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Research Article

Existence of Solutions to Boundary Value Problems for a Fourth-Order Difference Equation

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In this article, some new existence criteria of at least one solution to boundary value problems for a fourth-order difference equation are obtained by using the critical point theory. In a special case, a necessary and sufficient condition for the existence and uniqueness of solution is also established. An example of the main result is given.

1. Introduction

Throughout this article, the sets of all natural numbers, integers, and real numbers are defined as $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$, respectively. The transpose of a vector x is defined as x^*. For $a, b \in \mathbb{Z}$ with $a < b$, the discrete interval $\{a, a+1, \ldots, b\}$ is denoted as $[a, b]_{\mathbb{Z}}$.

Consider the existence of solutions to boundary value problem (BVP) for a fourth-order difference equation:

$$\begin{aligned} &\Delta^2\left(p(t-2)\Delta^2 x(t-2)\right) - \Delta\left(q(t-1)\Delta x(t-1)\right) \\ &\quad - r(t)x(t) + f(t) = 0, \quad t \in [1, T]_{\mathbb{Z}}, \end{aligned} \tag{1}$$

satisfying the boundary value conditions

$$\Delta^k x(-1) = \Delta^k x(T-1), \quad k = 0, 1, 2, 3, \tag{2}$$

where $T \in \mathbb{N}$ and $T \geq 1$, Δ is the forward difference operator denoted as $\Delta x(t) = x(t+1) - x(t)$, $\Delta^k x(t) = \Delta(\Delta^{k-1}x(t))$ $(k = 2, 3, 4)$, $\Delta^0 x(t) = x(t)$, $p(t) \in C([-1, T]_{\mathbb{Z}}, \mathbb{R})$ with $p(-1) = p(T-1)$, $p(0) = p(T)$, $q(t) \in C([0, T]_{\mathbb{Z}}, \mathbb{R})$ with $q(0) = q(T)$, $r(t) \in C([1, T]_{\mathbb{Z}}, \mathbb{R})$, $f(t) \in C([1, T]_{\mathbb{Z}}, \mathbb{R})$.

And (1), (2) can be regarded as a discrete analogue of

$$\begin{aligned} &\left[p(s)x''(s)\right]'' - \left[q(s)x'(s)\right]' - r(s)x(s) + f(s) = 0, \\ &\qquad s \in (0, 1), \end{aligned} \tag{3}$$

with boundary value conditions

$$x^{(i)}(0) = x^{(i)}(1), \quad i = 0, 1, 2, 3. \tag{4}$$

Equations similar in structure to (3) arise in the study of the existence of solutions to differential equations [1–11].

Difference equations are widely found in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology, and so on. Many authors were interested in difference equations and obtained some significant results [12–29].

Consider the fourth-order nonlinear difference equation

$$\Delta^2\left(p(t)\Delta^2 x(t)\right) + f(t, x(t)) = 0, \quad t \in \mathbb{Z}. \tag{5}$$

Thandapani and Arockiasamy [24] in 2001 established some new criteria for the oscillation and nonoscillation of solutions.

Xia [26] in 2017 considered the following second-order nonlinear difference equation with Jacobi operators

$$\begin{aligned} &Lx(t) - \omega x(t) \\ &\quad = f(t, x(t+\Gamma), \ldots, x(t), \ldots, x(t-\Gamma)), \quad t \in \mathbb{Z}, \end{aligned} \tag{6}$$

containing both many advances and retardations. By using variational methods and critical point theory, some new criteria are obtained for the existence of a nontrivial homoclinic solution.

By Krasnoselskii's fixed point theorems in cones, Cabada and Dimitrov [13] obtained some existence, multiplicity, and nonexistence results for the nonlinear singular and nonsingular fourth-order equation depending on a real parameter

$$\begin{aligned} &x(t+4) + Mx(t) = \lambda g(t) f(x(t)) + c(t), \\ &\qquad\qquad t \in \{0, 1, \ldots, T-1\}. \end{aligned} \tag{7}$$

In [18], a higher order nonlinear difference equation

$$\begin{aligned} &\sum_{i=0}^{n} r_i (X_{k-i} + X_{k+i}) + f(k, X_{k+\Gamma}, \ldots, X_k, \ldots, X_{k-\Gamma}) \\ &\quad = 0, \quad n \in \mathbb{N}, \ k \in \mathbb{Z}, \end{aligned} \tag{8}$$

is studied. By using critical point theory, sufficient conditions for the existence of periodic solutions are established.

