0) = 0$, $F \in C(\mathbb{N}_{\alpha-1, T+\alpha} \times \mathbb{R}, \mathbb{R})$, and $H \in C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

The article is organized as follows. In Section 2, we recall some definitions and basic lemmas used in this work. Then, we present the solutions of (6) and (7) by converting the problem to an equivalent summation equation. In Sections 3 and 4, we prove existence results of problems (6) and (7) by employing the hybrid fixed point theorem for three operators in a Banach algebra and the generalized Krasnoselskii's fixed point theorem, respectively. We end with some examples to illustrate our results in the last section.

2. Preliminaries

In what follows are the notations, definitions, and lemmas which are used in the main results.

Definition 1 (see [6]). Define the generalized falling function by $t^\alpha := \Gamma(t + 1)/\Gamma(t + 1 - \alpha)$, for any t and α for which the right-hand side is defined. If $t + 1 - \alpha$ is a pole of the Gamma function and $t + 1$ is not a pole, then $t^\alpha = 0$.

Lemma 2 (see [4]). Assume that the factorial functions are well defined. If $t \leq r$, then $t^\alpha \leq r^\alpha$ for any $\alpha > 0$.

Definition 3 (see [6]). For $\alpha > 0$ and f defined on $\mathbb{N}_a := \{a, a + 1, \dots\}$, the α -order fractional sum of f is defined by

$$\Delta^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t - \sigma(s))^{\alpha-1} f(s), \quad (8)$$

where $t \in \mathbb{N}_{a+\alpha}$ and $\sigma(s) = s + 1$.

Definition 4 (see [6]). For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the α -order Riemann-Liouville fractional difference of f is defined by

$$\begin{aligned} \Delta^\alpha f(t) &:= \Delta^N \Delta^{-(N-\alpha)} f(t) \\ &= \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t+\alpha} (t - \sigma(s))^{-\alpha-1} f(s), \end{aligned} \quad (9)$$

where $t \in \mathbb{N}_{a+N-\alpha}$ and $N \in \mathbb{N}$ is satisfied with $0 \leq N - 1 < \alpha < N$.

Lemma 5 (see [5]). For any real number ν and any positive integer p , the following equality holds:

$$\Delta^{-\nu} \Delta^p f(t) = \Delta^p \Delta^{-\nu} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\nu-p+k}}{\Gamma(\nu-p+k+1)} \Delta^k f(a), \quad (10)$$

or f is defined on $\mathbb{N}_a$.

Lemma 6 (see [4]). Let $0 \leq N-1 < \alpha \leq N$. Then,

$$\Delta^{-\alpha} \Delta^\alpha y(t) = y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \quad (11)$$

for some $C_i \in \mathbb{R}$, with $1 \leq i \leq N$.

We provide the following lemma dealing with linear variant of the boundary value problems (6) and (7) and give a representation of the solution.

Lemma 7. Let $\alpha, \gamma \in (0, 1]$, $p \in C(\mathbb{N}_{\alpha-1, T+\alpha} \times \mathbb{R}, \mathbb{R})$ with $p(\alpha-1, 0) = 0$, $y \in C(\mathbb{N}_{\alpha-1, T+\alpha}, \mathbb{R} - \{0\})$, and $h \in C(\mathbb{N}_{\alpha-1, T+\alpha}, \mathbb{R})$. Then, for $t \in \mathbb{N}_{0, T}$, the problem

$$\Delta^\alpha \left[\frac{u(t) - \Delta^{-\gamma} p(t+\gamma, u(t+\gamma))}{y(t)} \right] = h(t+\alpha), \quad (12)$$

$$u(\alpha-1) = 0,$$

has the unique solution

$$u(t) = \frac{y(t)}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-\sigma(s))^{\alpha-1} h(s+\alpha) + \frac{1}{\Gamma(\gamma)} \sum_{s=\alpha-\gamma-1}^{t-\gamma} (t-\sigma(s))^{\gamma-1} p(s+\gamma, u(s+\gamma)), \quad (13)$$

for $t \in \mathbb{N}_{\alpha-1, T+\alpha}$.

Proof. Using the fractional sum of order α : $\Delta^{-\alpha}$ for (12) and Lemmas 5 and 6, we obtain

$$\begin{aligned} \Delta^{-\alpha} h(t+\alpha) &= \left[\frac{u(t) - \Delta^{-\gamma} p(t+\gamma, u(t+\gamma))}{y(t)} \right] - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad \cdot \Delta^{-(1-\alpha)} \left[\frac{u(\alpha-1) - \Delta^{-\gamma} p(\alpha-1+\gamma, u(\alpha-1+\gamma))}{y(\alpha-1)} \right] \\ &= \left[\frac{u(t) - \Delta^{-\gamma} p(t+\gamma, u(t+\gamma))}{y(t)} \right] - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \end{aligned}$$

$$\begin{aligned} &\cdot \Delta^{-(1-\alpha)} \left\{ \frac{1}{y(\alpha-1)} \times \left[u(\alpha-1) - \frac{1}{\Gamma(\gamma)} \sum_{s=\alpha-1}^{(\alpha+\gamma-1)-\gamma} (\alpha+\gamma-1-\sigma(s))^{\gamma-1} p(s, u(s)) \right] \right\} \\ &= \left[\frac{u(t) - \Delta^{-\gamma} p(t+\gamma, u(t+\gamma))}{y(t)} \right] - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad \cdot \Delta^{-(1-\alpha)} \left\{ \frac{1}{y(\alpha-1)} [u(\alpha-1) - \gamma p(\alpha-1, u(\alpha-1))] \right\}, \end{aligned} \quad (14)$$

for $t \in \mathbb{N}_{\alpha-1, T+\alpha}$.

