Research Article

Anti-Periodic Dynamics of Quaternion-Valued Fuzzy Cellular Neural Networks with Time-Varying Delays on Time Scales

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A class of quaternion-valued fuzzy cellular neural networks with time-varying delays on time scales is proposed. Based on inequality analysis techniques on time scales, a fixed point theorem and the theory of calculus on time scales, the existence, and global exponential stability of anti-periodic solutions for this class of neural networks are established. The obtained results are completely new and supplement to the known results. Finally, a numerical example is given to illustrate the feasibility of our results.

1. Introduction

Since Yang and Yang [1] first introduced fuzzy cellular neural networks (FCNNs) combining fuzzy operations (fuzzy AND and fuzzy OR) with cellular neural networks, FCNNs have been successfully applied in many fields such as physics, chemistry, biology, economics, sociology, medicine, and meteorology [2]. Because all of their applications heavily rely on their dynamics, in recent years, a lot of meaningful results regarding the dynamics of them are obtained by many researchers (see [3–11] and reference therein). For instance, authors in [8] studied the global stability of equilibria of FCNNs, authors in [9] obtained some sufficient conditions for the existence and stability of a unique periodic solution of FCNNs, authors in [10] investigated the existence and global exponential stability of anti-periodic solutions for neutral type FCNNs with time-varying delays and D operator on time scales, the author in [11] studied the almost periodicity of FCNNs with multi-proportional delays, and in other accounts authors investigated other behaviors of FCNNs with time delays.

On the one hand, because quaternion-valued neural networks (QVNNs) as an extension of the real-valued neural networks and complex-valued neural networks can be extensively applied to the fields of robotics, attitude control of satellites, computer graphics, ensemble control, color

night vision, and image compression ([12–14]) and one of the benefits by using quaternion is the three-dimensional geometrical affine transformation that can be represented efficiently and compactly, the study of dynamical behaviors for QVNNs has received much attention of many scholars and some good results have been obtained for the stability [15–19], dissipativity [20], periodicity [21], pseudo almost periodicity [22], and synchronization of QVNNs [23, 24].

On the other hand, as we know, a special case of the quasi-periodicity of functions is the anti-periodicity and ω -anti-periodic functions are 2ω -periodic functions, but not all periodic functions are anti-periodic ones. Since the signal transmission process of neural networks can often be described as an anti-periodic process, the problem of anti-periodic solutions for various types of neural networks has been investigated by many authors ([10, 25–34]).

Moreover, continuous time and discrete time systems are very important in implementation and applications. In addition, in a realistic system, the interaction among agents can happen at any time, and maybe some continuous time intervals accompany some discrete moments. So it is necessary and significant to consider both continuous time and discrete time cases at the same time in networked systems. Fortunately, the time scale theory, which was introduced by Hilger [35], can unify the study of continuous and discrete analysis, and the study of dynamic equations on time scales

can contain, link, and extend the classical theory of differential and difference equations [36]. Recently, the theory of time scale calculus has been applied in real-valued neural networks [37–43] and complex-valued networks [44]. However, to the best of our knowledge, the existence and global stability of anti-periodic solutions of quaternion-valued fuzzy cellular neural networks (QVFCNNs) on time scales have not been considered yet. Besides, it is well known that time delays are unavoidable in real neural network systems and they may cause the changes of the dynamical behaviors of neural networks [11, 25, 40, 41].

Motivated by the above discussion, our main aim of this paper is to study the existence and global stability of anti-periodic solutions of QVFCNNs with time-varying delays. The innovation points of this paper are summarized as follows:

- (1) We propose a class of QVFCNNs with time-varying delays on time scales which can unify the continuous time and discrete time cases of QVFCNNs and, what is more, which can contain the QVFCNNs that their time argument may vary in some continuous time intervals accompanying some discrete moments.
- (2) The QVFCNNs proposed in this paper contain real-valued FCNNs and complex-valued FCNNs as their special cases.
- (3) Our methods of establishing the existence of anti-periodic solutions of QVFCNNs with time-varying delays on time scales are different from those used in [10, 25–34] and can be used to study other types of neural networks.
- (4) Our results are completely new and supplement to the known results, and our results show that if the coefficients of leakage terms of QVFCNNs with time-varying delays on time scales are positive regressive, then both the continuous time and discrete time QVFCNNs with time-varying delays have the same dynamics for the anti-periodicity.

This paper is organized as follows. In Section 2, we give the model description and introduce some definitions and preliminary lemmas and transform the quaternion-valued system (10) into an equivalent real-valued system. In Section 3, we establish the existence of anti-periodic solutions of the considered network based on a fixed point theorem. In Section 4, by using some inequality techniques, we derive some sufficient conditions for the global exponential stability of anti-periodic solutions of the considered network. In Section 5, we give an example to show the feasibility and effectiveness of our main results. This paper ends with a brief conclusion in Section 6.

2. Model Description and Preliminaries

The quaternion was invented in 1843 by Hamilton [45]. The skew field of quaternion is denoted by

$$\mathbb{Q} := \{q = q^R + q^I i + q^J j + q^K k\}, \quad (1)$$

where q^R, q^I, q^J, q^K are real numbers and the elements i, j , and k obey the Hamilton's multiplication rules:

$$\begin{aligned} ij &= -ji = k, \\ jk &= -kj = i, \\ ki &= -ik = j, \\ i^2 &= j^2 = k^2 = ijk = -1. \end{aligned} \quad (2)$$

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward and the backward jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ are defined by

$$\begin{aligned} \sigma(t) &= \inf \{s \in \mathbb{T}, s > t\}, \\ \rho(t) &= \sup \{s \in \mathbb{T}, s < t\}, \end{aligned} \quad (3)$$

$\forall t \in \mathbb{T}.$

The graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. The point $t \in \mathbb{T}$ is called left-dense, left-scattered, right-dense, or right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$ or $\sigma(t) > t$, respectively. Points that are right-dense and left-dense at the same time are called dense. If $\mathbb{T}$ has a left-scattered maximum m , define $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered maximum m , define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$.

Assume that $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \quad (4)$$

for all $s \in U$. We call $f^\Delta(t)$ the delta derivative of f at t .

Lemma 1 (see [36]). *Let $a \in \mathbb{T}^\kappa$, $b \in \mathbb{T}$ and assume that $f : \mathbb{T} \times \mathbb{T}^\kappa \rightarrow \mathbb{R}$ is continuous at (t, t) , where $t \in \mathbb{T}^\kappa$ with $t > a$. Also assume that $f^\Delta(t, \cdot)$ is rd-continuous on $[a, \sigma(t)]$. Suppose that for each $\varepsilon > 0$ there is a neighborhood U of t , independent of $\tau \in [a, \sigma(t)]$, such that*

$$\begin{aligned} &|[f(\sigma(t), \tau) - f(s, \tau)] - f^\Delta(t, \tau)[\sigma(t) - s]| \\ &\leq \varepsilon |\sigma(t) - s| \end{aligned} \quad (5)$$

for all $s \in U$, where f^Δ denotes the derivative of f with respect to the first variable. Then

$$\begin{aligned} \text{(i)} \quad g(t) &:= \int_a^t f(t, \tau) \Delta \tau \text{ implies } g^\Delta(t) := \int_a^t f^\Delta(t, \tau) \Delta \tau + f(\sigma(t), t); \\ \text{(ii)} \quad g(t) &:= \int_t^b f(t, \tau) \Delta \tau \text{ implies } g^\Delta(t) := \int_t^b f^\Delta(t, \tau) \Delta \tau - f(\sigma(t), t). \end{aligned}$$

A function $p : \mathbb{T} \rightarrow \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$, $\forall t \in \mathbb{T}^\kappa$ and is said to be positive regressive provided $1 + \mu(t)p(t) > 0$, $\forall t \in \mathbb{T}^\kappa$. The set of all regressive

and rd-continuous functions $p : \mathbb{T} \rightarrow \mathbb{R}$ are denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$ and the set of all positive regressive and rd-continuous functions $p : \mathbb{T} \rightarrow \mathbb{R}$ are denoted by $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall t \in \mathbb{T}\}$.

If $p \in \mathcal{R}^+$, then the exponential function is defined by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau\right), \quad \forall t, s \in \mathbb{T}, \quad (6)$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\log(1 + hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases} \quad (7)$$

Let $p, q : \mathbb{T} \rightarrow \mathbb{R}$ be two regressive functions; define

$$\begin{aligned} p \oplus q &= p + q + \mu p q, \\ \ominus p &= -\frac{p}{1 + \mu p}, \\ p \ominus q &= p \oplus (\ominus q). \end{aligned} \quad (8)$$

Lemma 2 (see [36]). *If $p \in \mathcal{R}^+$, then*

- (i) $e_p(t, s) > 0$, for all $t, s \in \mathbb{T}$;
- (ii) if $p(t) \leq q(t)$, then $e_p(t, s) \leq e_q(t, s)$ for all $t \geq s, s \in \mathbb{T}$.

Definition 3 (see [46]). We say that a time scale $\mathbb{T}$ is periodic if there exists $p > 0$ such that if $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of the time scale.

Definition 4 (see [46]). Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p . We say that the function $f : \mathbb{T} \rightarrow \mathbb{R}$ is periodic with period ω if there exists a natural number n such that $\omega = np$, $f(t + \omega) = f(t)$ for all $t \in \mathbb{T}$ and ω is the smallest positive number such that $f(t + \omega) = f(t)$.

Remark 5. From [47], we know that if $\mathbb{T}$ is ω -periodic, then the graininess function μ is ω -periodic.