In [23], Raafat determined the forbidden set, introduced an explicit formula for the solutions, and discussed the global behavior of solutions of the difference equation

$$x(t+1) = \frac{ax(t-3)}{b - cx(t-1)x(t-3)}, \quad t = 0, 1, 2, \ldots, \tag{9}$$

where a, b, c are positive real numbers and the initial conditions $x(3), x(2), x(1), x(0)$ are real numbers.

By using the symmetric mountain pass theorem, Chen and Tang [15] established some existence criteria to guarantee the fourth-order difference equation of the form

$$\begin{aligned} &\Delta^4 x(t-2) + q(t) x(t) \\ &\quad = f(t, x(t+1), x(t), x(t-1)), \quad t \in \mathbb{Z}, \end{aligned} \tag{10}$$

having infinitely many homoclinic orbits.

Using the direct method of the calculus of variations and the mountain pass technique, Leszczyński [20] obtained the existence of at least one and at least two solutions of the difference equation:

$$\begin{aligned} &\Delta^2 \left(\gamma(t-1) \phi_{p(t)} \left(\Delta^2 x(t-2)\right)\right) \\ &\quad = f(t, x(t+1), x(t), x(t-1)), \quad t \in [1, T]_{\mathbb{Z}}, \end{aligned} \tag{11}$$

with boundary value conditions

$$\begin{aligned} \Delta x(-1) &= \Delta x(0) = 0, \\ x(T+1) &= x(T+2) = 0. \end{aligned} \tag{12}$$

Our purpose in this article is to use the critical point theory to explore some existence criteria of solutions to the boundary value problem (1), (2) for a fourth-order nonlinear difference equation. In a special case, a necessary and sufficient condition for the existence and uniqueness of solution is also established. Results obtained here are motivated by the recent papers [18, 25].

Define

$$\begin{aligned} r_{\min} &= \min\{r(1), r(2), \ldots, r(T)\}, \\ r_{\max} &= \max\{r(1), r(2), \ldots, r(T)\}. \end{aligned} \tag{13}$$

The rest of this article is organized as follows. In Section 2, we shall introduce some basic notations, set up the variational framework of BVP (1), (2), and give a lemma which will be useful in the proofs of our theorems. Section 3 contains our results of at least one solution. In Section 4, we shall finish proving the main results. In Section 5, we shall provide an example to illustrate our main theorem.

2. Preliminaries

Let X be a vector space:

$$\begin{aligned} X := \Big\{ &x : [-1, T+2]_{\mathbb{Z}} \longrightarrow \mathbb{R} \mid \Delta^k x(-1) \\ &= \Delta^k x(T-1), \ k = 0, 1, 2, 3 \Big\}. \end{aligned} \tag{14}$$

For any $x, y \in X$, denote

$$\begin{aligned} (x, y) &:= \sum_{t=1}^{T} x(t) y(t), \\ \|x\| &:= \left(\sum_{t=1}^{T} x^2(t)\right)^{1/2}. \end{aligned} \tag{15}$$

Remark 1. It is obvious that

$$\begin{aligned} x(-1) &= x(T-1), \\ x(0) &= x(T), \\ x(1) &= x(T+1), \\ x(2) &= x(T+2), \\ &\qquad \forall x \in X. \end{aligned} \tag{16}$$

In reality, X is isomorphic to $\mathbb{R}^T$. In the following and in the sequel, when we write $x = (x(1), x(2), \ldots, x(T)) \in \mathbb{R}^T$, we always mean that x can be extended to a vector in X so that (16) is satisfied.