Since $u(\alpha-1) = 0$, $p(\alpha-1, 0) = 0$, and $y(\alpha-1) \neq 0$, it follows that

$$\left[\frac{u(t) - \Delta^{-\gamma} p(t+\gamma, u(t+\gamma))}{y(t)} \right] = \Delta^{-\alpha} h(t+\alpha). \quad (15)$$

Thus, (13) holds. The proof is completed. $\square$

Lemma 8. Let $\alpha, \beta, \gamma \in (0, 1]$, $1 < \alpha + \beta \leq 2$, $g \in C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta} \times \mathbb{R}, \mathbb{R})$ with $g(\alpha+\beta-2, 0) = 0$, $x \in C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}, \mathbb{R} - \{0\})$, and $k \in C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}, \mathbb{R})$ be given. Then, for $t \in \mathbb{N}_{0, T}$, the problem

$$\begin{aligned} \Delta^\alpha \left[\frac{\Delta^\beta u(t) - \Delta^{-\gamma} g(t+\beta+\gamma-1, u(t+\beta+\gamma-1))}{x(t)} \right] &= k(t+\alpha+\beta-1), \\ u(\alpha+\beta-2) &= \Delta^\beta u(\alpha-1) = 0, \end{aligned} \quad (16)$$

has the unique solution

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \sum_{s=\alpha-1}^{t-\beta} (t-\sigma(s))^{\beta-1} x(s+\beta) \\ &\quad \cdot \sum_{\xi=0}^{s-\alpha} (s-\sigma(\xi))^{\alpha-1} k(\xi+\alpha+\beta-1) + \frac{1}{\Gamma(\gamma)} \\ &\quad \cdot \sum_{s=\alpha+\beta-\gamma-2}^{t-\gamma} (t-\sigma(s))^{\gamma-1} g(s+\beta+\gamma, u(s+\beta+\gamma)), \end{aligned} \quad (17)$$

for $t \in \mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}$.

Proof. Using the fractional sum of order $\alpha : \Delta^{-\alpha}$ for (16) and Lemmas 5 and 6, we obtain

$$\begin{aligned}
& \Delta^{-\alpha} k(t + \alpha + \beta - 1) \\
&= \left[\frac{\Delta^\beta u(t) - \Delta^{-\gamma} g(t + \beta + \gamma - 1, u(t + \beta + \gamma - 1))}{x(t)} \right] \\
&\quad - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \Delta^{-(1-\alpha)} \left[\frac{\Delta^\beta u(\alpha - 1) - \Delta^{-\gamma} g(\alpha - 1 + \beta + \gamma - 1, u(\alpha - 1 + \beta + \gamma - 1))}{x(\alpha - 1)} \right] \\
&= \left[\frac{\Delta^\beta u(t) - \Delta^{-\gamma} g(t + \beta + \gamma - 1, u(t + \beta + \gamma - 1))}{x(t)} \right] \\
&\quad - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \Delta^{-(1-\alpha)} \left\{ \frac{1}{x(\alpha - 1)} \times \left[\Delta^\beta u(\alpha - 1) - \frac{1}{\Gamma(\gamma)} \sum_{s=\alpha+\beta-2}^{(\alpha+\beta+\gamma-2)-\gamma} (\alpha + \beta + \gamma - 2 - \sigma(s))^{\gamma-1} g(s, u(s)) \right] \right\} \\
&= \left[\frac{\Delta^\beta u(t) - \Delta^{-\gamma} g(t + \beta + \gamma - 1, u(t + \beta + \gamma - 1))}{x(t)} \right] \\
&\quad - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \Delta^{-(1-\alpha)} \left\{ \frac{1}{x(\alpha - 1)} [\Delta^\beta u(\alpha - 1) - \gamma g(\alpha + \beta - 2, u(\alpha + \beta - 2))] \right\},
\end{aligned} \tag{18}$$

for $t \in \mathbb{N}_{\alpha-1, T+\alpha}$.

Since $u(\alpha + \beta - 2) = \Delta^\beta u(\alpha - 1) = 0$, $g(\alpha + \beta - 2, 0) = 0$, and $x(\alpha - 1) \neq 0$, we have

$$\begin{aligned}
& \left[\frac{\Delta^\beta u(t) - \Delta^{-\gamma} g(t + \beta + \gamma - 1, u(t + \beta + \gamma - 1))}{x(t)} \right] \\
&= \Delta^{-\alpha} k(t + \alpha + \beta - 1).
\end{aligned} \tag{19}$$

That can be arranged in the form

$$\begin{aligned}
\Delta^\beta u(t) &= x(t) \Delta^{-\alpha} k(t + \alpha + \beta - 1) \\
&\quad + \Delta^{-\gamma} g(t + \beta + \gamma - 1, u(t + \beta + \gamma - 1)).
\end{aligned} \tag{20}$$

Using the fractional sum of order β for (20), we obtain

$$\begin{aligned}
u(t) &= \Delta^{-\beta} [x(t) \Delta^{-\alpha} k(t + \alpha + \beta - 1)] \\
&\quad + \Delta^{-\beta-\gamma} g(t + \beta + \gamma - 1, u(t + \beta + \gamma - 1)),
\end{aligned} \tag{21}$$

for $t \in \mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}$.

Thus (17) holds. Our proof is completed. $\square$

Lemma 9 (Arzelà-Ascoli theorem [66]). *A set of functions in $C[a, b]$ with the sup norm is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a, b]$.*

Lemma 10 (see [66]). *If a set is closed and relatively compact, then it is compact.*

3. Hybrid Fractional Sum-Difference Initial Value Problem (6)

In this section, we aim to show the existence results for problem (6). To accomplish this, we let $E = C(\mathbb{N}_{\alpha-1, T+\alpha}, \mathbb{R})$ be a space of all functions u and defined a norm and a multiplication in E by

$$\|u\| = \max_{t \in \mathbb{N}_{\alpha-1, T+\alpha}} |u(t)|, \tag{22}$$

$$(uv)(t) = u(t) v(t).$$

In addition, we define operator $\mathcal{F} : E \rightarrow E$ by

$$\begin{aligned}
& (\mathcal{F}u)(t) \\
&= \frac{f(t, u(t))}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t - \sigma(s))^{\alpha-1} F((s + \alpha), u(s + \alpha)) \\
&\quad + \frac{1}{\Gamma(\gamma)} \sum_{s=\alpha-\gamma-1}^{t-\gamma} (t - \sigma(s))^{\gamma-1} p(s + \gamma, u(s + \gamma)).
\end{aligned} \tag{23}$$

Clearly, problem (6) has solutions if and only if operator $\mathcal{F}$ has fixed points. The first shows the existence and uniqueness of a solution to problem (6) by using the Banach contraction principle.