Lemma 6 (see [47]). *Let $\mathbb{T}$ be ω -periodic and suppose $f : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$ satisfies the assumptions of Lemma 1. Define $g(t) = \int_t^{t+\omega} f(t, s) \Delta s$. If $f^\Delta(t, s)$ denotes the derivative of f with respect to t , then*

$$\begin{aligned} g^\Delta(t) &= \int_t^{t+\omega} f^\Delta(t, s) \Delta s + f(\sigma(t), t + \omega) \\ &\quad - f(\sigma(t), t). \end{aligned} \quad (9)$$

Definition 7. Let $\mathbb{T}$ be an ω -periodic time scale. A function $u : \mathbb{T} \rightarrow \mathbb{R}$ is said to be ω -anti-periodic on $\mathbb{T}$ if $u(t + \omega) = -u(t)$ for all $t \in \mathbb{T}$, $\omega > 0$ is a constant.

Definition 8. Let $\mathbb{T}$ be an ω -periodic time scale. A function $f = f^R + f^I i + f^J j + f^K k : \mathbb{T} \rightarrow \mathbb{Q}^n$ is called an ω -anti-periodic, where $f^m : \mathbb{T} \rightarrow \mathbb{R}^n$, $m \in \{R, I, J, K\} := M$ if for every $m \in M$, f^m is ω -anti-periodic.

In this paper, we consider the following QVFCNN with time-varying delays on time scales:

$$\begin{aligned} x_l^\Delta(t) &= -a_l(t) x_l(t) + \sum_{h=1}^n b_{lh}(t) f_h(x_h(t - \tau_{lh}(t))) \\ &\quad + \sum_{h=1}^n d_{lh}(t) \mu_h(t) \\ &\quad + \bigwedge_{h=1}^n \alpha_{lh}(t) \int_{t-\eta_{lh}(t)}^t g_h(x_h(s)) \Delta s \\ &\quad + \bigvee_{h=1}^n \beta_{lh}(t) \int_{t-\xi_{lh}(t)}^t g_h(x_h(s)) \Delta s \\ &\quad + \bigwedge_{h=1}^n T_{lh}(t) \mu_h(t) + \bigvee_{h=1}^n S_{lh}(t) \mu_h(t) + I_l(t), \end{aligned} \quad (10)$$

where $l \in \{1, 2, \dots, n\} := \Lambda$, $t \in \mathbb{T}$, $\mathbb{T}$ is a periodic time scale; n is the number of neurons in layers; $x_l(t) \in \mathbb{Q}$ and $\mu_h(t) \in \mathbb{Q}$ are the state of the l th neuron at time t and the deviations of the h th neuron at time t , respectively; $a_l(t) > 0$ represents the rate with which the l th neuron will reset its potential to the resting state in isolation when they are disconnected from the network and the external inputs at time t , $\alpha_{lh}(t) \in \mathbb{Q}$, $\beta_{lh}(t) \in \mathbb{Q}$, $T_{lh}(t) \in \mathbb{Q}$, and $S_{lh}(t) \in \mathbb{Q}$ are the elements of fuzzy feedback MIN template, fuzzy feedback MAX template, fuzzy feed forward MIN template, and fuzzy feed forward MAX template, respectively; $b_{lh}(t) \in \mathbb{Q}$ and $d_{lh}(t) \in \mathbb{Q}$ are the elements of feedback template and feed forward template, $\bigwedge, \bigvee$ denote the fuzzy AND and fuzzy OR operations, respectively; $f_h(\cdot)$ and $g_h(\cdot) : \mathbb{Q} \rightarrow \mathbb{Q}$ are the activation functions; $\tau_{lh}(t) \geq 0$, $\eta_{lh}(t) \geq 0$ and $\xi_{lh}(t) \geq 0$ correspond to transmission delays at time t and satisfy $t - \tau_{lh}(t) \in \mathbb{T}$, $t - \eta_{lh}(t) \in \mathbb{T}$, $t - \xi_{lh}(t) \in \mathbb{T}$ for $t \in \mathbb{T}$; $I_l(t) \in \mathbb{Q}$ denotes the input of the l th neuron at time t , $l \in \Lambda$.

Throughout the rest of the paper, we denote $[a, b]_{\mathbb{T}} = \{t \mid t \in [a, b] \cap \mathbb{T}\}$. For convenience, for a bounded and continuous function $f : \mathbb{T} \rightarrow \mathbb{R}$, we denote $\underline{f} = \inf_{t \in \mathbb{T}} |f(t)|$ and $\overline{f} = \sup_{t \in \mathbb{T}} |f(t)|$.

The initial conditions of system (10) are

$$x_l(s) = \varphi_l(s), \quad s \in [t_0 - \nu, t_0]_{\mathbb{T}}, \quad (11)$$

where $\nu = \max_{l, h \in \Lambda} \{\overline{\tau}_{lh}, \overline{\eta}_{lh}, \overline{\xi}_{lh}\}$, $\varphi_l \in C_{rd}([t_0 - \nu, t_0]_{\mathbb{T}}, \mathbb{Q})$, $l \in \Lambda$.

Lemma 9 (see [8]). *Suppose x and y are two states of system (15). Then we have*

$$\begin{aligned} &\left| \bigwedge_{j=1}^n \alpha_{ij}(t) f_j(x) - \bigwedge_{j=1}^n \alpha_{ij}(t) f_j(y) \right| \\ &\leq \sum_{j=1}^n |\alpha_{ij}(t)| |f_j(x) - f_j(y)|, \quad i \in \Lambda, \end{aligned}$$

$$\begin{aligned}
& \left| \bigvee_{j=1}^n \beta_{ij}(t) f_j(x) - \bigvee_{j=1}^n \beta_{ij}(t) f_j(y) \right| \\
& \leq \sum_{j=1}^n |\beta_{ij}(t)| |f_j(x) - f_j(y)|, \quad i \in \Lambda.
\end{aligned} \tag{12}$$

In order to overcome the inconvenience of the noncommutativity of quaternion multiplication, in the following, we will first decompose system (10) into the vector form of the four real-valued systems. To do so, for $x_h = x_h^R + x_h^I i + x_h^J j + x_h^K k \in \mathbb{Q}$, where $x_h^R, x_h^I, x_h^J, x_h^K \in \mathbb{R}$, we assume that the activation functions $f_h(x_h)$ and $g_h(x_h)$ of (10) can be expressed as

$$\begin{aligned}
f_h(x_h) &= f_h^R(x_h^R, x_h^I, x_h^J, x_h^K) + i f_h^I(x_h^R, x_h^I, x_h^J, x_h^K) \\
&\quad + j f_h^J(x_h^R, x_h^I, x_h^J, x_h^K) \\
&\quad + k f_h^K(x_h^R, x_h^I, x_h^J, x_h^K), \\
g_h(x_h) &= g_h^R(x_h^R, x_h^I, x_h^J, x_h^K) + i g_h^I(x_h^R, x_h^I, x_h^J, x_h^K) \\
&\quad + j g_h^J(x_h^R, x_h^I, x_h^J, x_h^K) \\
&\quad + k g_h^K(x_h^R, x_h^I, x_h^J, x_h^K),
\end{aligned} \tag{13}$$

where $f_h^m, g_h^m : \mathbb{R}^4 \rightarrow \mathbb{R}, h \in \Lambda, m \in M$.

Remark 10. In system (10), if $x_h = x_h^R + x_h^I i \in \mathbb{C}$, where $x_h^R, x_h^I \in \mathbb{R}$ and the activation functions $f_h, g_h : \mathbb{C} \rightarrow \mathbb{C}$ are complex variable functions, that is,

$$\begin{aligned}
f_h(x_h) &= f_h^R(x_h^R, x_h^I) + i f_h^I(x_h^R, x_h^I), \\
g_h(x_h) &= g_h^R(x_h^R, x_h^I) + i g_h^I(x_h^R, x_h^I),
\end{aligned} \tag{14}$$

where $f_h^m, g_h^m : \mathbb{R}^2 \rightarrow \mathbb{R}, h \in \Lambda, m = R, I$, and all the quaternion-valued coefficients of (10) are transformed to the complex-valued coefficients, then system (10) degenerates to a complex-valued system; if all of the activation functions and coefficients of (10) are real variable functions, then system (10) degenerates to a real-valued system.

It follows from Hamilton's multiplication rules that the equivalent real-valued system of QVFCNN (10) can be written as follows:

$$\begin{aligned}
X_l^\Delta(t) &= -a_l(t) X_l(t) + \sum_{h=1}^n B_{lh}(t) \tilde{F}_h[x_h(t - \tau_{lh}(t))] \\
&\quad + \sum_{h=1}^n D_{lh}(t) U_h(t) \\
&\quad + \bigwedge_{h=1}^n A_{lh}(t) \int_{t-\eta_{lh}(t)}^t \tilde{G}_h[x_h(s)] \Delta s \\
&\quad + \bigvee_{h=1}^n C_{lh}(t) \int_{t-\xi_{lh}(t)}^t \tilde{G}_h[x_h(s)] \Delta s
\end{aligned}$$

$$\begin{aligned}
& + \bigwedge_{h=1}^n T_{lh}^*(t) U_h(t) + \bigvee_{h=1}^n S_{lh}^*(t) U_h(t) + I_l^*(t), \\
& l \in \Lambda, t \in \mathbb{T},
\end{aligned} \tag{15}$$

where

$$\tilde{f}_h^m[x_h] = f_h^m(x_h^R, x_h^I, x_h^J, x_h^K),$$

$$\tilde{g}_h^m[x_h] = g_h^m(x_h^R, x_h^I, x_h^J, x_h^K),$$

$$X_l(t) = \begin{pmatrix} x_l^R(t) \\ x_l^I(t) \\ x_l^J(t) \\ x_l^K(t) \end{pmatrix},$$

$$I_l^*(t) = \begin{pmatrix} I_l^R(t) \\ I_l^I(t) \\ I_l^J(t) \\ I_l^K(t) \end{pmatrix},$$