For any $x \in X$, define the functional I as

$$\begin{aligned} I(x) := {} & \frac{1}{2}\sum_{t=1}^{T} p(t-2)\left(\Delta^2 x(t-2)\right)^2 \\ & + \frac{1}{2}\sum_{t=1}^{T} q(t)(\Delta x(t))^2 - \frac{1}{2}\sum_{t=1}^{T} r(t)(x(t))^2 \\ & + \sum_{t=1}^{T} f(t)x(t). \end{aligned} \tag{17}$$

It is easy to see that $I \in C^1(X, \mathbb{R})$ and

$$\begin{aligned} \frac{\partial I}{\partial x(t)} = {} & \Delta^2\left(p(t-2)\Delta^2 x(t-2)\right) \\ & - \Delta\left(q(t-1)\Delta x(t-1)\right) - r(t)x(t) \\ & + f(t), \quad t \in [1, T]_{\mathbb{Z}}. \end{aligned} \tag{18}$$

For this reason, $I'(x) = 0$ when and only when

$$\begin{aligned} & \Delta^2\left(p(t-2)\Delta^2 x(t-2)\right) - \Delta\left(q(t-1)\Delta x(t-1)\right) \\ & \quad - r(t)x(t) + f(t) = 0, \quad t \in [1, T]_{\mathbb{Z}}. \end{aligned} \tag{19}$$

Thereby a function $x \in X$ is a critical point of the functional I on X when and only when x is a solution of the BVP (1), (2).

Let P and Q be the $T \times T$ matrices as follows.

If $T = 1$, let $P = Q = (0)$.

If $T = 2$, let

$$\begin{aligned} P &= \begin{pmatrix} p(-1) + 4p(0) + 3p(1) & -2p(0) - 4p(1) - 2p(2) \\ -2p(0) - 4p(1) - 2p(2) & p(0) + 4p(1) + 3p(2) \end{pmatrix}, \\ Q &= \begin{pmatrix} q(0) + q(1) & -q(0) - q(1) \\ -q(0) - q(1) & q(0) + q(1) \end{pmatrix}. \end{aligned} \tag{20}$$

If $T = 3$, let

$$P = \begin{pmatrix} p(-1) + 4p(0) + p(1) & p(2) - 2\left(p(0) + p(1)\right) & p(1) - 2\left(p(2) + p(3)\right) \\ p(2) - 2\left(p(0) + p(1)\right) & p(0) + 4p(1) + p(2) & p(3) - 2\left(p(1) + p(2)\right) \\ p(1) - 2\left(p(2) + p(3)\right) & p(3) - 2\left(p(1) + p(2)\right) & p(1) + 4p(2) + p(3) \end{pmatrix}. \tag{21}$$

If $T = 4$, let

$$P = \begin{pmatrix} p(-1) + 4p(0) + p(1) & -2\left(p(0) + p(1)\right) & p(1) + p(3) & -2\left(p(3) + p(4)\right) \\ -2\left(p(0) + p(1)\right) & p(0) + 4p(1) + p(2) & -2\left(p(1) + p(2)\right) & p(2) + p(4) \\ p(1) + p(3) & -2\left(p(1) + \mathrm{p}(2)\right) & p(1) + 4p(2) + p(3) & -2\left(p(2) + p(3)\right) \\ -2\left(p(3) + p(4)\right) & p(2) + p(4) & -2\left(p(2) + p(3)\right) & p(2) + 4p(3) + p(4) \end{pmatrix}. \tag{22}$$

If $T \geq 5$, let

$$P = \begin{pmatrix} n(1) & m(1) & p(1) & 0 & \cdots & 0 & p(T-1) & m(T) \\ m(1) & n(2) & m(2) & p(2) & \cdots & 0 & 0 & p(T) \\ p(1) & m(2) & n(3) & m(3) & \cdots & 0 & 0 & 0 \\ 0 & p(2) & m(3) & n(4) & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & m(\mathrm{T}-3) & p(T-3) & 0 \\ 0 & 0 & 0 & 0 & \cdots & n(T-2) & m(T-2) & p(T-2) \\ p(T-1) & 0 & 0 & 0 & \cdots & m(T-2) & n(T-1) & m(T-1) \\ m(T) & p(T) & 0 & 0 & \cdots & p(T-2) & m(T-1) & n(T) \end{pmatrix}, \tag{23}$$

where $n(k) = p(k) + 4p(k-1) + p(k-2)$, $m(k) = -2(p(k-1) + p(k))$, $k = 1, 2, \ldots, T$.