Theorem 11. *Assume that $f \in C(\mathbb{N}_{\alpha-1, T+\alpha} \times \mathbb{R}, \mathbb{R} - \{0\})$, $p \in C(\mathbb{N}_{\alpha-1, T+\alpha} \times \mathbb{R}, \mathbb{R})$ with $p(\alpha - 1, 0) = 0$, and $H \in C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. In addition, suppose that*

(A₁) there exist constants $L_1, L_2 > 0$ such that for each $t \in \mathbb{N}_{\alpha-1, T+\alpha}$ and $u, v \in \mathbb{R}$

$$\begin{aligned} |F(t, u) - F(t, v)| &\leq L_1 |u - v|, \\ |p(t, u) - p(t, v)| &\leq L_2 |u - v|, \end{aligned} \quad (24)$$

(A₂) there exists a positive function θ with bound $\|\theta\|$ such that for each $t \in \mathbb{N}_{\alpha-1, T+\alpha}$ and $u, v \in \mathbb{R}$

$$|f(t, u) - f(t, v)| \leq \theta(t). \quad (25)$$

If $L_1 \|\theta\| (T^\alpha / \Gamma(\alpha + 1)) + L_2 ((T + \alpha)^\gamma / \Gamma(\gamma + 1)) < 1$, then problem (6) has a unique solution.

Proof. We shall show that $\mathcal{F}$ is a contraction. For any $u, v \in \mathcal{E}$ and for each $t \in \mathbb{N}_{\alpha-1, T+\alpha}$, we have

$$\begin{aligned} |(\mathcal{F}u)(t) - (\mathcal{F}v)(t)| &\leq \frac{\|f\|}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t - \sigma(s))^{\alpha-1} \\ &\quad \cdot |F((s + \alpha), u(s + \alpha)) - F((s + \alpha), v(s + \alpha))| \\ &\quad + \frac{1}{\Gamma(\gamma)} \sum_{s=\alpha-\gamma-1}^{t-\gamma} (t - \sigma(s))^{\gamma-1} \\ &\quad \cdot |p(s + \gamma, u(s + \gamma)) - p(s + \gamma, v(s + \gamma))| \\ &\leq \frac{\|\theta\|}{\Gamma(\alpha)} L_1 \|u - v\| \sum_{s=0}^T (T + \alpha - \sigma(s))^{\alpha-1} + \frac{1}{\Gamma(\gamma)} \\ &\quad \cdot L_2 \|u - v\| \sum_{s=\alpha-\gamma-1}^{T+\alpha-\gamma} (T + \alpha - \sigma(s))^{\gamma-1} \leq \|u - v\| \\ &\quad \cdot \left\{ L_1 \|\theta\| \frac{T^\alpha}{\Gamma(\alpha + 1)} + L_2 \frac{(T + \alpha)^\gamma}{\Gamma(\gamma + 1)} \right\}. \end{aligned} \quad (26)$$

Thus, we have $\|(\mathcal{F}u) - (\mathcal{F}v)\| < \|u - v\|$.

Consequently, $\mathcal{F}$ is a contraction. Therefore, by the Banach fixed point theorem, we get that $\mathcal{F}$ has a fixed point which is a unique solution of problem (6). $\square$

In the second result, we deduce the existence of at least one solution of the initial value problem (6) by using the hybrid fixed point theorem for three operators in a Banach algebra. Clearly, E is a Banach algebra with respect to the above norm and multiplication in it.

Theorem 12 (hybrid fixed point theorem for three operators in a Banach algebra [67]). *Let S be a nonempty, closed convex, and bounded subset of the Banach algebra E , and let $A, C : E \rightarrow E$ and $B : S \rightarrow E$ be three operators such that*

- (i) *A and C are Lipschitzian with Lipschitz constants δ and ρ , respectively,*
- (ii) *B is completely continuous,*
- (iii) *$x = AxBy + Cx \Rightarrow x \in S$ for all $y \in S$,*
- (iv) *$\delta M + \rho < 1$, where $M = \|B(S)\|$.*

Then, the operator equation $AxBx + Cx = x$ has a solution.

Theorem 13. *Assume that $f \in C(\mathbb{N}_{\alpha-1, T+\alpha} \times \mathbb{R}, \mathbb{R} - \{0\})$, $p \in C(\mathbb{N}_{\alpha-1, T+\alpha} \times \mathbb{R}, \mathbb{R})$ with $p(\alpha - 1, 0) = 0$, and $H \in C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. In addition, suppose that*

(H₁) *there exist two positive functions ϕ and ψ with bound $\|\phi\|$ and $\|\psi\|$, respectively, such that for each $t \in \mathbb{N}_{\alpha-1, T+\alpha}$ and $u, v \in \mathbb{R}$*

$$\begin{aligned} |f(t, u) - f(t, v)| &\leq \phi(t) |u - v|, \\ |p(t, u) - p(t, v)| &\leq \psi(t) |u - v|, \end{aligned} \quad (27)$$

(H₂) *there exists a function $\omega \in C(\mathbb{N}_{\alpha-1, T+\alpha}, \mathbb{R}^+)$ and a continuous nondecreasing function $\Psi : [0, \infty) \rightarrow (0, \infty)$ such that for each $(t, u) \in \mathbb{N}_{\alpha-1, T+\alpha} \times \mathbb{R}$*

$$|F(t, u)| \leq \omega(t) \Psi(|u|), \quad (28)$$

(H₃) *there exists a number*

$$\begin{aligned} R &\geq \frac{\hat{f}_0 (\|\omega\| \Psi(R) T^\alpha / \Gamma(\alpha + 1)) + \hat{p}_0 ((T + \alpha)^\gamma / \Gamma(\gamma + 1))}{1 - \|\phi\| (\|\omega\| \Psi(R) T^\alpha / \Gamma(\alpha + 1)) - \|\psi\| ((T + \alpha)^\gamma / \Gamma(\gamma + 1))}, \end{aligned} \quad (29)$$

where $\hat{f}_0 = \max_{t \in \mathbb{N}_{\alpha-1, T+\alpha}} |f(t, 0)|$, $\hat{p}_0 = \max_{t \in \mathbb{N}_{\alpha-1, T+\alpha}} |p(t, 0)|$, and

$$\|\phi\| \frac{\|\omega\| \Psi(R) T^\alpha}{\Gamma(\alpha + 1)} + \|\psi\| \frac{(T + \alpha)^\gamma}{\Gamma(\gamma + 1)} < 1. \quad (30)$$

Then problem (6) has a unique solution on $\mathbb{N}_{\alpha-1, T+\alpha}$.