$$\tilde{G}_h[x_h(t)] = \begin{pmatrix} \tilde{g}_h^R[x_h(t)] \\ \tilde{g}_h^I[x_h(t)] \\ \tilde{g}_h^J[x_h(t)] \\ \tilde{g}_h^K[x_h(t)] \end{pmatrix},$$

$$U_h(t) = \begin{pmatrix} \mu_h^R(t) \\ \mu_h^I(t) \\ \mu_h^J(t) \\ \mu_h^K(t) \end{pmatrix},$$

$$B_{lh}(t) = \begin{pmatrix} b_{lh}^R(t) & -b_{lh}^I(t) & -b_{lh}^J(t) & -b_{lh}^K(t) \\ b_{lh}^I(t) & b_{lh}^R(t) & -b_{lh}^K(t) & b_{lh}^J(t) \\ b_{lh}^J(t) & b_{lh}^K(t) & b_{lh}^R(t) & -b_{lh}^I(t) \\ b_{lh}^K(t) & -b_{lh}^I(t) & b_{lh}^J(t) & b_{lh}^R(t) \end{pmatrix},$$

$$D_{lh}(t) = \begin{pmatrix} d_{lh}^R(t) & -d_{lh}^I(t) & -d_{lh}^J(t) & -d_{lh}^K(t) \\ d_{lh}^I(t) & d_{lh}^R(t) & -d_{lh}^K(t) & d_{lh}^J(t) \\ d_{lh}^J(t) & d_{lh}^K(t) & d_{lh}^R(t) & -d_{lh}^I(t) \\ d_{lh}^K(t) & -d_{lh}^J(t) & d_{lh}^I(t) & d_{lh}^R(t) \end{pmatrix},$$

$$A_{lh}(t) = \begin{pmatrix} \alpha_{lh}^R(t) & -\alpha_{lh}^I(t) & -\alpha_{lh}^J(t) & -\alpha_{lh}^K(t) \\ \alpha_{lh}^I(t) & \alpha_{lh}^R(t) & -\alpha_{lh}^K(t) & \alpha_{lh}^J(t) \\ \alpha_{lh}^J(t) & \alpha_{lh}^K(t) & \alpha_{lh}^R(t) & -\alpha_{lh}^I(t) \\ \alpha_{lh}^K(t) & -\alpha_{lh}^J(t) & \alpha_{lh}^I(t) & \alpha_{lh}^R(t) \end{pmatrix},$$

$$\begin{aligned}
C_{lh}(t) &= \begin{pmatrix} \beta_{lh}^R(t) & -\beta_{lh}^I(t) & -\beta_{lh}^J(t) & -\beta_{lh}^K(t) \\ \beta_{lh}^I(t) & \beta_{lh}^R(t) & -\beta_{lh}^K(t) & \beta_{lh}^J(t) \\ \beta_{lh}^J(t) & \beta_{lh}^K(t) & \beta_{lh}^R(t) & -\beta_{lh}^I(t) \\ \beta_{lh}^K(t) & -\beta_{lh}^J(t) & \beta_{lh}^I(t) & \beta_{lh}^R(t) \end{pmatrix}, \\
S_{lh}^*(t) &= \begin{pmatrix} S_{lh}^R(t) & -S_{lh}^I(t) & -S_{lh}^J(t) & -S_{lh}^K(t) \\ S_{lh}^I(t) & S_{lh}^R(t) & -S_{lh}^K(t) & S_{lh}^J(t) \\ S_{lh}^J(t) & S_{lh}^K(t) & S_{lh}^R(t) & -S_{lh}^I(t) \\ S_{lh}^K(t) & -S_{lh}^J(t) & S_{lh}^I(t) & S_{lh}^R(t) \end{pmatrix}, \\
T_{lh}^*(t) &= \begin{pmatrix} T_{lh}^R(t) & -T_{lh}^I(t) & -T_{lh}^J(t) & -T_{lh}^K(t) \\ T_{lh}^I(t) & T_{lh}^R(t) & -T_{lh}^K(t) & T_{lh}^J(t) \\ T_{lh}^J(t) & T_{lh}^K(t) & T_{lh}^R(t) & -T_{lh}^I(t) \\ T_{lh}^K(t) & -T_{lh}^J(t) & T_{lh}^I(t) & T_{lh}^R(t) \end{pmatrix}, \\
\tilde{F}_h[x_h(t - \tau_{lh}(t))] &= \begin{pmatrix} \tilde{f}_h^R[x_h(t - \tau_{lh}(t))] \\ \tilde{f}_h^I[x_h(t - \tau_{lh}(t))] \\ \tilde{f}_h^J[x_h(t - \tau_{lh}(t))] \\ \tilde{f}_h^K[x_h(t - \tau_{lh}(t))] \end{pmatrix},
\end{aligned} \tag{16}$$

with the initial conditions

$$X_l(s) = \Phi_l(s), \quad s \in [t_0 - \nu, t_0]_{\mathbb{T}}, \tag{17}$$

where $\Phi_l(s) = (\varphi_l^R(s), \varphi_l^I(s), \varphi_l^J(s), \varphi_l^K(s))^T$, $\varphi_l^m(s) \in C_{rd}([t_0 - \nu, t_0]_{\mathbb{T}}, \mathbb{R})$, $l \in \Lambda$, $m \in M$.

Remark 11. It is easy to see that if $X(t) = (X_1(t), X_2(t), \dots, X_n(t))^T$ is a solution of system (15), where $X_l(t) = (x_l^R(t), x_l^I(t), x_l^J(t), x_l^K(t))$, $l \in \Lambda$, then $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$, where $x_l(t) = x_l^R(t) + ix_l^I(t) + jx_l^J(t) + kx_l^K(t)$, $l \in \Lambda$ is a solution of (10). Thus, the problem of finding an anti-periodic solution for (10) is reduced to finding one for system (15). For considering the stability of anti-periodic solutions of (10), we just need to consider the stability of anti-periodic solutions of system (15).

3. The Existence of Anti-Periodic Solutions

In this section, we will state and prove the sufficient conditions for the existence of anti-periodic solutions of (15).

Let $\mathbb{X} = \{\varphi \mid \varphi = (\varphi_1^R, \varphi_1^I, \varphi_1^J, \varphi_1^K, \varphi_2^R, \varphi_2^I, \varphi_2^J, \varphi_2^K, \dots, \varphi_n^R, \varphi_n^I, \varphi_n^J, \varphi_n^K)^T := (\varphi_1, \varphi_2, \dots, \varphi_n)^T \in C(\mathbb{T}, \mathbb{R}^{4n}) : \varphi(t + \omega) = -\varphi(t), \forall t \in \mathbb{T}\}$ with the norm $\|\varphi\|_{\mathbb{X}} = \max\{\max_{l \in \Lambda, m \in M} \{\sup_{t \in \mathbb{T}} |\varphi_l^m(t)|\}\}$, then $\mathbb{X}$ is a Banach space. Let $\varphi^0 = ((\varphi_1^0)^R, (\varphi_1^0)^I, (\varphi_1^0)^J, (\varphi_1^0)^K, (\varphi_2^0)^R, (\varphi_2^0)^I, (\varphi_2^0)^J, (\varphi_2^0)^K, \dots, (\varphi_n^0)^R, (\varphi_n^0)^I, (\varphi_n^0)^J, (\varphi_n^0)^K)^T := (\varphi_1^0, \varphi_2^0, \dots, \varphi_n^0)^T$ where $\varphi_l^0(t) = \int_t^{t+\omega} G_{-a_l}(t, \sigma(s)) (\sum_{h=1}^n D_{lh}(s) \times U_h(s) + \bigwedge_{h=1}^n T_{lh}^*(s) U_h(s) + \bigvee_{h=1}^n S_{lh}^*(s) U_h(s) + I_l^*(s)) \Delta s$, $l \in \Lambda$ and r be a constant satisfying $r \geq \|\varphi^0\|_{\mathbb{X}}$.

Throughout this paper, we assume that the following conditions hold:

(H₁) Functions $a_l, \tau_{lh}, \eta_{lh}, \xi_{lh} \in C(\mathbb{T}, \mathbb{R}^+)$ with $-a_l \in \mathcal{R}^+$, $b_{lh}^m, d_{lh}^m, \alpha_{lh}^m, \beta_{lh}^m, T_{lh}^m, S_{lh}^m \in C(\mathbb{T}, \mathbb{R})$ are ω -periodic, $\mu_h^m, I_l^m \in C(\mathbb{T}, \mathbb{R})$ are ω -anti-periodic, and functions $f_h^m, g_h^m \in C(\mathbb{R}^4, \mathbb{R})$ satisfy

$$\begin{aligned}
f_h^m(x_h^R, x_h^I, x_h^J, x_h^K) &= -f_h^m(-x_h^R, -x_h^I, -x_h^J, -x_h^K), \\
g_l^m(x_h^R, x_h^I, x_h^J, x_h^K) &= -g_l^m(-x_h^R, -x_h^I, -x_h^J, -x_h^K),
\end{aligned} \tag{18}$$

where $x_l^m \in \mathbb{R}$, $l, h \in \Lambda$, $m \in M$.