If $T \geq 3$, let

$$Q = \begin{pmatrix} q(0)+q(1) & -q(1) & 0 & \cdots & -q(0) \\ -q(1) & q(1)+q(2) & -q(2) & \cdots & 0 \\ 0 & -q(2) & q(2)+q(3) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -q(T-1) \\ -q(0) & 0 & 0 & \cdots & q(T-1)+q(0) \end{pmatrix}. \tag{24}$$

Let $M := P + Q$. Therefore, we rewrite $I(x)$ by

$$I(x) = \frac{1}{2}x^* Mx - \frac{1}{2}\sum_{t=1}^{T} r(t)(x(t))^2 + \sum_{t=1}^{T} f(t)x(t). \tag{25}$$

Assume that X is a real Banach space and $I \in C^1(X, \mathbb{R})$ is a continuously Fréchet differentiable functional defined on X. As usual, I is said to satisfy the Palais-Smale condition if any sequence $\{x_k\}_{k=1}^{\infty} \subset X$ for which $\{I(x_k)\}_{k=1}^{\infty}$ is bounded and $I'(x_k) \longrightarrow 0$ as $k \longrightarrow \infty$ possesses a convergent subsequence. Here, the sequence $\{x_k\}_{k=1}^{\infty}$ is called a Palais-Smale sequence.

Let B_ρ be the open ball in X about 0 of radius ρ, ∂B_ρ be its boundary, and $\overline{B}_\rho$ be its closure.

Lemma 2 (saddle point theorem [30]). *Suppose that X is a real Banach space, $X = X_1 \oplus X_2$, where $X_1 \neq \{0\}$ and is finite dimensional. Suppose that $I \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition and*

(I_1) there are constants $\varepsilon, \eta > 0$ such that $I|_{\partial B_\eta \cap X_1} \leq \varepsilon(I_2)$ there is $\xi \in B_\eta \cap X_1$ and a constant $\chi > \varepsilon$ such that $I_{\xi + X_2} \geq \chi$.

Then I admits a critical value $c \geq \chi$, where

$$c = \inf_{g \in \Omega} \max_{x \in B_\eta \cap X_1} I(g(x)), \Omega = \left\{ g \in C\left(\overline{B}_\eta \cap X_1, X\right) \mid g|_{\partial B_\eta \cap X_1} = id \right\} \tag{26}$$

and id is defined as the identity operator.

3. Main Theorems

Our main theorems are as follows.

Theorem 3. *Assume that the following assumptions are satisfied.*

(p) For any $t \in [-1, T]_{\mathbb{Z}}$, $p(t) > 0$.

(q) For any $t \in [0, T]_{\mathbb{Z}}$, $q(t) > 0$.

(r_1) For any $t \in [1, T]_{\mathbb{Z}}$, $r(t) < 0$.

(M) $n(1)+m(1)+p(1)+p(T-1)+m(T) = 0$, $n(2)+m(2)+p(2)+p(T)+m(1) = 0$, $n(k)+m(k)+p(k)+p(k-2)+m(k-1) = 0$, $k = 3, 4, \ldots, T$.

(f_1) For any $t \in [1, T]_{\mathbb{Z}}$,

$$-\sum_{t=1}^{T} f^2(t) \sum_{t=1}^{T} r(t) < (\lambda_{min} - r_{max}) \sum_{t=1}^{T} f(t), \tag{27}$$

where λ_{min} can be referred to (32).

Then the BVP (1), (2) has at least one solution.

If $r(t) > 0$, we obtain the theorem as follows.

Theorem 4. *Assume that (p), (q), (M) and the following assumptions are satisfied.*

(r_2) For any $t \in [1, T]_{\mathbb{Z}}$, $r(t) > 0$.

(r_3) $r_{min} > \lambda_{max}$, where λ_{max} can be referred to (33).

(f_2) For any $t \in [1, T]_{\mathbb{Z}}$,

$$\sum_{t=1}^{T} f^2(t) \sum_{t=1}^{T} r(t) < (r_{min} - \lambda_{max}) \left(\sum_{t=1}^{T} f(t) \right)^2. \tag{28}$$

Then the BVP (1), (2) has at least one solution.