Proof. Define subset S of E as

$$S = \{u \in E : \|u\| \leq R\}. \quad (31)$$

We see that S is closed, convex, and bounded subset of the Banach space $\mathcal{E}$. By Lemma 7, we define three operators $\mathcal{A} : E \rightarrow E$, $\mathcal{B} : S \rightarrow E$, and $\mathcal{C} : E \rightarrow E$ by

$$\mathcal{A}u(t) := f(t, u(t)), \quad t \in \mathbb{N}_{\alpha-1, T+\alpha},$$

$$\begin{aligned} \mathcal{B}u(t) &:= \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t - \sigma(s))^{\alpha-1} F(s + \alpha, u(s + \alpha)), \\ &\quad t \in \mathbb{N}_{\alpha-1, T+\alpha}, \end{aligned} \quad (32)$$

$$\mathcal{C}u(t)$$

$$:= \frac{1}{\Gamma(\gamma)} \sum_{s=\alpha-\gamma-1}^{t-\gamma} (t - \sigma(s))^{\gamma-1} p(s + \gamma, u(s + \gamma)),$$

$$t \in \mathbb{N}_{\alpha-1, T+\alpha}.$$

Note that problem (6) has solutions if and only if the operator $u = \mathcal{A}u\mathcal{B}v + \mathcal{C}u$ has fixed points.

To show that all operators satisfy all the conditions of Theorem 12, we proceed with the following steps.

Step 1. Prove that $\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian on E .

For any $u, v \in \mathbb{R}$ and for each $t \in \mathbb{N}_{\alpha-1, T+\alpha}$, then by (H_1) , we have

$$\begin{aligned} |\mathcal{A}u(t) - \mathcal{A}v(t)| &= |f(t, u(t)) - f(t, v(t))| \leq \phi(t) |u - v| \\ &\leq \|\phi\| \|u - v\|, \\ |\mathcal{C}u(t) - \mathcal{C}v(t)| &= \frac{1}{\Gamma(\gamma)} \sum_{s=\alpha-\gamma-1}^{t-\gamma} (t - \sigma(s))^{\gamma-1} \\ &\cdot |p(s + \gamma, u(s + \gamma)) - p(s + \gamma, v(s + \gamma))| \leq \frac{1}{\Gamma(\gamma)} \end{aligned} \quad (33)$$

$$\begin{aligned} &\cdot \sum_{s=\alpha-\gamma-1}^{T+\alpha-\gamma} (T + \alpha - \sigma(s))^{\gamma-1} \psi(s) |u - v| \\ &\leq \frac{\|\psi(s)\| (T + \alpha)^\gamma}{\Gamma(\gamma + 1)} \|u - v\|. \end{aligned}$$

This implies that, for all $u, v \in \mathbb{R}$,

$$\begin{aligned} \|\mathcal{A}u(t) - \mathcal{A}v(t)\| &\leq \|\phi\| \|u - v\|, \\ \|\mathcal{C}u(t) - \mathcal{C}v(t)\| &\leq \frac{\|\psi(s)\| (T + \alpha)^\gamma}{\Gamma(\gamma + 1)} \|u - v\|. \end{aligned} \quad (34)$$

Therefore, $\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian on E with Lipschitz constants $\|\phi\|$ and $\|\psi(s)\|(T + \alpha)^\gamma/\Gamma(\gamma + 1)$.

Step 2. We prove that $\mathcal{B}$ is completely continuous on S .

Since F is continuous, the operator $\mathcal{B}$ is continuous on S . Next, we will prove that the set $\mathcal{B}(S)$ is uniformly bounded in S . For any $u \in S$, we find that

$$\begin{aligned} |\mathcal{B}u(t)| &\leq \left| \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t - \sigma(s))^{\alpha-1} \omega(s) \Psi(R) \right| \\ &\leq \frac{\|\omega\| \Psi(R) T^\alpha}{\Gamma(\alpha + 1)} =: K, \end{aligned} \quad (35)$$

for all $t \in \mathbb{N}_{\alpha-1, T+\alpha}$. Therefore, $\|\mathcal{B}\| \leq K$, which shows that $\mathcal{B}$ is uniformly bounded on S .

Next we show that $\mathcal{B}(S)$ is an equicontinuous set in E . For any $\epsilon > 0$, there exists a positive constant δ such that, for $t_1, t_2 \in \mathbb{N}_{\alpha-1, T+\alpha}$,

$$|t_2^\alpha - t_1^\alpha| < \frac{\Gamma(\alpha + 1)}{\|\omega\| \Psi(R)}, \quad \text{whenever } |t_2 - t_1| < \delta. \quad (36)$$

Then we obtain

$$\begin{aligned} |(\mathcal{B}x)(t_2) - (\mathcal{B}x)(t_1)| &\leq \frac{\|\omega\| \Psi(R)}{\Gamma(\alpha)} \left| \sum_{s=0}^{t_2-\alpha} (t_2 - \sigma(s))^{\alpha-1} \right. \\ &\quad \left. - \sum_{s=0}^{t_1-\alpha} (t_1 - \sigma(s))^{\alpha-1} \right| \leq \frac{\|\omega\| \Psi(R)}{\Gamma(\alpha + 1)} |t_2^\alpha - t_1^\alpha| < \epsilon. \end{aligned} \quad (37)$$

This implies that the set $\mathcal{B}(S)$ is an equicontinuous set. From the Arzelà-Ascoli theorem, we find that $\mathcal{B}$ is completely continuous.

Step 3. $u = \mathcal{A}u\mathcal{B}v + \mathcal{C}u \Rightarrow u \in S$ for all $v \in S$.

