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Research Article

Optimal Control Strategy for a Discrete Time Smoking Model with Specific Saturated Incidence Rate

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The aim of this paper is to study and investigate the optimal control strategy of a discrete mathematical model of smoking with specific saturated incidence rate. The population that we are going to study is divided into five compartments: potential smokers, light smokers, heavy smokers, temporary quitters of smoking, and permanent quitters of smoking. Our objective is to find the best strategy to reduce the number of light smokers, heavy smokers, and temporary quitters of smoking. We use three control strategies which are awareness programs through media and education, treatment, and psychological support with follow-up. Pontryagins maximum principle in discrete time is used to characterize the optimal controls. The numerical simulation is carried out using MATLAB. Consequently, the obtained results confirm the performance of the optimization strategy.

1. Introduction

Following the WHO report on the global tobacco epidemic, which was published on 19 July 2017 in the United Nations High-Level Political Forum for Sustainable Development in New York, tobacco consumption is the world's leading cause of death with a rate of more than 7 million deaths a year [1]. Tobacco cosumption is known as the main cause of death of lethal diseases such as lung cancer, oral cavity, stomach ulcer, and a probable cause of death for cancers of the larynx, bladder, pancreas, and renal pelvis. Comparative data on smoking show that the risk of heart attack among smokers is 70% higher than that of nonsmokers. In addition to that, the economic costs are also enormous totaling more than 1400 billion dollars (US $) in health expenditure and loss of productivity [2].

Lung cancer in smokers is ten times higher than in nonsmokers, and one in ten smokers will die of lung cancer. In Spain, it is estimated that about 55,000 deaths a year are attributable to smoking [3]. However, smoking-related illnesses cause more than 440,000 deaths each year in the United States and more than 105,000 deaths in the United Kingdom each year. Moreover, about 4 million people die from smoking-related diseases worldwide and half of all smokers die from smoking-related diseases, while the number of new smokers continues to increase [4].

As far as Morocco is concerned, a new report by the World Health Organization revealed that the number of smokers has notably risen, expecting it to reach over 7 million by 2025 [5]. The WHO has urged the Moroccan government to increase taxes on cigarettes and other tobacco products to discourage use and decrease the rising number of smokers across the country. In its Global Report on Trends in Prevalence of Tobacco Smoking in 2015, WHO estimated that up to 21% of Morocco's population (approximately 4,820,500 persons) smoked in 2010.

More specifically, the report noted that about 42% of men and up to 2% of women smoked in Morocco in 2010. It goes on to add that the highest rate of smoking among men was seen in the 25–39 age groups and 15–24 age groups among women. WHO recommends that at least one adult survey and one youth survey be completed every five years. In the event that the Moroccan government does not adopt new measures to discourage the use of tobacco, the UN estimated that the number of smokers could be over 7 million by 2025. Member states, including Morocco, adopted a voluntary global target to reduce tobacco use 30% (smokers and smokeless) by 2025. However, based on the current smoking trend, Morocco will not achieve the target, WHO concluded [6].

Mathematical modeling of smoking has been studied by many researchers [3, 7–9]. We observe that most of those researchers focused on the continuous-time models described by the differential equations. It is noted that, in recent years, more and more attention has been given to discrete time models (see [10–13] and the references cited therein). The reasons for adopting discrete modeling are as follows: Firstly, the statistical data are collected at discrete moments (day, week, month, or year). So, it is more direct and more accurate and timely to describe the disease using discrete time models than continuous time models. Secondly, the use of discrete time models can avoid some mathematical complexities such as choosing a function space and regularity of the solution. Thirdly, the numerical simulations of continuous time models are obtained by the way of discretization.

Based on the aforementioned reasons, we will develop in this paper a discrete time model studying the dynamics of smokers and introduce a saturated incidence rate to be analysed in detail in the next section. Also, we add to our model two elements which were not taken into consideration in the most previous researches. Those two elements are a group of light smokers who quit smoking permanently and a group of heavy smokers who died due to diseases generated by the excess of smoking.

In addition, in order to find the best strategy to reduce the number of light smokers, heavy smokers, and temporary quitters of smoking we will use three control srategies which are awareness programs through media and education, treatment, and psychological support with follow-up.