(H₂) Functions $f_h^m, g_h^m \in C(\mathbb{R}^4, \mathbb{R})$ and there exist positive constants L_h^m, H_h^m such that for all $x_h^m, y_h^m \in \mathbb{R}$,

$$\begin{aligned}
&|f_h^m(x_h^R, x_h^I, x_h^J, x_h^K) - f_h^m(y_h^R, y_h^I, y_h^J, y_h^K)| \\
&\leq L_h^R |x_h^R - y_h^R| + L_h^I |x_h^I - y_h^I| + L_h^J |x_h^J - y_h^J| \\
&\quad + L_h^K |x_h^K - y_h^K|, \\
&|g_h^m(x_h^R, x_h^I, x_h^J, x_h^K) - g_h^m(y_h^R, y_h^I, y_h^J, y_h^K)| \\
&\leq H_h^R |x_h^R - y_h^R| + H_h^I |x_h^I - y_h^I| + H_h^J |x_h^J - y_h^J| \\
&\quad + H_h^K |x_h^K - y_h^K|
\end{aligned} \tag{19}$$

and $f_h^m(0, 0, 0, 0) = 0$, $g_h^m(0, 0, 0, 0) = 0$, $h \in \Lambda$, $m \in M$.

(H₃)

$$\max_{l \in \Lambda} \left\{ \frac{\omega \kappa_l P_l}{1 + \kappa_l} \right\} := \xi < 1, \tag{20}$$

where $\kappa_l = e_{-a_l}(0, \omega)$, $l \in \Lambda$,

$$\begin{aligned}
P_l &= \sum_{h=1}^n L_h \left(\bar{b}_{lh}^R + \bar{b}_{lh}^I + \bar{b}_{lh}^J + \bar{b}_{lh}^K \right) \\
&\quad + \sum_{h=1}^n \bar{\eta}_{lh} H_h \left(\bar{\alpha}_{lh}^R + \bar{\alpha}_{lh}^I + \bar{\alpha}_{lh}^J + \bar{\alpha}_{lh}^K \right) \\
&\quad + \sum_{h=1}^n \bar{\xi}_{lh} H_h \left(\bar{\beta}_{lh}^R + \bar{\beta}_{lh}^I + \bar{\beta}_{lh}^J + \bar{\beta}_{lh}^K \right), \quad l \in \Lambda,
\end{aligned} \tag{21}$$

$$L_h = L_h^R + L_h^I + L_h^J + L_h^K,$$

$$H_h = H_h^R + H_h^I + H_h^J + H_h^K, \quad h \in \Lambda.$$

Lemma 12. Suppose that (H₁) holds. Then $X(t) = (X_1(t), X_2(t), \dots, X_n(t))^T$ is an ω -anti-periodic solution of system (15) if and only if $X_l(t)$ satisfies

$$X_l(t) = \int_t^{t+\omega} G_l(t, \sigma(s)) K_l(s, x) \Delta s, \quad l \in \Lambda, \quad t \in \mathbb{T}, \tag{22}$$

where $G_l(t, \sigma(s)) = -e_{-a_l}(t, \sigma(s))/(1 + e_{-a_l}(0, \omega))$,

$$\begin{aligned} K_l(s, x) = & \sum_{h=1}^n B_{lh}(s) \tilde{F}_h[x_h(s - \tau_{lh}(s))] \\ & + \sum_{h=1}^n D_{lh}(s) U_h(s) \\ & + \bigwedge_{h=1}^n A_{lh}(s) \int_{s-\eta_{lh}(s)}^s \tilde{G}_h[x_h(u)] \Delta u \\ & + \bigvee_{h=1}^n C_{lh}(s) \int_{s-\xi_{lh}(s)}^s \tilde{G}_h[x_h(u)] \Delta u \\ & + \bigwedge_{h=1}^n T_{lh}^*(s) U_h(s) + \bigvee_{h=1}^n S_{lh}^*(s) U_h(s) \\ & + I_l^*(s). \end{aligned} \quad (23)$$

Proof. Let $X(t)$ be an ω -anti-periodic solution of system (15). Set $Y_l(s) = e_{-a_l}(t, s)X_l(s)$, then

$$\begin{aligned} Y_l^\Delta(s) = & a_l(s) e_{-a_l}(t, \sigma(s)) X_l(s) \\ & + e_{-a_l}(t, \sigma(s)) X_l^\Delta(s) \\ = & e_{-a_l}(t, \sigma(s)) (a_l(s) X_l(s) + X_l^\Delta(s)) \\ = & e_{-a_l}(t, \sigma(s)) K_l(s, x). \end{aligned} \quad (24)$$

Integrating (24) from t to $t + \omega$, we have

$$Y(t + \omega) - Y(t) = \int_t^{t+\omega} e_{-a_l}(t, \sigma(s)) K_l(s, x) \Delta s. \quad (25)$$

Note that $Y_l(t) = X_l(t)$, thus

$$\begin{aligned} & e_{-a_l}(t, t + \omega) X_l(t + \omega) \\ = & X_l(t) + \int_t^{t+\omega} e_{-a_l}(t, \sigma(s)) K_l(s, x) \Delta s. \end{aligned} \quad (26)$$

Since $X_l(t + \omega) = -X_l(t)$, we have

$$X_l(t) = \int_t^{t+\omega} G_l(t, \sigma(s)) K_l(s, x) \Delta s, \quad l \in \Lambda, \quad t \in \mathbb{T}, \quad (27)$$

that is, $X(t)$ is a solution of (22).

On the other hand, let $X(t)$ be a solution of (22). By Lemma 6, we obtain

$$X_l^\Delta(t) = \left(- \int_t^{t+\omega} \frac{e_{-a_l}(t, \sigma(s))}{1 + e_{-a_l}(0, \omega)} K_l(s, x) \Delta s \right)^\Delta$$

$$\begin{aligned} & = \int_t^{t+\omega} \frac{a_l(t) e_{-a_l}(t, \sigma(s))}{1 + e_{-a_l}(0, \omega)} K_l(s, x) \Delta s \\ & \quad - \frac{e_{-a_l}(\sigma(t), \sigma(t) + \omega)}{1 + e_{-a_l}(0, \omega)} K_l(t + \omega, x) \\ & \quad + \frac{e_{-a_l}(\sigma(t), \sigma(t))}{1 + e_{-a_l}(0, \omega)} K_l(t, x) \\ & = \int_t^{t+\omega} \frac{a_l(t) e_{-a_l}(t, \sigma(s))}{1 + e_{-a_l}(0, \omega)} (X_l^\Delta(s) \\ & \quad + a_l(s) X_l(s)) \Delta s - \frac{e_{-a_l}(\sigma(t), \sigma(t) + \omega)}{1 + e_{-a_l}(0, \omega)} \\ & \quad \cdot K_l(t + \omega, x) + \frac{e_{-a_l}(\sigma(t), \sigma(t))}{1 + e_{-a_l}(0, \omega)} K_l(t, x) \\ & = \frac{a_l(t)}{1 + e_{-a_l}(0, \omega)} \int_t^{t+\omega} e_{-a_l}(t, \sigma(s)) (X_l^\Delta(s) \\ & \quad + a_l(s) X_l(s)) \Delta s + K_l(t, x) = \frac{a_l(t)}{1 + e_{-a_l}(0, \omega)} \\ & \quad \cdot \int_t^{t+\omega} (e_{-a_l}(t, s) X_l(s))^\Delta \Delta s + K_l(t, x) = -a_l(t) \\ & \quad \cdot X_l(t) + K_l(t, x), \quad l \in \Lambda, \quad t \in \mathbb{T}. \end{aligned} \quad (28)$$

Therefore, $(X_1(t), X_2(t), \dots, X_n(t))^T$ is an ω -anti-periodic solution of system (15). The proof of Lemma 12 is completed. $\square$

Remark 13. It is easy to see that

$$\begin{aligned} \frac{1}{1 + \kappa_l} & \leq |G_l(t, \sigma(s))| \leq \frac{\kappa_l}{1 + \kappa_l}, \\ \sigma(s) & \in [t, t + \omega]_{\mathbb{T}}, \end{aligned} \quad (29)$$

$$l \in \Lambda, \quad t \in \mathbb{T}.$$

Theorem 14. Let (H_1) – (H_3) hold. Then, system (15) has a unique ω -anti-periodic solution in the region $\mathbb{X}^* = \{\varphi \in \mathbb{X} : \|\varphi - \varphi^0\|_{\mathbb{X}} \leq \xi r / (1 - \xi)\}$.

Proof. Define an operator $\Phi : \mathbb{X}^* \rightarrow \mathbb{X}$ by

$$\begin{aligned} \Phi \varphi = & ((\Phi \varphi)_1, (\Phi \varphi)_2, \dots, (\Phi \varphi)_n)^T \\ \rightarrow & (X_1^\varphi, X_2^\varphi, \dots, X_n^\varphi)^T, \end{aligned} \quad (30)$$

where $X_l^\varphi(t) = \int_t^{t+\omega} G_l(t, \sigma(s)) K_l(s, \varphi) \Delta s$, $t \in \mathbb{T}$, $l \in \Lambda$ and G_l, K_l are defined the same as those in Lemma 12.