Let $u \in E$ and $v \in S$ be arbitrary elements such that $u = \mathcal{A}u\mathcal{B}v + \mathcal{C}u$. Then,

$$\begin{aligned} |u(t)| &\leq |\mathcal{A}u(t)| |\mathcal{B}v(t)| + |\mathcal{C}u(t)| \leq \frac{|f(t, u(t))|}{\Gamma(\alpha)} \\ &\cdot \sum_{s=0}^{t-\alpha} (t - \sigma(s))^{\alpha-1} |F(s + \alpha, u(s + \alpha))| + \frac{1}{\Gamma(\gamma)} \\ &\cdot \sum_{s=\alpha-1}^t (t + \gamma - \sigma(s))^{\gamma-1} |p(s, u(s))| \\ &\leq \frac{|f(t, u(t)) - f(t, 0)| + |f(t, 0)|}{\Gamma(\alpha)} \\ &\cdot \sum_{s=0}^T (T + \alpha - \sigma(s))^{\alpha-1} \|\omega\| \Psi(R) + \frac{1}{\Gamma(\gamma)} \\ &\cdot \sum_{s=\alpha-1}^{T+\alpha} (T + \alpha + \gamma - \sigma(s))^{\gamma-1} \\ &\cdot (|p(s, u(s)) - p(t, 0)| + |p(t, 0)|) \leq (R \|\phi\| \\ &+ \hat{f}_0) \frac{\|\omega\| \Psi(R) T^\alpha}{\Gamma(\alpha + 1)} + (R \|\psi\| + \hat{p}_0) \frac{(T + \alpha)^\gamma}{\Gamma(\gamma + 1)}. \end{aligned} \quad (38)$$

We find that

$$\begin{aligned} \|u\| &\leq (R \|\phi\| + \hat{f}_0) \frac{\|\omega\| \Psi(R) T^\alpha}{\Gamma(\alpha + 1)} \\ &+ (R \|\psi\| + \hat{p}_0) \frac{(T + \alpha)^\gamma}{\Gamma(\gamma + 1)} \leq R. \end{aligned} \quad (39)$$

Therefore, $u \in S$.

Step 4. We prove that $\delta M + \rho < 1$. Since

$$\begin{aligned} M = \|\mathcal{B}(S)\| &= \sup_{u \in S} \left\{ \max_{t \in \mathbb{N}_{\alpha-1, T+\alpha}} |\mathcal{B}u(t)| \right\} \\ &\leq \frac{\|\omega\| \Psi(R) T^\alpha}{\Gamma(\alpha + 1)}, \end{aligned} \quad (40)$$

and by (H_3) , we have

$$\|\phi\| M + \|\psi\| \frac{(T + \alpha)^\gamma}{\Gamma(\gamma + 1)} < 1, \quad (41)$$

with $\delta = \|\phi\|$ and $\rho = \|\psi\|((T + \alpha)^\gamma/\Gamma(\gamma + 1))$.

We see that all the conditions of Theorem 12 are satisfied. Hence, the operator equation $u = \mathcal{A}u\mathcal{B}v + \mathcal{C}u$ has a solution in S . In consequence, problem (6) has a solution on $\mathbb{N}_{\alpha-1, T+\alpha}$. This completes the proof. $\square$

4. Hybrid Fractional Sequential Sum-Difference Initial Value Problem (7)

In this section, we prove existence results of problem (7). To accomplish this, we denote that $X = C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}, \mathbb{R})$ and define the space of all functions u with the norm by

$$\|u\| = \max_{t \in \mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}} |u(t)|. \quad (42)$$

In addition, we define the operator $\mathcal{H} : X \rightarrow X$ by

$$\begin{aligned} (\mathcal{H}u)(t) = & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \sum_{s=\alpha-1}^{t-\beta} \left[(t-\sigma(s))^{\beta-1} \right. \\ & \cdot h((s+\beta), u(s+\beta)) \times \sum_{\xi=0}^{s-\alpha} (s-\sigma(\xi))^{\alpha-1} \\ & \cdot H((\xi+\alpha+\beta-1), u(\xi+\alpha+\beta-1)) \left. \right] + \frac{1}{\Gamma(\gamma)} \\ & \cdot \sum_{s=\alpha+\beta-\gamma-2}^{t-\gamma} (t-\sigma(s))^{\gamma-1} g(s+\beta+\gamma, u(s+\beta+\gamma)). \end{aligned} \quad (43)$$

Clearly, problem (7) has solutions if and only if the operator $\mathcal{H}$ has fixed points. The first shows the existence and uniqueness of a solution to problem (7) by using the Banach contraction principle.

Theorem 14. Assume that $h \in C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta} \times \mathbb{R}, \mathbb{R} - \{0\})$, $g \in C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta} \times \mathbb{R}, \mathbb{R})$ with $g(\alpha+\beta-2, 0) = 0$, and $H \in C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. In addition, suppose that

$$\begin{aligned} |(\mathcal{H}u)(t) - (\mathcal{H}v)(t)| \leq & \frac{\|h\|}{\Gamma(\alpha)\Gamma(\beta)} \sum_{s=\alpha-1}^{t-\beta} \left[(t-\sigma(s))^{\beta-1} \right. \\ & \cdot \sum_{\xi=0}^{s-\alpha} (s-\sigma(\xi))^{\alpha-1} |H((\xi+\alpha+\beta-1), u(\xi+\alpha+\beta-1)) - H((\xi+\alpha+\beta-1), v(\xi+\alpha+\beta-1))| \left. \right] + \frac{1}{\Gamma(\gamma)} \\ & \cdot \sum_{s=\alpha+\beta-\gamma-2}^{t-\gamma} (t-\sigma(s))^{\gamma-1} |g(s+\beta+\gamma, u(s+\beta+\gamma)) - g(s+\beta+\gamma, v(s+\beta+\gamma))| \leq \frac{\|\mathcal{H}\|}{\Gamma(\alpha)\Gamma(\beta)} \\ & \cdot \sum_{s=\alpha-1}^{T+\alpha} \sum_{\xi=0}^{s-\alpha} (T+\alpha+\beta-\sigma(s))^{\beta-1} (s-\sigma(\xi))^{\alpha-1} [M_1 |u-v| + M_2 |\Delta^{-\omega}u - \Delta^{-\omega}v|] + \frac{N}{\Gamma(\gamma)} \|u-v\| \sum_{s=\alpha+\beta-\gamma-2}^{T+\alpha+\beta-\gamma} (T+\alpha+\beta \\ & - \sigma(s))^{\gamma-1} \leq \frac{\|\mathcal{H}\|}{\Gamma(\alpha)\Gamma(\beta)} \left[M_1 + M_2 \frac{(T+\alpha+\beta)^\omega}{\Gamma(\omega+1)} \right] \|u-v\| (T+\beta)^{\beta-1} (T+\alpha+\beta)^{\alpha-1} + \frac{N}{\Gamma(\gamma)} \|u-v\| (T+\alpha+\beta)^\gamma \\ & \leq \|u-v\| \left\{ \frac{\|\mathcal{H}\|}{\Gamma(\alpha)\Gamma(\beta)} (T+\beta)^{\beta-1} (T+\alpha+\beta)^{\alpha-1} \left[M_1 + M_2 \frac{(T+\alpha+\beta)^\omega}{\Gamma(\omega+1)} \right] + \frac{N}{\Gamma(\gamma+1)} \right\} = \|u-v\| \Omega. \end{aligned} \quad (47)$$

Thus, we have $\|(\mathcal{H}u) - (\mathcal{H}v)\| < \|u-v\|$.