In this paper, we construct a discrete $\mathrm{PLSQ}^t\mathrm{Q}^p$ Mathematical Smoking Model with Specific Saturated Incidence Rate and introduce the control of awareness measures. In Section 2, the mathematical model is proposed. In Section 3, we investigate the optimal control problem for the proposed discrete mathematical model. Section 4 consists of numerical simulation through MATLAB. The conclusion is given in Section 5.

2. Formulation of the Mathematical Model

In this section, we present a discrete $\mathrm{PLSQ}^t\mathrm{Q}^p$ Mathematical Smoking Model. The population under invistegation is divided into five compartments: potential smokers (nonsmokers) P_k, light smokers L_k, heavy smokers S_k, smokers who temporarily quit smoking Q_k^t, and smokers who permanently quit smoking Q_k^p, respectively. Following what has been done in many works [2], we introduce a saturated incidence rate $\beta_i(X_kY_k/(1+m_jY_k))$ in discete time (where $X_kY_k \in \{P_kL_k, P_kS_k, L_kS_k\}$, β_i –for $i = 1, 2,$ – is the contact rate between X_k and Y_k, m_j –for $i = 1, 2, 3$– is a positive constant) to describe the crowding effect among potential smokers, light smokers, and heavy smokers. $\beta_i X_kY_k$ measures the infection force of the smoking and $1/(1+m_iY_k)$ describes the crowding effect and the "psychological" effect from the behavioral change of the individuals Y_k when their number increases.

2.1. Description of the Model. **The compartment P:** the potential smokers (nonsmokers), poeple who have not smoked yet but might become smokers in the future. This compartment is increased by the recruitment of individuals at rate Λ and P is decreased with the rates $\beta_1(1-\rho)(P_kL_k/(1+m_1L_k))$, $\beta_1\rho(P_kS_k/(1+m_2S_k))$ and some of the poeple vacate at a constant death rate of μ due to the total natural death rate μP_k.

The compartment L: the occasional smokers whose number increases when the potential smokers start to smoke with a saturated incidence rate $\beta_1(1-\rho)(P_kL_k/(1+m_1L_k))$. Some other individials will leave the compartment with the saturated incidence rate $\beta_2(L_kS_k/(1+m_3S_k))$, the rate β_3L_k, and μL_k. Here, β_3 is the rate of light smokers who permanently quit smoking.

The compartment S: the people who are heavy smokers and whose number increases by the saturated incidence rate $\beta_1\rho(P_kS_k/(1+m_2S_k))$, $\beta_2(L_kS_k/(1+m_3S_k))$ and the rate α of temporary quitters who revert back to smoking. Some others will leave at the rates γS_k, λS_k, and μS_k. Here, λ is the rate of death due to heavy somking and γ is the rate of quitting smoking.

The compartment $\mathbf{Q}^t$: the individuals who temporarily quit smoking, whose number increases at the rate $\gamma(1-\sigma)S_k$ and decreases at the rates μQ_k^t and αQ_k^t, where $(1-\sigma)$ is the fraction of heavy smokers who temporarily quit smoking (at a rate γ).

The compartment $\mathbf{Q}^p$: the individuals who permamently quit smoking, whose number increases with the rates $\gamma\sigma S_k$ and β_3L_k. Some people of this compartment will die with the rate μQ_k^p, where σ is the remaining fraction of heavy smokers who permanently quit smoking (at a rate γ).

The following diagram will demonstrate the flow directions of individuals among the compartments. These directions are going to be represented by directed arrows in Figure 1.

2.2. Model Equations. Through the addition of the rates at which individuals enter the compartment and also by subtructing the rates at which people vacate the compartment, we obtain an equation of difference for the rate at which the individuals of each compartment change over discrete time. Hence, we present the smoking