If $r(t) = 0$, consider the following equation:

$$\begin{aligned} &\Delta^2\left(p(t-2)\Delta^2 x(t-2)\right) - \Delta\left(q(t-1)\Delta x(t-1)\right) \\ &\quad = f(t), \quad t \in [1, T]_{\mathbb{Z}}, \end{aligned} \tag{29}$$

with boundary value condition (2).

By a similar argument to that in Section 2, we define the functional I as

$$I(x) = \frac{1}{2}x^* Mx - \left(\widetilde{f}, x\right), \tag{30}$$

where $\widetilde{f} = (f(1), f(2), \ldots, f(T))^*$. Hence, a function $x \in X$ is a critical point of the functional I on X when and only when x is a solution of the BVP (29), (2).

It is obvious to find that the critical point of I is just the solution of the following linear equation:

$$Mx = \tilde{f}. \tag{31}$$

Let $\Omega = (M, \tilde{f})$. Making use of the linear algebraic theory, we can obtain the following necessary and sufficient conditions.

Theorem 5. *The BVP (29), (2) has at least one solution when and only when $rank(M) = rank(\Omega)$, where $rank(M)$ defines the rank of the matrix M.*

Theorem 6. *The BVP (29), (2) has a unique solution when and only when $rank(M) = T$.*

4. Proofs of the Theorems

In this section, we shall finish proofs of the theorems by using critical point theory.

By (M), it is evident that M is positive semidefinite. $\lambda_1 = 0$ is an eigenvalue of M and $(1, 1, \ldots, 1)^*$ is an eigenvector corresponding to 0. Let $\lambda_2, \lambda_3, \ldots, \lambda_T$ be the other eigenvalues of M.

Define

$$\lambda_{\min} = \min\{\lambda_2, \lambda_3, \ldots, \lambda_T\}, \tag{32}$$

$$\lambda_{\max} = \max\{\lambda_2, \lambda_3, \ldots, \lambda_T\}. \tag{33}$$

Let the space $X_2 = \{(x_1, x_2, \ldots, x_T)^* \in X : x_j \equiv c, c \in \mathbb{R}, j = 1, 2, \ldots, T\}$ and $X_2^{\perp} = X_1$ such that

$$X = X_1 \oplus X_2. \tag{34}$$

Proof of Theorem 3. Let $\{x_k\}_{k\in\mathbb{N}} \subset X$ be such that $\{I(x_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(x_k) \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, for any $k \in \mathbb{N}$, there exists a positive constant K_1 such that

$$|I(x_k)| \le K_1. \tag{35}$$

It comes from Hölder inequality that

$$\sum_{t=1}^{T} |f(t)\, x_k(t)| \le \left(\sum_{t=1}^{T} f^2(t)\right)^{1/2} \left(\sum_{t=1}^{T} (x_k(t))^2\right)^{1/2}. \tag{36}$$

By (25) and (36), we have

$$\begin{aligned} K_1 &\ge I(x_k) \\ &= \frac{1}{2}(x_k)^* M x_k - \frac{1}{2}\sum_{t=1}^{T} r(t)(x_k(t))^2 \\ &\quad + \sum_{t=1}^{T} f(t)\, x_k(t) \\ &\ge -\frac{1}{2}\sum_{t=1}^{T} r(t)(x_k(t))^2 - \sum_{t=1}^{T} |f(t)\, x_k(t)| \\ &\ge -\frac{r_{\max}}{2}\sum_{t=1}^{T} (x_k(t))^2 \\ &\quad - \left(\sum_{t=1}^{T} f^2(t)\right)^{1/2} \left(\sum_{t=1}^{T} (x_k(t))^2\right)^{1/2} \\ &= -\frac{r_{\max}}{2}\|x_k\|^2 - \left(\sum_{t=1}^{T} f^2(t)\right)^{1/2} \|x_k\|. \end{aligned} \tag{37}$$

That is,

$$-\frac{r_{\max}}{2}\|x_k\|^2 - \left(\sum_{t=1}^{T} f^2(t)\right)^{1/2} \|x_k\| \le K_1. \tag{38}$$

From the assumption (r_1), (38) implies that $\{x_k\}_{k\in\mathbb{N}}$ is bounded in X. Since the dimension of X is finite, $\{x_k\}_{k\in\mathbb{N}}$ has a convergent subsequence. As a result, $I(x)$ satisfies the Palais-Smale condition. Hence, it is sufficient to prove that $I(x)$ satisfies the conditions (I_1) and (I_2) of Lemma 2.