For $\varphi \in \mathbb{X}^*$, then $\|\varphi\|_{\mathbb{X}} \leq \|\varphi - \varphi^0\|_{\mathbb{X}} + \|\varphi^0\|_{\mathbb{X}} \leq \xi r / (1 - \xi) + r = r / (1 - \xi)$. First, we show that for any $\varphi \in \mathbb{X}^*$, $\Phi \varphi \in \mathbb{X}^*$. From (H_2) , we have

$$(\Phi \varphi)_l(t + \omega) = \left\{ \int_t^{t+\omega} \left[G_l(t + \omega, \sigma(s)) \left(\sum_{h=1}^n B_{lh}(s) \tilde{F}_h[\varphi_h(s - \tau_{lh}(s))] + \sum_{h=1}^n D_{lh}(s) U_h(s) + \bigwedge_{h=1}^n A_{lh}(s) \int_{s-\eta_{lh}(s)}^s \tilde{G}_h[\varphi_h(u)] \Delta u + \bigvee_{h=1}^n C_{lh}(s) \int_{s-\xi_{lh}(s)}^s \tilde{G}_h[\varphi_h(u)] \Delta u + \bigwedge_{h=1}^n T_{lh}^*(s) U_h(s) + \bigvee_{h=1}^n S_{lh}^*(s) U_h(s) + I_l^*(s) \right) \right] \Delta s \right\}$$

$$\begin{aligned}
&= \left\{ \int_t^{t+\omega} \left[G_l(t+\omega, \sigma(s) + \omega) \right. \right. \\
&\quad \cdot \left(\sum_{h=1}^n B_{lh}(s+\omega) \bar{F}_h[\varphi_h(s+\omega - \tau_{lh}(s+\omega))] + \sum_{h=1}^n D_{lh}(s+\omega) U_h(s+\omega) + \sum_{h=1}^n A_{lh}(s+\omega) \int_{s+\omega-\eta_{lh}(s+\omega)}^{s+\omega} \bar{G}_h[\varphi_h(u)] \Delta u + \sum_{h=1}^n C_{lh}(s+\omega) \int_{s+\omega-\xi_{lh}(s+\omega)}^{s+\omega} \bar{G}_h[\varphi_h(u)] \Delta u + \sum_{h=1}^n T_{lh}^*(s+\omega) U_h(s+\omega) + \sum_{h=1}^n S_{lh}^*(s+\omega) U_h(s+\omega) + I_l^*(s+\omega) \right) \Delta s \Bigg] \\
&\quad \left. - \int_t^{t+\omega} \left[G_l(t, \sigma(s)) \left(\sum_{h=1}^n B_{lh}(s) \bar{F}_h[\varphi_h(s - \tau_{lh}(s))] + \sum_{h=1}^n D_{lh}(s) U_h(s) + \sum_{h=1}^n A_{lh}(s) \int_{s-\eta_{lh}(s)}^s \bar{G}_h[\varphi_h(u)] \Delta u + \sum_{h=1}^n C_{lh}(s) \int_{s-\xi_{lh}(s)}^s \bar{G}_h[\varphi_h(u)] \Delta u + \sum_{h=1}^n T_{lh}^*(s) U_h(s) + \sum_{h=1}^n S_{lh}^*(s) U_h(s) + I_l^*(s) \right) \Delta s \right] \right\} = -(\Phi\varphi)_l(t), \quad l \in \Lambda, \quad t \in \mathbb{T}.
\end{aligned} \tag{31}$$

From (H_1) – (H_3) , we have

$$\begin{aligned}
&\left| (\Phi\varphi)_l^R(t) - (\varphi^0)_l^R(t) \right| = \left| \int_t^{t+\omega} G_l(t, \sigma(s)) \right. \\
&\quad \cdot \left(\sum_{j=1}^n (b_{lh}^R(s) \tilde{f}_h^R[\varphi_h(s - \tau_{lh}(s))] - b_{lh}^I(s) \tilde{f}_h^I[\varphi_h(s - \tau_{lh}(s))] - b_{lh}^J(s) \tilde{f}_h^J[\varphi_h(s - \tau_{lh}(s))] - b_{lh}^K(s) \tilde{f}_h^K[\varphi_h(s - \tau_{lh}(s))]) \right. \\
&\quad + \sum_{h=1}^n \alpha_{lh}^R(s) \int_{s-\eta_{lh}(s)}^s \bar{g}_h^R[\varphi_h(u)] \Delta u - \sum_{h=1}^n \alpha_{lh}^I(s) \int_{s-\eta_{lh}(s)}^s \bar{g}_h^I[\varphi_h(u)] \Delta u - \sum_{h=1}^n \alpha_{lh}^J(s) \int_{s-\eta_{lh}(s)}^s \bar{g}_h^J[\varphi_h(u)] \Delta u \\
&\quad - \sum_{h=1}^n \alpha_{lh}^K(s) \int_{s-\eta_{lh}(s)}^s \bar{g}_h^K[\varphi_h(u)] \Delta u + \sum_{h=1}^n \beta_{lh}^R(s) \int_{s-\xi_{lh}(s)}^s \bar{g}_h^R[\varphi_h(u)] \Delta u - \sum_{h=1}^n \beta_{lh}^I(s) \int_{s-\xi_{lh}(s)}^s \bar{g}_h^I[\varphi_h(u)] \Delta u \\
&\quad \left. - \sum_{h=1}^n \beta_{lh}^J(s) \int_{s-\xi_{lh}(s)}^s \bar{g}_h^J[\varphi_h(u)] \Delta u - \sum_{h=1}^n \beta_{lh}^K(s) \int_{s-\xi_{lh}(s)}^s \bar{g}_h^K[\varphi_h(u)] \Delta u \right) \Delta s \Bigg| \leq \int_t^{t+\omega} G_l(t, \sigma(s)) \\
&\quad \cdot \left(\sum_{h=1}^n L_h (\bar{b}_{lh}^R + \bar{b}_{lh}^I + \bar{b}_{lh}^J + \bar{b}_{lh}^K) + \sum_{h=1}^n \bar{\eta}_{lh} H_h (\bar{\alpha}_{lh}^R + \bar{\alpha}_{lh}^I + \bar{\alpha}_{lh}^J + \bar{\alpha}_{lh}^K) + \sum_{h=1}^n \bar{\xi}_{lh} H_h (\bar{\beta}_{lh}^R + \bar{\beta}_{lh}^I + \bar{\beta}_{lh}^J + \bar{\beta}_{lh}^K) \right) \Delta s \|\varphi\|_{\mathbb{X}} \\
&\leq \frac{\omega \kappa_l}{1 + \kappa_l} \left(\sum_{h=1}^n L_h (\bar{b}_{lh}^R + \bar{b}_{lh}^I + \bar{b}_{lh}^J + \bar{b}_{lh}^K) + \sum_{h=1}^n \bar{\eta}_{lh} H_h (\bar{\alpha}_{lh}^R + \bar{\alpha}_{lh}^I + \bar{\alpha}_{lh}^J + \bar{\alpha}_{lh}^K) + \sum_{h=1}^n \bar{\xi}_{lh} H_h (\bar{\beta}_{lh}^R + \bar{\beta}_{lh}^I + \bar{\beta}_{lh}^J + \bar{\beta}_{lh}^K) \right) \frac{r}{1 - \xi} \\
&= \frac{\omega \kappa_l P_l}{1 + \kappa_l} \frac{r}{1 - \xi} \leq \frac{\xi r}{1 - \xi}, \quad l \in \Lambda.
\end{aligned} \tag{32}$$

In a similar way, one can obtain

$$\begin{aligned}
&\left| (\Phi\varphi)_l^m(t) - (\varphi^0)_l^m(t) \right| \leq \frac{\xi r}{1 - \xi}, \\
&m = I, J, K, \quad l \in \Lambda.
\end{aligned} \tag{33}$$

Therefore,

$$\|\Phi\varphi - \varphi^0\| \leq \frac{\xi r}{1 - \xi}, \tag{34}$$

which implies that $\Phi\varphi \in \mathbb{X}^*$, so the mapping Φ is a self-mapping from $\mathbb{X}^*$ to $\mathbb{X}^*$. Next, we shall prove that Φ is a contraction mapping. In fact, for any $\varphi, \psi \in \mathbb{X}^*$, we can get

$$\begin{aligned}
&\left| (\Phi\varphi - \Phi\psi)_l^R(t) \right| = \left| \int_t^{t+\omega} G_l(t, \sigma(s)) \right. \\
&\quad \cdot \left(\sum_{h=1}^n (b_{lh}^R(s) (\tilde{f}_h^R[\varphi_h(s - \tau_{lh}(s))] - \tilde{f}_h^R[\psi_h(s - \tau_{lh}(s))]) - b_{lh}^I(s) (\tilde{f}_h^I[\varphi_h(s - \tau_{lh}(s))] - \tilde{f}_h^I[\psi_h(s - \tau_{lh}(s))]) - b_{lh}^J(s) (\tilde{f}_h^J[\varphi_h(s - \tau_{lh}(s))] - \tilde{f}_h^J[\psi_h(s - \tau_{lh}(s))]) - b_{lh}^K(s) (\tilde{f}_h^K[\varphi_h(s - \tau_{lh}(s))] - \tilde{f}_h^K[\psi_h(s - \tau_{lh}(s))]) \right. \\
&\quad + \sum_{h=1}^n \alpha_{lh}^R(s) \int_{s-\eta_{lh}(s)}^s (\bar{g}_h^R[\varphi_h(u)] - \bar{g}_h^R[\psi_h(u)]) \Delta u - \sum_{h=1}^n \alpha_{lh}^I(s) \int_{s-\eta_{lh}(s)}^s (\bar{g}_h^I[\varphi_h(u)] - \bar{g}_h^I[\psi_h(u)]) \Delta u - \sum_{h=1}^n \alpha_{lh}^J(s) \int_{s-\eta_{lh}(s)}^s (\bar{g}_h^J[\varphi_h(u)] - \bar{g}_h^J[\psi_h(u)]) \Delta u - \sum_{h=1}^n \alpha_{lh}^K(s) \int_{s-\eta_{lh}(s)}^s (\bar{g}_h^K[\varphi_h(u)] - \bar{g}_h^K[\psi_h(u)]) \Delta u \\
&\quad + \sum_{h=1}^n \beta_{lh}^R(s) \int_{s-\xi_{lh}(s)}^s (\bar{g}_h^R[\varphi_h(u)] - \bar{g}_h^R[\psi_h(u)]) \Delta u - \sum_{h=1}^n \beta_{lh}^I(s) \int_{s-\xi_{lh}(s)}^s (\bar{g}_h^I[\varphi_h(u)] - \bar{g}_h^I[\psi_h(u)]) \Delta u - \sum_{h=1}^n \beta_{lh}^J(s) \int_{s-\xi_{lh}(s)}^s (\bar{g}_h^J[\varphi_h(u)] - \bar{g}_h^J[\psi_h(u)]) \Delta u - \sum_{h=1}^n \beta_{lh}^K(s) \int_{s-\xi_{lh}(s)}^s (\bar{g}_h^K[\varphi_h(u)] - \bar{g}_h^K[\psi_h(u)]) \Delta u \Bigg) \Delta s \Bigg| \\
&\leq \int_t^{t+\omega} G_l(t, \sigma(s)) \left(\sum_{h=1}^n L_h (\bar{b}_{lh}^R + \bar{b}_{lh}^I + \bar{b}_{lh}^J + \bar{b}_{lh}^K) + \sum_{h=1}^n \bar{\eta}_{lh} H_h (\bar{\alpha}_{lh}^R + \bar{\alpha}_{lh}^I + \bar{\alpha}_{lh}^J + \bar{\alpha}_{lh}^K) + \sum_{h=1}^n \bar{\xi}_{lh} H_h (\bar{\beta}_{lh}^R + \bar{\beta}_{lh}^I + \bar{\beta}_{lh}^J + \bar{\beta}_{lh}^K) \right) \Delta s \|\varphi - \psi\|_{\mathbb{X}} \leq \frac{\omega \kappa_l}{1 + \kappa_l} \left(\sum_{h=1}^n L_h (\bar{b}_{lh}^R + \bar{b}_{lh}^I + \bar{b}_{lh}^J + \bar{b}_{lh}^K) + \sum_{h=1}^n \bar{\eta}_{lh} H_h (\bar{\alpha}_{lh}^R + \bar{\alpha}_{lh}^I + \bar{\alpha}_{lh}^J + \bar{\alpha}_{lh}^K) + \sum_{h=1}^n \bar{\xi}_{lh} H_h (\bar{\beta}_{lh}^R + \bar{\beta}_{lh}^I + \bar{\beta}_{lh}^J + \bar{\beta}_{lh}^K) \right) \frac{r}{1 - \xi} \\
&\quad + \sum_{h=1}^n \bar{\xi}_{lh} H_h (\bar{\beta}_{lh}^R + \bar{\beta}_{lh}^I + \bar{\beta}_{lh}^J + \bar{\beta}_{lh}^K) \|\varphi - \psi\|_{\mathbb{X}} \leq \xi \|\varphi - \psi\|_{\mathbb{X}}, \quad l \in \Lambda.
\end{aligned} \tag{35}$$