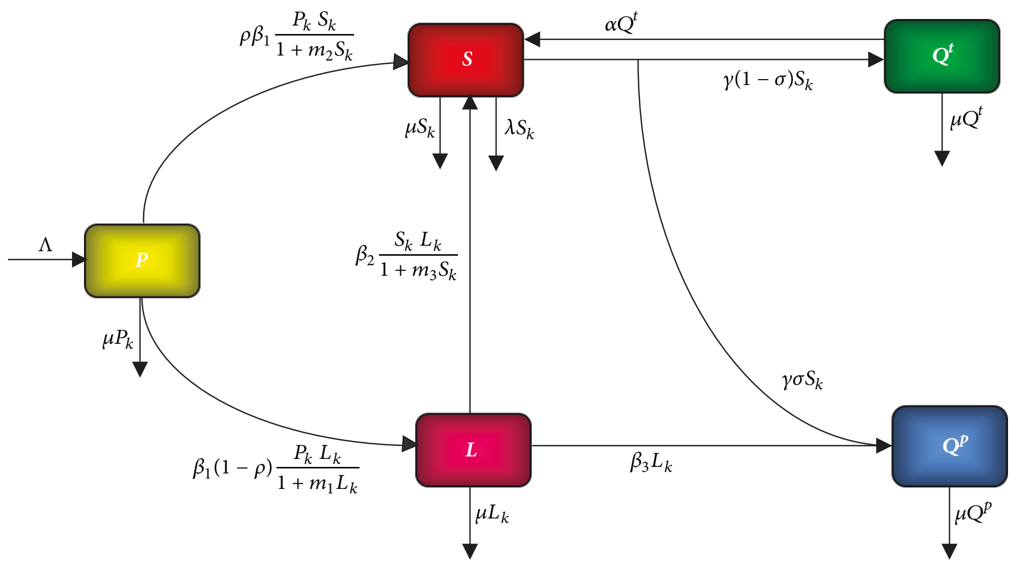


Figure 1: The flow between the five compartments $PLSQ^tQ^p$.

infection model by the following system of difference equations:

$$
\begin{aligned}
P_{k+1} &= \Lambda + (1-\mu) P_k - \beta_1 (1-\rho) \frac{P_k L_k}{1+m_1 L_k} \\
&\quad - \beta_1 \rho \frac{P_k S_k}{1+m_2 S_k} \\
L_{k+1} &= (1-\mu) L_k + \beta_1 (1-\rho) \frac{P_k L_k}{1+m_1 L_k} \\
&\quad - \beta_2 \frac{L_k S_k}{1+m_3 S_k} - \beta_3 L_k \\
S_{k+1} &= (1-\mu-\lambda-\gamma) S_k + \beta_1 \rho \frac{P_k S_k}{1+m_2 S_k} \\
&\quad + \beta_2 \frac{L_k S_k}{1+m_3 S_k} + \alpha Q_k^t \\
Q_{k+1}^t &= (1-\mu-\alpha) Q_k + \gamma (1-\sigma) S_k \\
Q_{k+1}^p &= (1-\mu) Q_k^p + \gamma\sigma S_k + \beta_3 L_k
\end{aligned}
\tag{1}
$$

3. The Optimal Control Problem

The strategies of control that we adopt consist of an awareness program through media and education, treatment, and psychological support with follow-up. Our main goal in adopting those strategies is to minimize the number of occasional smokers, heavy smokers, and the temporarily quitters of smoking during the time steps $k = 0$ to T and also minimizing the cost spent in apllying the three strategies. In this model, we include the three controls u_k, v_k, and w_k that represent consecutively the awareness program through media and education, treatment, and psychological support with follow-up as measures at time k. So, the controlled mathematical system is given by the following system of difference equations:

$$
\begin{aligned}
P_{k+1} &= \Lambda + (1-\mu) P_k - \beta_1 (1-\rho) \frac{P_k L_k}{1+m_1 L_k} \\
&\quad - \beta_1 \rho \frac{P_k S_k}{1+m_2 S_k} \\
L_{k+1} &= (1-\mu) L_k + \beta_1 (1-\rho) \frac{P_k L_k}{1+m_1 L_k} \\
&\quad - \beta_2 \frac{L_k S_k}{1+m_3 S_k} - \beta_3 L_k - c_1 u_k L_k \\
S_{k+1} &= (1-\mu-\lambda-\gamma) S_k + \beta_1 \rho \frac{P_k S_k}{1+m_2 S_k} \\
&\quad + \beta_2 \frac{L_k S_k}{1+m_3 S_k} + \alpha Q_k^t - c_2 v_k S_k \\
Q_{k+1}^t &= (1-\mu-\alpha) Q_k^t + \gamma (1-\sigma) S_k - c_3 w_k Q_k^t \\
Q_{k+1}^p &= (1-\mu) Q_k^p + \gamma\sigma S_k + \beta_3 L_k + c_1 u_k L_k + c_2 v_k S_k \\
&\quad + c_3 w_k Q_k^t,
\end{aligned}
\tag{2}
$$

where

$$
c_i = \begin{cases} 1 & \\ 0 & \end{cases} \qquad \text{for } i = 1, 2, 3. \tag{3}
$$