Consequently, $\mathcal{H}$ is a contraction. Therefore, by the Banach fixed point theorem, we get that $\mathcal{H}$ has a fixed point which is a unique solution of problem (7). $\square$

In the second result, we deduce the existence of at least one solution of the initial value problem (7) by using the generalized Krasnoselskii's fixed point theorem.

Theorem 15 (generalized Krasnoselskii's fixed point theorem [68]). Let M be a nonempty, closed convex, and bounded

(B_1) there exist constants $M_1, M_2, N > 0$ such that for each $t \in \mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}$ and $u, v, u^*, v^* \in \mathbb{R}$

$$\begin{aligned} |H(t, u, u^*) - H(t, v, v^*)| \\ \leq M_1 |u-v| + M_2 |u^* - v^*|, \\ |g(t, u) - g(t, v)| \leq N |u-v|, \end{aligned} \quad (44)$$

(B_2) there exists a positive function ϑ with bound $\|\vartheta\|$ such that for each $t \in \mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}$ and $u, v \in \mathbb{R}$

$$|h(t, u) - h(t, v)| \leq \vartheta(t). \quad (45)$$

If $\Omega < 1$ then problem (7) has a unique solution, where

$$\begin{aligned} \Omega := & \frac{\|\mathcal{H}\|}{\Gamma(\alpha)\Gamma(\beta)} (T+\beta)^{\beta-1} (T+\alpha+\beta)^{\alpha-1} \\ & \cdot \left[M_1 + M_2 \frac{(T+\alpha+\beta)^\omega}{\Gamma(\omega+1)} \right] + \frac{N (T+\alpha+\beta)^\gamma}{\Gamma(\gamma+1)}. \end{aligned} \quad (46)$$

Proof. We shall show that $\mathcal{H}$ is a contraction. For any $u, v \in \mathcal{C}$ and for each $t \in \mathbb{N}_{\alpha-1, T+\alpha}$, we have

subset of the Banach space X . Let $A : X \rightarrow X$ and $B : M \rightarrow X$ be two operators such that

- (i) A is a contraction,
- (ii) B is completely continuous,
- (iii) $x = Ax + By$ for all $y \in M \Rightarrow x \in M$.

Then, the operator equation $Ax + Bx = x$ has a solution.

Theorem 16. Assume that $h \in C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta} \times \mathbb{R}, \mathbb{R} - \{0\})$, $g \in C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta} \times \mathbb{R}, \mathbb{R})$ with $g(\alpha + \beta - 2, 0) = 0$, and $H \in C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. In addition, suppose that

(K₁) there exist two positive functions ϑ and θ with bound $\|\vartheta\|$ and $\|\theta\|$, respectively, such that, for each $t \in \mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}$ and $u, v, \Delta^{-\omega}u, \Delta^{-\omega}v \in \mathbb{R}$,

$$\begin{aligned} |h(t, u) - h(t, v)| &\leq \vartheta(t) |u - v|, \\ |H[t, u, \Delta^{-\omega}u] - H[t, u, \Delta^{-\omega}v]| & \\ &\leq \theta(t) (|u - v| - |\Delta^{-\omega}u - \Delta^{-\omega}v|), \end{aligned} \quad (48)$$

(K₂) there exist functions $\mu, \nu, \kappa \in C(\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}, \mathbb{R}^+)$ such that

$$\begin{aligned} |h(t, u)| &\leq \mu(t) \quad \forall (t, u) \in \mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta} \times \mathbb{R}, \\ |H(t, u, \Delta^{-\omega}u)| &\leq \nu(t) \\ \forall (t, u, \Delta^{-\omega}u) &\in \mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta} \times \mathbb{R} \times \mathbb{R}, \\ |g(t, u)| &\leq \kappa(t) \quad \forall (t, u) \in \mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta} \times \mathbb{R}. \end{aligned} \quad (49)$$

If

$$\begin{aligned} \frac{(T + \alpha + \beta)^\beta}{\Gamma(\beta + 1)} \left\{ \frac{\|\vartheta\| \|\nu\| (T + \alpha + \beta)^\alpha}{\Gamma(\alpha + 1)} \right. \\ \left. + \|\mu\| \|\theta\| \left(1 + \frac{(T + \alpha + \beta)^\omega}{\Gamma(\omega + 1)} \right) \right\} < 1, \end{aligned} \quad (50)$$

then problem (7) has a unique solution on $\mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}$.

Proof. Let $\max_{t \in \mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}} |\mu(t)| = \|\mu\|$, $\max_{t \in \mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}} |\nu(t)| = \|\nu\|$, and $\max_{t \in \mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}} |\kappa(t)| = \|\kappa\|$, and choose a constant

$$R \geq \frac{\|\kappa\| (T + \alpha + \beta)^\gamma}{\Gamma(\gamma + 1)} + \frac{\|\mu\| \|\nu\| (T + \alpha + \beta)^{\alpha+\beta}}{\Gamma(\alpha + 1) \Gamma(\beta + 1)}. \quad (51)$$

We consider $B_R = \{u \in X : \|u\| \leq R\}$. Define four operators $\mathcal{P} : X \rightarrow X$, $\mathcal{Q} : B_R \rightarrow X$, and $\mathcal{S} : X \rightarrow X$ by

$$\begin{aligned} \mathcal{P}u(t) &:= h(t, u(t)), \\ \mathcal{Q}u(t) &:= \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t - \sigma(s))^{\alpha-1} \\ &\quad \cdot H(s + \alpha + \beta - 1, u(s + \alpha + \beta - 1)), \\ \mathcal{S}u(t) &:= \frac{1}{\Gamma(\gamma)} \sum_{s=\alpha+\beta-2}^{t-\gamma} (t - \sigma(s))^{\gamma-1} \\ &\quad \cdot g(s + \beta + \gamma, u(s + \beta + \gamma)), \end{aligned} \quad (52)$$

for $t \in \mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}$, and

$$\mathcal{T}u(t) := \frac{1}{\Gamma(\beta)} \sum_{s=0}^{t-\beta} (t - \sigma(s))^{\beta-1} \mathcal{P}u(s) \mathcal{Q}u(s). \quad (53)$$

Problem (7) has solutions if and only if the operator $u = \mathcal{S}u + \mathcal{T}u$ has fixed points.