In a similar way, we can obtain

$$\begin{aligned} |(\Phi\varphi - \Phi\psi)_l^m(t)| &\leq \xi \|\varphi - \psi\|_{\mathbb{X}}, \\ m &= I, J, K, \quad l \in \Lambda. \end{aligned} \quad (36)$$

Hence,

$$\|\Phi\varphi - \Phi\psi\|_{\mathbb{X}} \leq \xi \|\varphi - \psi\|_{\mathbb{X}}. \quad (37)$$

Therefore, by (H_3) , Φ is a contraction mapping. Thus, system (15) has a unique ω -anti-periodic solution in the region $\mathbb{X}^* = \{\varphi \in \mathbb{X} : \|\varphi - \varphi^0\|_{\mathbb{X}} \leq \xi r / (1 - \xi)\}$. This completes the proof of Theorem 14. $\square$

4. Global Exponential Stability of Anti-Periodic Solutions

In this section, we will study the global exponential stability of anti-periodic solutions of (15).

Theorem 15. Assume that (H_1) – (H_3) hold. Suppose further that

$$(H_4) \quad \underline{a}_l > \max\{P_l\}, \quad l \in \Lambda,$$

then system (15) has an ω -anti-periodic solution that is globally exponentially stable.

Proof. From Theorem 14, we see that system (15) has an ω -anti-periodic solution $Z^*(t) = (X_1^*(t), X_2^*(t), \dots, X_n^*(t))^T$ with initial value $\varphi^*(s) = (\varphi_1^*(s), \varphi_2^*(s), \dots, \varphi_n^*(s))^T$. Suppose that $Z(t) = (X_1(t), X_2(t), \dots, X_n(t))^T$ is an arbitrary solution of system (15) with initial value $\varphi(s) = (\varphi_1(s), \varphi_2(s), \dots, \varphi_n(s))^T$. Then

$$\begin{aligned} (Z(t) - Z^*(t))_l^\Delta(t) &= -a_l(t) (X_l(t) - X_l^*(t)) \\ &+ \sum_{h=1}^n B_{lh}(t) \\ &\cdot (\tilde{F}_h[x_h(t - \tau_{lh}(t))] - \tilde{F}_h[x_h^*(t - \tau_{lh}(t))]) \\ &+ \bigwedge_{h=1}^n A_{lh}(t) \int_{t-\eta_{lh}(t)}^t (\tilde{G}_h[x_h(s)] - \tilde{G}_h[x_h^*(s)]) \Delta s \\ &+ \bigvee_{h=1}^n C_{lh}(t) \int_{t-\xi_{lh}(t)}^t (\tilde{G}_h[x_h(s)] - \tilde{G}_h[x_h^*(s)]) \Delta s, \end{aligned} \quad (38)$$

$$l \in \Lambda, \quad t \in \mathbb{T}.$$

Let Θ_l be defined by

$$\begin{aligned} \Theta_l(\theta) &= \underline{a}_l - \theta - \exp\left(\theta \sup_{s \in \mathbb{T}} \mu(s)\right) \\ &\cdot \left(\sum_{h=1}^n \exp(\theta \bar{\tau}_{lh}) L_h(\bar{b}_{lh}^R + \bar{b}_{lh}^I + \bar{b}_{lh}^J + \bar{b}_{lh}^K) \right. \\ &+ \sum_{h=1}^n \bar{\eta}_{lh} H_h(\bar{\alpha}_{lh}^R + \bar{\alpha}_{lh}^I + \bar{\alpha}_{lh}^J + \bar{\alpha}_{lh}^K) \\ &+ \left. \sum_{h=1}^n \bar{\xi}_{lh} H_h(\bar{\beta}_{lh}^R + \bar{\beta}_{lh}^I + \bar{\beta}_{lh}^J + \bar{\beta}_{lh}^K) \right), \quad l \in \Lambda. \end{aligned} \quad (39)$$

By (H_5) , for $l \in \Lambda$, we find

$$\Theta_l(0) = \underline{a}_l - P_l > 0. \quad (40)$$

Since Θ_l is continuous on $[0, +\infty)$ and $\Theta_l(\theta) \rightarrow -\infty$ as $\theta \rightarrow +\infty$, so there exists θ_l such that $\Theta_l(\theta_l) = 0$ and $\Theta_l(\theta) > 0$ for $\theta \in (0, \theta_l)$, $l \in \Lambda$. By choosing $c = \min_{l \in \Lambda} \{\theta_l\}$, we have $\Theta_l(c) \geq 0$, $l \in \Lambda$. So, we can choose a positive constant $0 < \lambda < \min\{c, \min_{l \in \Lambda} \{\underline{a}_l\}\}$ such that

$$\Theta_l(\lambda) > 0, \quad l \in \Lambda, \quad (41)$$

which imply that

$$\begin{aligned} &\frac{\exp(\lambda \sup_{s \in \mathbb{T}} \mu(s))}{\underline{a}_l - \lambda} \left(\sum_{h=1}^n \exp(\lambda \bar{\tau}_{lh}) \right. \\ &\cdot L_h(\bar{b}_{lh}^R + \bar{b}_{lh}^I + \bar{b}_{lh}^J + \bar{b}_{lh}^K) + \sum_{h=1}^n \bar{\eta}_{lh} H_h(\bar{\alpha}_{lh}^R + \bar{\alpha}_{lh}^I \\ &+ \bar{\alpha}_{lh}^J + \bar{\alpha}_{lh}^K) + \sum_{h=1}^n \bar{\xi}_{lh} H_h(\bar{\beta}_{lh}^R + \bar{\beta}_{lh}^I + \bar{\beta}_{lh}^J + \bar{\beta}_{lh}^K) \left. \right) \\ &< 1, \end{aligned} \quad (42)$$

where $l \in \Lambda$. Let

$$M_* = \min \left\{ \frac{\underline{a}_l}{P_l} \right\}, \quad l \in \Lambda, \quad (43)$$

by (H_4) , we have $M_* > 1$. Thus,

$$\begin{aligned} \frac{1}{M_*} &< \frac{\exp(\lambda \sup_{s \in \mathbb{T}} \mu(s))}{\underline{a}_l - \lambda} \left(\sum_{h=1}^n \exp(\lambda \bar{\tau}_{lh}) \right. \\ &\cdot L_h(\bar{b}_{lh}^R + \bar{b}_{lh}^I + \bar{b}_{lh}^J + \bar{b}_{lh}^K) + \sum_{h=1}^n \bar{\eta}_{lh} H_h(\bar{\alpha}_{lh}^R + \bar{\alpha}_{lh}^I \\ &+ \bar{\alpha}_{lh}^J + \bar{\alpha}_{lh}^K) + \sum_{h=1}^n \bar{\xi}_{lh} H_h(\bar{\beta}_{lh}^R + \bar{\beta}_{lh}^I + \bar{\beta}_{lh}^J + \bar{\beta}_{lh}^K) \left. \right). \end{aligned} \quad (44)$$