For any $u = (c, c, \ldots, c)^* \in X_2$, we have

$$I(u) = -\frac{1}{2}\sum_{t=1}^{T} r(t)\, c^2 + \sum_{t=1}^{T} f(t)\, c. \tag{39}$$

Choose

$$c = \frac{\sum_{t=1}^{T} f(t)}{\sum_{t=1}^{T} r(t)}, \tag{40}$$

$$\eta = \|u\| = \frac{\sqrt{T}\sum_{t=1}^{T} f(t)}{\sum_{t=1}^{T} r(t)}. \tag{41}$$

Therefore,

$$I(u) = \frac{\left(\sum_{t=1}^{T} f(t)\right)^2}{2\sum_{t=1}^{T} r(t)}. \tag{42}$$

Let

$$\varepsilon = \frac{\left(\sum_{t=1}^{T} f(t)\right)^2}{2\sum_{t=1}^{T} r(t)}. \tag{43}$$

Thus,

$$I(u) = \varepsilon, \quad \forall u \in \partial B_\eta \cap X_1. \tag{44}$$

The condition (I_1) of Lemma 2 is proved.

Then, we prove the condition (I_2) of Lemma 2. For any $v \in X_1$, by Hölder inequality, we have

$$\begin{aligned} I(v) &= \frac{1}{2}v^* M v - \frac{1}{2}\sum_{t=1}^{T} r(t)(v(t))^2 + \sum_{t=1}^{T} f(t)v(t) \\ &\geq \frac{\lambda_{\min}}{2}\|v\|^2 - \frac{r_{\max}}{2}\sum_{t=1}^{T}(v(t))^2 - \sum_{t=1}^{T}\left|f(t)v(t)\right| \\ &\geq \frac{\lambda_{\min}}{2}\|v\|^2 - \frac{r_{\max}}{2}\|v\|^2 - \left(\sum_{t=1}^{T} f^2(t)\right)^{1/2}\|v\| \\ &= \frac{\lambda_{\min} - r_{\max}}{2}\|v\|^2 - \left(\sum_{t=1}^{T} f^2(t)\right)^{1/2}\|v\| \\ &\geq -\frac{\sum_{t=1}^{T} f^2(t)}{2(\lambda_{\min} - r_{\max})}, \end{aligned} \tag{45}$$

as one finds by minimization with respect to $\|v\|$. In other words,

$$I(v) \geq -\frac{\sum_{t=1}^{T} f^2(t)}{2(\lambda_{\min} - r_{\max})}. \tag{46}$$

Let

$$\chi = -\frac{\sum_{t=1}^{T} f^2(t)}{2(\lambda_{\min} - r_{\max})}. \tag{47}$$

It follows from (f_1) that

$$I(v) \geq \chi > \varepsilon. \tag{48}$$

All the assumptions of Lemma 2 are proved. According to saddle point theorem, the proof of Theorem 3 is complete. □

Proof of Theorem 4. Let $\{x_k\}_{k\in\mathbb{N}} \subset X$ be such that $\{I(x_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(x_k) \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, for any $k \in \mathbb{N}$, there exists a positive constant K_2 such that

$$\left|I(x_k)\right| \leq K_2. \tag{49}$$

By (25), (33) and Hölder inequality, we have

$$\begin{aligned} -K_2 &\leq I(x_k) \\ &= \frac{1}{2}(x_k)^* M x_k - \frac{1}{2}\sum_{t=1}^{T} r(t)(x_k(t))^2 \\ &\quad + \sum_{t=1}^{T} f(t)x_k(t) \\ &\leq \frac{\lambda_{\max}}{2}\|x_k\|^2 - \frac{r_{\min}}{2}\|x_k\|^2 + \sum_{t=1}^{T}\left|f(t)x_k(t)\right| \\ &\leq \frac{\lambda_{\max} - r_{\min}}{2}\|x_k\|^2 \\ &\quad + \left(\sum_{t=1}^{T} f^2(t)\right)^{1/2}\left(\sum_{t=1}^{T}(x_k(t))^2\right)^{1/2} \\ &= \frac{\lambda_{\max} - r_{\min}}{2}\|x_k\|^2 + \left(\sum_{t=1}^{T} f^2(t)\right)^{1/2}\|x_k\|. \end{aligned} \tag{50}$$