The proof is divided into three steps as follows.

Step 1. Verify $u = \mathcal{S}u + \mathcal{T}u$ map bounded sets into bounded sets in B_R .

For each $v \in B_R$, we obtain

$$\begin{aligned} |u(t)| &= |\mathcal{S}u(t) + \mathcal{T}u(t)| \leq \frac{1}{\Gamma(\gamma)} \\ &\quad \cdot \sum_{s=\alpha+\beta-\gamma-2}^{t-\gamma} (t - \sigma(s))^{\gamma-1} |g(s + \beta + \gamma, u(s + \beta + \gamma))| \\ &\quad + \frac{1}{\Gamma(\beta)} \sum_{s=0}^{t-\beta} (t - \sigma(s))^{\beta-1} |\mathcal{P}u(s)| |\mathcal{Q}u(s)| \leq \frac{\|\kappa\|}{\Gamma(\gamma)} \\ &\quad \cdot \sum_{s=\alpha+\beta-\gamma-2}^{T+\alpha+\beta-\gamma} (T + \alpha + \beta - \sigma(s))^{\gamma-1} + \frac{\|\mu\| \|\nu\|}{\Gamma(\alpha) \Gamma(\beta)} \\ &\quad \cdot \sum_{s=0}^{T+\alpha} \left[(T + \alpha + \beta - \sigma(s))^{\beta-1} \sum_{\xi=0}^{s-\alpha} (\xi - \sigma(\xi))^{\alpha-1} \right] \\ &\leq \frac{\|\kappa\| (T + \alpha + \beta)^\gamma}{\Gamma(\gamma + 1)} + \frac{\|\mu\| \|\nu\| (T + \alpha)^{\alpha+\beta}}{\Gamma(\alpha + 1) \Gamma(\beta + 1)} \leq R. \end{aligned} \quad (54)$$

Thus, $\|u\| \leq R$. This implies that $\mathcal{S}u + \mathcal{T}u$ is uniformly bounded.

Hence, condition (iii) of Theorem 15 holds.

Step 2. Check that $\mathcal{T}$ is contraction mapping.

For any $u, v \in \mathbb{R}$ and for each $t \in \mathbb{N}_{\alpha+\beta-2, T+\alpha+\beta}$, by (K₁), we have

$$\begin{aligned} |\mathcal{T}u(t) - \mathcal{T}v(t)| &\leq \frac{1}{\Gamma(\beta)} \left| \sum_{s=0}^{t-\beta} (t - \sigma(s))^{\beta-1} [|\mathcal{P}u(s)| \right. \\ &\quad \cdot |\mathcal{Q}u(s)| - |\mathcal{P}v(s)| |\mathcal{Q}v(s)|] \Big| = \frac{1}{\Gamma(\beta)} \left| \sum_{s=0}^{t-\beta} (t \right. \\ &\quad \left. - \sigma(s))^{\beta-1} [|\mathcal{Q}u(s)| (|\mathcal{P}u(s)| - |\mathcal{P}v(s)|) + |\mathcal{P}v(s)| \right. \\ &\quad \left. \cdot (|\mathcal{Q}u(s)| - |\mathcal{Q}v(s)|)] \right| = \frac{1}{\Gamma(\beta)} \left| \sum_{s=0}^{t-\beta} (t - \sigma(s))^{\beta-1} \right. \\ &\quad \cdot \left\{ \frac{\|\vartheta\| \|\nu\| (T + \alpha + \beta)^\alpha}{\Gamma(\alpha + 1)} \|u - v\| + \|\mu\| \|\theta\| \right. \\ &\quad \cdot \left(\|u - v\| + \|u - v\| \frac{1}{\Gamma(\omega)} \sum_{s=\alpha+\beta-2}^t (t - \sigma(s))^{\omega-1} \right) \Big| \\ &\leq \|u - v\| \frac{(T + \alpha + \beta)^\beta}{\Gamma(\beta + 1)} \left\{ \frac{\|\vartheta\| \|\nu\| (T + \alpha + \beta)^\alpha}{\Gamma(\alpha + 1)} \right. \\ &\quad \left. + \|\mu\| \|\theta\| \left(1 + \frac{(T + \alpha + \beta)^\omega}{\Gamma(\omega + 1)} \right) \right\}. \end{aligned} \quad (55)$$

By (50), $\mathcal{F}$ is a contraction mapping. Hence, condition (i) of Theorem 15 holds.

Step 3. Check that $\mathcal{S}$ is completely continuous on B_R .

The operator $\mathcal{S}$ is obviously continuous on B_R . Furthermore, $\mathcal{S}$ is uniformly bounded on B_R since

$$\|\mathcal{S}\| \leq \frac{\|\kappa\| (T + \alpha + \beta)^\gamma}{\Gamma(\gamma + 1)}. \quad (56)$$

For any $\epsilon > 0$, there exists a positive constant δ such that for $t_1, t_2 \in \mathbb{N}_{\alpha+\beta, T+\alpha+\beta}$

$$|t_2^\gamma - t_1^\gamma| < \frac{\Gamma(\gamma + 1)}{\|\kappa\|}, \quad \text{whenever } |t_2 - t_1| < \delta. \quad (57)$$

Then,

$$\begin{aligned} & |(\mathcal{S}x)(t_2) - (\mathcal{S}x)(t_1)| \\ & \leq \frac{1}{\Gamma(\gamma)} \left| \sum_{s=\alpha+\beta-\gamma-2}^{t_2-\gamma} (t_2 - \sigma(s))^{\gamma-1} \right. \\ & \quad \cdot |g(s + \beta + \gamma, u(s + \beta + \gamma))| \\ & \quad - \sum_{s=\alpha+\beta-\gamma-2}^{t_1-\gamma} (t_1 - \sigma(s))^{\gamma-1} \\ & \quad \cdot |g(s + \beta + \gamma, u(s + \beta + \gamma))| \Big| \leq \frac{\|\kappa\|}{\Gamma(\gamma + 1)} |t_2^\gamma \\ & \quad - t_1^\gamma| < \epsilon. \end{aligned} \quad (58)$$

This implies that the set $\mathcal{S}(B_R)$ is an equicontinuous set. Therefore, by the Arzelà-Ascoli theorem, we find that $\mathcal{S}$ is completely continuous. Hence, condition (ii) of Theorem 15 holds.