Let $\|Z(t) - Z^*(t)\| = \max_{l \in \Lambda} \{\max_{m \in M} \{|x_l^m(t) - (x_l^*)^m(t)|\}\}$, $\|\omega\|_0 = \max_{l \in \Lambda} \{\max_{m \in M} \{\sup_{s \in [t_0 - \nu, t_0]_{\mathbb{T}}} |\varphi_l^m(s) - (\varphi_l^*)^m(s)|\}\}$. For any $\varepsilon > 0$, it is obvious that

$$\|Z(t_0) - Z^*(t_0)\| < \|\omega\|_0 + \varepsilon \quad (45)$$

and

$$\begin{aligned} \|Z(t) - Z^*(t)\| &< (\|\omega\|_0 + \varepsilon) e_{\Theta\lambda}(t, t_0) \\ &< M_* (\|\omega\|_0 + \varepsilon) e_{\Theta\lambda}(t, t_0), \end{aligned} \quad (46)$$

$$\forall t \in [t_0 - \nu, t_0]_{\mathbb{T}}.$$

We claim that

$$\begin{aligned} \|Z(t) - Z^*(t)\| &< M_* (\|\omega\|_0 + \varepsilon) e_{\Theta\lambda}(t, t_0), \\ &\forall t \in (t_0, +\infty)_{\mathbb{T}}. \end{aligned} \quad (47)$$

If (47) is not true, then there must be a first time $t_1 \in (t_0, +\infty)_{\mathbb{T}}$ such that

$$\begin{aligned} \|Z(t_1) - Z^*(t_1)\| &\geq M_* (\|\omega\|_0 + \varepsilon) e_{\Theta\lambda}(t_1, t_0), \\ \|Z(t) - Z^*(t)\| &< M_* (\|\omega\|_0 + \varepsilon) e_{\Theta\lambda}(t, t_0), \\ &t \in [t_0 - \nu, t_1]_{\mathbb{T}}. \end{aligned} \quad (48)$$

Therefore, there must exist a constant $C \geq 1$ such that

$$\begin{aligned} \|Z(t_1) - Z^*(t_1)\| &= CM_* (\|\omega\|_0 + \varepsilon) e_{\Theta\lambda}(t_1, t_0), \\ \|Z(t) - Z^*(t)\| &< CM_* (\|\omega\|_0 + \varepsilon) e_{\Theta\lambda}(t, t_0), \end{aligned}$$

Multiplying both sides of (38) by $e_{-d_l}(t_0, \sigma(t))$ and integrating over $[t_0, t]_{\mathbb{T}}$, we get

$$\begin{aligned} (Z(t) - Z^*(t))_l &= (Z(t_0) - Z^*(t_0))_l e_{-a_l}(t, t_0) \\ &+ \int_{t_0}^t e_{-a_l}(t, \sigma(s)) \times \left(\sum_{h=1}^n B_{lh}(s) \right. \\ &\cdot (\tilde{F}_h[x_h(s - \tau_{lh}(s))] - \tilde{F}_h[x_h^*(s - \tau_{lh}(s))]) \\ &+ \bigwedge_{h=1}^n A_{lh}(s) \int_{s-\eta_{lh}(s)}^s (\tilde{G}_h[x_h(u)] - \tilde{G}_h[x_h^*(u)]) \Delta u \\ &+ \bigvee_{h=1}^n C_{lh}(s) \\ &\cdot \int_{s-\xi_{lh}(s)}^s (\tilde{G}_h[x_h(u)] - \tilde{G}_h[x_h^*(u)]) \Delta u \Big) \Delta s, \end{aligned} \quad (50)$$

$$t \in \mathbb{T}, l \in \Lambda.$$

In view of (45), (46), (49) and $M_* > 1$, we have

$$\begin{aligned} |x_l^R(t_1) - (x^*)^R_l(t_1)| &= \left| (x_l^R(t_0) - (x^*)^R_l(t_0)) e_{-a_l}(t_1, t_0) + \int_{t_0}^{t_1} e_{-a_l}(t_1, \sigma(s)) \right. \\ &\cdot \left(\sum_{h=1}^n (b_{lh}^R(s) (\tilde{f}_h^R[x_h(s - \tau_{lh}(s))] - \tilde{f}_h^R[x_h^*(s - \tau_{lh}(s))]) - b_{lh}^L(s) (\tilde{f}_h^L[x_h(s - \tau_{lh}(s))] - \tilde{f}_h^L[x_h^*(s - \tau_{lh}(s))]) \right. \\ &- b_{lh}^K(s) (\tilde{f}_h^K[x_h(s - \tau_{lh}(s))] - \tilde{f}_h^K[x_h^*(s - \tau_{lh}(s))]) \\ &+ \bigwedge_{h=1}^n \alpha_{lh}^R(s) \int_{s-\eta_{lh}(s)}^s (\tilde{g}_h^R[x_h(u)] - \tilde{g}_h^R[x_h^*(u)]) \Delta u - \bigwedge_{h=1}^n \alpha_{lh}^L(s) \int_{s-\eta_{lh}(s)}^s (\tilde{g}_h^L[x_h(u)] - \tilde{g}_h^L[x_h^*(u)]) \Delta u \\ &- \bigwedge_{h=1}^n \alpha_{lh}^K(s) \int_{s-\eta_{lh}(s)}^s (\tilde{g}_h^K[x_h(u)] - \tilde{g}_h^K[x_h^*(u)]) \Delta u \\ &+ \bigvee_{h=1}^n \beta_{lh}^R(s) \int_{s-\xi_{lh}(s)}^s (\tilde{g}_h^R[x_h(u)] - \tilde{g}_h^R[x_h^*(u)]) \Delta u - \bigvee_{h=1}^n \beta_{lh}^L(s) \int_{s-\xi_{lh}(s)}^s (\tilde{g}_h^L[x_h(u)] - \tilde{g}_h^L[x_h^*(u)]) \Delta u \\ &- \bigvee_{h=1}^n \beta_{lh}^K(s) \int_{s-\xi_{lh}(s)}^s (\tilde{g}_h^K[x_h(u)] - \tilde{g}_h^K[x_h^*(u)]) \Delta u \Big) \Delta s \Big| \\ &\leq \left\{ \frac{e_{-\Theta\lambda}(t_1, t_0)}{CM_*} + \int_{t_0}^{t_1} e_{-\Theta\lambda}(t_1, \sigma(s)) \left[e_{\lambda}(\sigma(s), s - \tau_{lh}(s)) \sum_{h=1}^n L_h (\tilde{b}_{lh}^R + \tilde{b}_{lh}^L + \tilde{b}_{lh}^K) + e_{\lambda}(\sigma(s), s) \sum_{h=1}^n \tilde{\eta}_{lh} H_h (\tilde{\alpha}_{lh}^R + \tilde{\alpha}_{lh}^L + \tilde{\alpha}_{lh}^K) \right. \right. \\ &+ e_{\lambda}(\sigma(s), s) \times \sum_{h=1}^n \tilde{\xi}_{lh} H_h (\tilde{\beta}_{lh}^R + \tilde{\beta}_{lh}^L + \tilde{\beta}_{lh}^K) \Big] \Delta s \Big\} CM_* (\|\omega\|_0 + \varepsilon) e_{\Theta\lambda}(t_1, t_0) \\ &\leq \left\{ \frac{1}{M_*} - \frac{\exp(\lambda \sup_{s \in \mathbb{T}} \mu(s))}{d_l - \lambda} \left[\exp(\lambda \tau_{lh}) \sum_{h=1}^n L_h (\tilde{b}_{lh}^R + \tilde{b}_{lh}^L + \tilde{b}_{lh}^K) + \sum_{h=1}^n \tilde{\eta}_{lh} H_h (\tilde{\alpha}_{lh}^R + \tilde{\alpha}_{lh}^L + \tilde{\alpha}_{lh}^K) \right. \right. \\ &+ \sum_{h=1}^n \tilde{\xi}_{lh} H_h (\tilde{\beta}_{lh}^R + \tilde{\beta}_{lh}^L + \tilde{\beta}_{lh}^K) \Big] e_{-(d_l - \lambda)}(t_1, t_0) + \frac{\exp(\lambda \sup_{s \in \mathbb{T}} \mu(s))}{d_l - \lambda} \left[\exp(\lambda \tau_{lh}) \sum_{h=1}^n L_h (\tilde{b}_{lh}^R + \tilde{b}_{lh}^L + \tilde{b}_{lh}^K) \right. \\ &+ \sum_{h=1}^n \tilde{\eta}_{lh} H_h (\tilde{\alpha}_{lh}^R + \tilde{\alpha}_{lh}^L + \tilde{\alpha}_{lh}^K) + \sum_{h=1}^n \tilde{\xi}_{lh} H_h (\tilde{\beta}_{lh}^R + \tilde{\beta}_{lh}^L + \tilde{\beta}_{lh}^K) \Big] \Big\} CM_* (\|\omega\|_0 + \varepsilon) e_{\Theta\lambda}(t_1, t_0) < CM_* (\|\omega\|_0 + \varepsilon) e_{\Theta\lambda}(t_1, t_0), \quad l \in \Lambda. \end{aligned} \quad (51)$$

Similarly, we have

$$\begin{aligned} |x_l^m(t_1) - (x^*)^m_l(t_1)| &< CM_* (\|\omega\|_0 + \varepsilon) e_{\Theta\lambda}(t_1, t_0), \\ m &= I, J, K, l \in \Lambda. \end{aligned} \quad (52)$$

Hence

$$\begin{aligned} \|Z(t_1) - Z^*(t_1)\| &= \max \{ |x_l^m(t_1) - (x^*)^m_l(t_1)|, m \in M, l \in \Lambda \} \\ &< CM_* (\|\omega\|_0 + \varepsilon) e_{\Theta\lambda}(t_1, t_0), \end{aligned} \quad (53)$$

which contradicts the first equation of (48). Therefore, (47) holds. Letting $\varepsilon \rightarrow 0^+$ leads to

$$\begin{aligned} \|Z(t) - Z^*(t)\| &\leq M_* \|\omega\|_0 e_{\Theta\lambda}(t, t_0), \\ &\forall t \in (t_0, +\infty)_{\mathbb{T}}. \end{aligned} \quad (54)$$

Hence, the ω -anti-periodic solution of system (15) is globally exponentially stable. The proof of Theorem 15 is completed. $\square$

Remark 16. In view of Remark 11, we have that, under conditions of Theorems 14 and 15, system (10) has a unique anti-periodic solution that is globally exponentially stable.