That is,

$$\frac{r_{\min} - \lambda_{\max}}{2}\|x_k\|^2 - \left(\sum_{t=1}^{T} f^2(t)\right)^{1/2}\|x_k\| \leq K_2. \tag{51}$$

By condition (r_3), (51) implies that $\{x_k\}_{k\in\mathbb{N}}$ is a bounded in X. Since the dimension of X is finite, $\{x_k\}_{k\in\mathbb{N}}$ has a convergent subsequence. Consequently, $I(x)$ satisfies the Palais-Smale condition. Hence, it is sufficient to prove that $I(x)$ satisfies conditions (I_1) and (I_2) of Lemma 2.

For any $u = (c, c, \ldots, c)^* \in X_2$, let

$$\varepsilon = \frac{\left(\sum_{t=1}^{T} f(t)\right)^2}{2\sum_{t=1}^{T} r(t)}, \tag{52}$$

and similar to the proof of Theorem 1.1 we have

$$I(u) = \varepsilon, \quad \forall u \in \partial B_\eta \cap X_1. \tag{53}$$

Condition (I_1) of Lemma 2 is proved.

Then, we prove the condition (I_2) of Lemma 2. For any $v \in X_1$, by Hölder inequality, we have

$$\begin{aligned} I(v) &= \frac{1}{2}v^* M v - \frac{1}{2}\sum_{t=1}^{T} r(t)(v(t))^2 + \sum_{t=1}^{T} f(t)v(t) \\ &\leq \frac{\lambda_{\max}}{2}\|v\|^2 - \frac{r_{\min}}{2}\sum_{t=1}^{T}(v(t))^2 + \sum_{t=1}^{T}\left|f(t)v(t)\right| \\ &\leq \frac{\lambda_{\max}}{2}\|v\|^2 - \frac{r_{\min}}{2}\|v\|^2 + \left(\sum_{t=1}^{T} f^2(t)\right)^{1/2}\|v\| \\ &= \frac{\lambda_{\max} - r_{\min}}{2}\|v\|^2 + \left(\sum_{t=1}^{T} f^2(t)\right)^{1/2}\|v\| \\ &\leq -\frac{\sum_{t=1}^{T} f^2(t)}{2(\lambda_{\max} - r_{\min})}, \end{aligned} \tag{54}$$

as one finds by maximization with respect to $\|v\|$. That is to say,

$$I(v) \leq -\frac{\sum_{t=1}^{T} f^2(t)}{2(\lambda_{\max} - r_{\min})}. \tag{55}$$

Let

$$\chi = -\frac{\sum_{t=1}^{T} f^2(t)}{2(\lambda_{\max} - r_{\min})}. \tag{56}$$

By (f_2), we have

$$I(v) \le \chi < \varepsilon. \tag{57}$$

Therefore, $-I$ satisfies the condition of saddle point theorem. By Lemma 2, the proof of Theorem 4 is finished. □

5. Examples

In this section, we shall provide two examples to illustrate our main theorem.

Example 1. For $t \in [1,3]_{\mathbb{Z}}$, assume that

$$\begin{aligned} &\Delta^2\left((t-2)\Delta^2 x(t-2)\right) - \Delta\left((t-1)^2 \Delta x(t-1)\right) \\ &\quad + t^3 x(t) + 0.1t = 0, \end{aligned} \tag{58}$$

satisfying the boundary value conditions

$$\begin{aligned} x(-1) &= x(2), \\ \Delta x(-1) &= \Delta x(2), \\ \Delta^2 x(-1) &= \Delta^2 x(2), \\ \Delta^3 x(-1) &= \Delta^3 x(2). \end{aligned} \tag{59}$$