We see that all the assumptions of Theorem 15 are satisfied. Therefore, we can conclude that problem (7) has at least one solution. The proof is completed. $\square$

5. Examples

In this section, we provide some examples to illustrate our results.

Example 1. Consider the following fractional difference initial value problem:

$$\begin{aligned} & \Delta^{1/2} \left[\frac{u(t) - \Delta^{-1/3} p(t + 1/3, u(t + 1/3))}{f(t, u(t))} \right] \\ & = F \left[t + \frac{1}{2}, u \left(t + \frac{1}{2} \right) \right], \quad t \in \mathbb{N}_{0,10}, \\ & u \left(-\frac{1}{2} \right) = 0, \end{aligned} \quad (59)$$

where

$$\begin{aligned} p(t, u(t)) &= \frac{u^2(t) + 4|u(t)|}{3 + |u(t)|} \cdot \frac{t \sin^2 2\pi t}{5000(2 + e^{-t})}, \\ f(t, u(t)) &= \frac{10(2 \cos^2 2\pi t + 3t)}{3(100\pi - 2t)^2} \cdot \frac{u^2(t) + 3|u(t)|}{|u(t)| + 10}, \\ F[t, u(t)] &= \frac{(t^2 + 5)(8|u(t)| + 10\pi)}{80(100\pi - t)}. \end{aligned} \quad (60)$$

We set $\alpha = 1/2$, $\gamma = 1/3$, and $T = 10$.

Noticing that (H_1) – (H_2) hold, for each $t \in \mathbb{N}_{-1/2, 21/2}$, we have

$$\begin{aligned} |p(t, u) - p(t, v)| &= \frac{4t}{15000(2 + e^{-t})} |u - v|, \\ |f(t, u) - f(t, v)| &= \frac{2 + 3t}{(100\pi - 2t)^2} |u - v|, \\ |F[t, u] - F[t, v]| &= \left(\frac{t^2 + 5}{100\pi - t} \right) \left(\frac{|u|}{10} + \frac{\pi}{8} \right). \end{aligned} \quad (61)$$

Thus,

$$\begin{aligned} \|\phi\| &= 0.000389, \\ \|\psi\| &= 0.00139, \\ \|\omega\| &= 0.441. \end{aligned} \quad (62)$$

Finally, we find that

$$\begin{aligned} \hat{f}_0 &= 0.00389, \\ \hat{p}_0 &= 0.00105. \end{aligned} \quad (63)$$

So, (H_3) holds with a number $R \in [730.5887, 892.9406]$.

Hence, by Theorem 13, problem (59) has a unique solution on $\mathbb{N}_{-1/2, 21/2}$.

Example 2. Consider the following fractional difference boundary value problem:

$$\begin{aligned} & \Delta^{1/2} \left[\frac{\Delta^{2/5} u(t) - \Delta^{-1/3} g(t - 4/15, u(t - 4/15))}{h(t, u(t))} \right] \\ & = H \left[t - \frac{1}{10}, u \left(t - \frac{1}{10} \right), \Delta^{-3/4} u \left(t + \frac{13}{20} \right) \right], \\ & u \left(-\frac{11}{10} \right) = \Delta^{2/5} u \left(-\frac{1}{2} \right) = 0, \end{aligned} \quad (64)$$

for $t \in \mathbb{N}_{0,10}$, where

$$\begin{aligned} g(t, u(t)) &= \log \left(1 + \frac{|u(t)| e^t \sin 5\pi t}{1 + |u(t)|} \right), \\ h(t, u(t)) &= \left(\frac{|u(t)| + 1}{|u(t)| + 2} \right) \frac{e^{-\sin^2 \pi t}}{100e - t}, \end{aligned}$$

$$\begin{aligned}
& H \left[t, u(t), \Delta^{-3/4} u \left(t + \frac{3}{4} \right) \right] \\
&= \frac{1}{3} \arctan \left(\frac{3 |u(t)| \sin^2 5\pi t}{10 + |u(t)|} \right) \\
&\quad - \frac{1}{3} \arctan \left(\frac{3 \left| \Delta^{-3/4} u(t + 3/4) \right| \sin^2 5\pi t}{10 + \left| \Delta^{-3/4} u(t + 3/4) \right|} \right). \quad (65)
\end{aligned}$$

We let $\alpha = 1/2$, $\beta = 2/5$, $\gamma = 1/3$, $\omega = 3/4$, and $T = 10$.

Noticing that (K_1) – (K_2) hold, for each $t \in \mathbb{N}_{-11/10, 109/10}$, we obtain

$$\begin{aligned}
& |h(t, u) - h(t, v)| = \left(\frac{e^{-\sin^2 \pi t}}{400e - 4t} \right) |u - v|, \\
& \left| H \left[t, u, \Delta^{-3/4} u \right] - H \left[t, v, \Delta^{-3/4} v \right] \right| \\
&= \frac{1}{10} \sin^2 5\pi t \left(|u - v| + \left| \Delta^{-3/4} u - \Delta^{-3/4} v \right| \right), \quad (66) \\
& |h(t, u)| = \frac{e^{-\sin^2 \pi t}}{100e - t} + e^{-\pi t}, \\
& \left| H \left[t, u, \Delta^{-3/4} u \right] \right| = 3 \sin^2 5\pi t.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\|\vartheta\| &= 0.000923, \\
\|\theta\| &= 0.1, \\
\|\mu\| &= 0.00369, \\
\|\nu\| &= 3. \quad (67)
\end{aligned}$$

Finally, we find that

$$\begin{aligned}
& \frac{(T + \alpha + \beta)^\beta}{\Gamma(\beta + 1)} \left\{ \frac{\|\vartheta\| \|\nu\| (T + \alpha + \beta)^\alpha}{\Gamma(\alpha + 1)} \right. \\
& \left. + \|\mu\| \|\theta\| \left(1 + \frac{(T + \alpha + \beta)^\omega}{\Gamma(\omega + 1)} \right) \right\} \approx 0.2277 < 1. \quad (68)
\end{aligned}$$

Hence, by Theorem 16, problem (64) has a unique solution on $\mathbb{N}_{-11/10, 109/10}$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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