Remark 17. In view of Theorem 14, Theorem 15, and Remark 16, we see that under condition $-a_l \in \mathcal{R}^+$, $l \in \Lambda$ both the continuous time and the discrete time cases of system (10) have the same dynamics for the anti-periodicity.

5. An Example

In this section, we give an example to illustrate the feasibility and effectiveness of our results obtained in Sections 3 and 4.

Example 18. In system (10), let $n = 2$, $\omega = \pi$. The coefficients are taken as follows:

$$\begin{aligned}
 \tilde{f}_h(x) &= \tilde{g}_h(x) \\
 &= \frac{1}{80h} (|x^R + 1| - |x^J - 1|) \\
 &\quad + i \frac{h}{80} \sin \frac{\sqrt{2}}{2} (x^I + x^K) - j \frac{h}{80} \sin x^J \\
 &\quad + \frac{h}{80} k \tanh (x^I + x^K + x^R), \\
 a_1(t) &= 0.8 + 0.02 |\cos t|, \\
 a_2(t) &= 0.95 + 0.01 |\cos t|, \\
 I_l(t) &= 0.2l \cos t + i(0.1 + 0.2 \sin t) + 0.2j \sin t \\
 &\quad + 0.1k \cos t, \\
 b_{lh}(t) &= 0.4 \sin 2t + 0.1hi \cos 2t + 0.1lj \cos 2t \\
 &\quad + 0.3k \sin 2t, \\
 d_{lh}(t) &= 0.1l \cos 2t + 0.3i \sin 2t + 0.1hj \cos 2t \\
 &\quad + 0.1k \sin 2t, \\
 \alpha_{lh}(t) &= 0.1h \cos 2t + 0.2hi \sin 2t + 0.2j \sin 2t \\
 &\quad + 0.1k \sin 2t, \\
 \beta_{lh}(t) &= 0.3 \sin 2t + 0.1i \cos 2t + 0.2hj \cos 2t \\
 &\quad + 0.2k \sin 2t, \\
 T_{lh}(t) &= 0.2 + 0.1h \sin 2t + 0.1i \cos 2t + 0.1j \sin 2t \\
 &\quad + 0.1lk \cos 2t, \\
 S_{lh}(t) &= 0.1 \cos 2t + 0.1hi \sin 2t + 0.1j \cos 2t \\
 &\quad + 0.2lk \sin 2t, \\
 \mu_h(t) &= 0.1l \sin t + 0.2hi \cos t + 0.2j \cos t \\
 &\quad + 0.1k \sin t,
 \end{aligned} \tag{55}$$

if $\mathbb{T} = \mathbb{R}$, then we take

$$\begin{aligned}
 \tau_{lh}(t) &= \cos^2 t, \\
 \eta_{lh}(t) &= \frac{h}{2} |\cos t|, \\
 \xi_{lh}(t) &= \frac{l}{2} \sin^2 t,
 \end{aligned} \tag{56}$$

and if $\mathbb{T} = \mathbb{Z}$, then we take

$$\begin{aligned}
 \tau_{lh}(t) &= 1 + \cos^2 \left(\frac{\pi}{2} + \pi t \right), \\
 \eta_{lh}(t) &= \frac{h}{2} |\sin \pi t|, \\
 \xi_{lh}(t) &= \frac{l}{2} \sin^2 \pi t,
 \end{aligned} \tag{57}$$

$l, h = 1, 2$.

By calculating, we have

$$L_1^m = H_1^m = 0.0125,$$

$$L_2^m = H_2^m = 0.025,$$

$$\bar{a}_1 = 0.82,$$

$$\bar{a}_2 = 0.96,$$

$m \in M$,

$$\underline{a}_1 = 0.8,$$

$$\underline{a}_2 = 0.95,$$

$$\bar{b}_{11}^R = 0.4,$$

$$\bar{b}_{11}^I = \bar{b}_{11}^J = \bar{b}_{12}^J = 0.1,$$

$$\bar{b}_{11}^K = 0.3,$$

$$\bar{b}_{12}^R = \bar{b}_{21}^R = 0.4,$$

$$\bar{b}_{12}^I = 0.2,$$

$$\bar{b}_{12}^K = 0.3,$$

$$\bar{b}_{21}^I = 0.1,$$

$$\bar{b}_{21}^J = 0.2,$$

$$\bar{b}_{21}^K = 0.3,$$

$$\bar{b}_{22}^R = 0.4,$$

$$\bar{b}_{22}^I = \bar{b}_{22}^J = 0.2,$$

$$\bar{b}_{22}^K = \bar{d}_{11}^I = 0.3,$$

$$\bar{d}_{11}^R = \bar{d}_{11}^J = \bar{d}_{11}^K = 0.1,$$

$$\bar{d}_{12}^R = \bar{d}_{12}^K = 0.1,$$

$$\bar{d}_{12}^I = 0.3,$$

$$\bar{d}_{12}^J = \bar{d}_{21}^R = 0.2,$$

FIGURE 1: Responses of the four parts of x with continuous time t .

$$\begin{aligned}
 \bar{d}_{21}^I &= 0.1, & \bar{\alpha}_{21}^K &= \bar{\alpha}_{22}^K = 0.1, \\
 \bar{d}_{21}^I &= \bar{d}_{21}^K = 0.1, & \bar{\alpha}_{22}^I &= 0.4, \\
 \bar{d}_{22}^R &= \bar{d}_{22}^I = 0.2, & \bar{\beta}_{11}^R &= \bar{\beta}_{21}^R = \bar{\beta}_{12}^R = \bar{\beta}_{22}^R = 0.3, \\
 \bar{d}_{22}^I &= 0.3, & \bar{\beta}_{11}^I &= \bar{\beta}_{12}^I = \bar{\beta}_{21}^I = \bar{\beta}_{22}^I = 0.1, \\
 \bar{d}_{22}^K &= 0.1, & \bar{\beta}_{12}^I &= \bar{\beta}_{22}^I = 0.4, \\
 \bar{\tau}_{11} &= \bar{\tau}_{12} = \bar{\tau}_{21} = \bar{\tau}_{22} = 1, & \bar{\beta}_{11}^I &= \bar{\beta}_{21}^I = 0.2, \\
 \bar{\alpha}_{11}^R &= \bar{\alpha}_{11}^K = 0.1, & \bar{\beta}_{12}^K &= \bar{\beta}_{11}^K = \bar{\beta}_{21}^K = \bar{\beta}_{22}^K = 0.2, \\
 \bar{\alpha}_{11}^I &= \bar{\alpha}_{11}^I = \bar{\alpha}_{12}^R = \bar{\alpha}_{12}^I = 0.2, & & \\
 \bar{\alpha}_{12}^I &= 0.4, & & \\
 \bar{\alpha}_{12}^K &= 0.1, & & \\
 \bar{\alpha}_{21}^R &= 0.1, & & \\
 \bar{\alpha}_{21}^I &= \bar{\alpha}_{21}^I = \bar{\alpha}_{22}^I = \bar{\alpha}_{22}^R = 0.2, & &
 \end{aligned}
 \tag{58}$$

when $\mathbb{T} = \mathbb{R}$,

$$\begin{aligned}
 \bar{\xi}_{11} &= \bar{\xi}_{12} = 0.5, \\
 \bar{\xi}_{21} &= \bar{\xi}_{22} = 1,
 \end{aligned}$$

FIGURE 2: Responses of the four parts of x with discrete time t .

$$\begin{aligned}\bar{\eta}_{11} &= \bar{\eta}_{21} = 0.5, \\ \bar{\eta}_{12} &= \bar{\eta}_{22} = 1,\end{aligned}\quad (59)$$

when $\mathbb{T} = \mathbb{Z}$,

$$\bar{\xi}_{11} = \bar{\xi}_{12} = \bar{\xi}_{21} = \bar{\xi}_{22} = \bar{\eta}_{11} = \bar{\eta}_{21} = \bar{\eta}_{12} = \bar{\eta}_{22} = 0. \quad (60)$$

Therefore, whether $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, we have $-a_i \in \mathcal{R}^+$ and it is easy to verify that conditions (H_1) and (H_2) are satisfied. By a simple calculation, we have

$$\xi \leq \max \left\{ \frac{P_1}{a_1}, \frac{P_2}{a_2} \right\} = \max \{0.4, 0.5\} = 0.5 < 1, \quad (61)$$

which implies that conditions (H_3) and (H_4) are also satisfied. Therefore, according to Theorems 14 and 15, (10) has a unique π -anti-periodic solution, which is globally exponentially stable (Figures 1 and 2).

6. Conclusion

In this paper, we have proposed a class of QVFCNNs with time-varying delays on time scales which can unify the

continuous time and discrete time cases of QVFCNNs and contain real-valued FCNNs and complex-valued FCNNs as their special cases. Based on a novel method that is different from those used in [10, 25–34], we established the existence and global exponential stability of anti-periodic solutions of the QVFCNNs. To the best of our knowledge, this is the first paper to study the existence of anti-periodic solutions for QVFCNNs with time-varying delays on time scales. The method of this paper may be applied to study other types of the QVNNs on times scales such as quaternion-valued recurrent neural networks, quaternion-valued BAM neural networks, quaternion-valued shunting inhibitory cellular neural networks, quaternion-valued Cohen-Grossberg neural networks, and so on.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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