We have

$$\begin{aligned} p(t) &= t > 0, \\ q(t) &= t^2 > 0, \\ r(t) &= -t^3 < 0, \\ f(t) &= 0.1t, \\ &\qquad t \in [1,3]_{\mathbb{Z}}, \end{aligned} \tag{60}$$

with

$$\begin{aligned} p(-1) &= 2, \\ p(0) &= 3, \\ q(0) &= 9, \\ M &= \begin{pmatrix} 25 & -7 & -18 \\ -7 & 14 & -7 \\ -18 & -7 & 25 \end{pmatrix}, \end{aligned} \tag{61}$$

and the eigenvalues of M are $\lambda_1 = 0, \lambda_2 = 21$, and $\lambda_3 = 43$. Thus, $\lambda_{\min} = 21, r_{\max} = -1$ and

$$\begin{aligned} -\sum_{t=1}^{3} f^2(t) \sum_{t=1}^{3} r(t) &= -(0.01 + 0.04 + 0.09) \times (-36) \\ &= 5.04 \\ &< (\lambda_{\min} - r_{\max}) \sum_{t=1}^{3} f(t) \\ &= [21 - (-1)] \times (0.1 + 0.2 + 0.3) \\ &= 13.2. \end{aligned} \tag{62}$$

From the above argument, we see that all the suppositions of Theorem 3 are satisfied; then the BVP (58), (59) has at least one solution.

Example 2. For $t \in [1,4]_{\mathbb{Z}}$, assume that

$$\begin{aligned} &\Delta^2\left((t-2)\Delta^2 x(t-2)\right) - \Delta\left((t-1)^3 \Delta x(t-1)\right) \\ &\quad + 2tx(t) + 0.2t = 0, \end{aligned} \tag{63}$$

satisfying the boundary value conditions

$$\begin{aligned} x(-1) &= x(3), \\ \Delta x(-1) &= \Delta x(3), \\ \Delta^2 x(-1) &= \Delta^2 x(3), \\ \Delta^3 x(-1) &= \Delta^3 x(3). \end{aligned} \tag{64}$$

We have

$$\begin{aligned} p(t) &= t > 0, \\ q(t) &= t^3 > 0, \\ r(t) &= -2t < 0, \\ f(t) &= 0.2t, \\ &\qquad t \in [1,4]_{\mathbb{Z}}, \end{aligned} \tag{65}$$

with

$$\begin{aligned} p(-1) &= 3, \\ p(0) &= 4, \\ q(0) &= 256, \\ M &= \begin{pmatrix} 277 & -11 & 4 & -270 \\ -11 & 19 & -14 & 6 \\ 4 & -14 & 84 & -74 \\ -270 & 6 & -74 & 338 \end{pmatrix}, \end{aligned} \tag{66}$$

and the eigenvalues of M are $\lambda_1 = 0, \lambda_2 \approx 23.9633, \lambda_3 \approx 107.7880$ and $\lambda_4 \approx 586.2487$. Thus, $\lambda_{\min} \approx 23.9633, r_{\max} = -2$ and

$$\begin{aligned} -\sum_{t=1}^{4} f^2(t) \sum_{t=1}^{4} r(t) &= -(0.04 + 0.16 + 0.36 + 0.64) \\ &\quad \times (-20) = 24 \\ &< (\lambda_{\min} - r_{\max}) \sum_{t=1}^{4} f(t) \\ &\approx [23.9633 - (-2)] \\ &\quad \times (0.2 + 0.4 + 0.6 + 0.8) \\ &= 51.9266. \end{aligned} \tag{67}$$

From the above argument, we see that all the suppositions of Theorem 3 are satisfied; then the BVP (63), (64) has at least one solution.

6. Conclusions

Difference equations, the discrete analogue of differential equations, occur widely in numerous settings and forms, both in mathematics itself and in its applications in theory and practice. The boundary value problem discussed in this paper has important analogue in the continuous case of the fourth-order differential equation. Such problem is of special significance for the study of beam equations which are used to describe the bending of an elastic beam. The problem discussed in this paper can be extended to $2n$th-order difference boundary value problem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly to writing of this article. All authors read and approved the final manuscript.

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