Table 1: Interpretations according to the values of c_i.

c_1	c_2	c_3	Interpretations
0	0	0	Discrete smoking model (without controls)
1	0	0	Discrete smoking model with awareness program: (u)
0	1	0	Discrete smoking model with treatment: (v)
0	0	1	Discrete smoking model with psychological support along follow-up: (w)
1	1	0	Discrete smoking model with (u) and (v)
1	0	1	Discrete smoking model with (u) and (w)
0	1	1	Discrete smoking model with (v) and (w)
1	1	1	Discrete smoking model with (u), (v) and (w)

There are three controls $u = (u_{0,}u_{1,} \dots u_T)$, $v = (v_{0,}v_{1,} \dots v_T)$, and $w = (w_{0,}w_{1,} \dots w_T)$. The first control can be interpreted as the proportion to be adopted for the awareness program through media and education. So, we note that $u_k L_k$ is the proportion of the light smoker individuals who moved to the individuals who permanently quit smoking at time step k. The second control can be interpreted as the proportion to be subjected to treatment. So, we note that $v_k S_k$ is the proportion of the individuals who will move from the class of heavy smokers towards the class of the individuals who permanently quit smoking at time step k. The third control can also be interpreted as the proportion to get psychological support with follow-up. So, we note that $w_k Q_k^t$ is the proportion of the individuals who temporarily quit smoking and who will transform into the individuals who permanently quit smoking at time step k. Indeed, the system above (2) presents eight different models as Table 1 explains.

The problem that we face here is how to minimize the objective functional:

$$J(u,v,w) = A_T S_T + B_T L_T + C_T Q_T^t + \sum_{k=0}^{T-1} \Big(A_k S_k + B_k L_k + C_k Q_k^t + \frac{D_k}{2} c_1 u_k^2 + \frac{E_k}{2} c_2 v_k^2 + \frac{F_k}{2} c_3 w_k^2 \Big), \tag{4}$$

where the parameters $A_k > 0$, $B_k > 0$, $C_k > 0$, $D_k > 0$, $E_k > 0$, and $F_k > 0$ are the cost coefficients; they are selected to weigh the relative importance of S_k, L_k, Q_k^t, u_k, v_k, and w_k at time k. T is the final time.

In other words, we seek the optimal controls u_k, v_k, and w_k such that

$$J(u^*, v^*, w^*) = \min_{(u,v,w) \in U_{ad}} J(u,v,w), \tag{5}$$

where U_{ad} is the set of admissible controls defined by

$$U_{ad} = \{(u_k, v_k, w_k) : a \le u_k \le b, c \le v_k \le d, e \le w_k \le f;\ k = 0, 1, 2 \dots T-1\} \tag{6}$$

The sufficient condition for the existence of optimal controls (u, v, w) for problem (2) and (4) comes from the following theorem.

Theorem 1. *There exists an optimal control (u^*, v^*, w^*) such that*

$$J(u^*, v^*, w^*) = \min_{(u,v,w) \in U_{ad}} J(u,v,w) \tag{7}$$

subjet to the control system (2) with initial conditions.

Proof. Since the coefficients of the state equations are bounded and there are a finite number of time steps, $P = (P_0, P_1, \dots, P_T)$, $L = (L_0, L_1, \dots, L_T)$, $S = (S_0, S_1, \dots, S_T)$, $Q^t = (Q_0^t, Q_1^t, \dots, Q_T^t)$, and $Q^p = (Q_0^p, Q_1^p, \dots, Q_T^p)$ are uniformly bounded for all $(u; v; w)$ in the control set U_{ad}; thus $J(u; v; w)$ is bounded for all $(u; v; w) \in U_{ad}$.Since $J(u; v; w)$ is bounded, $\inf_{(u,v,w)\in U_{ad}} J(u, v, w)$ is finite, and there exists a sequence $(u^j; v^j; w^j) \in Uad$ such that $\lim_{j \longrightarrow +\infty} J(u^j, v^j, w^j) = \inf_{(u,v,w)\in U_{ad}} J(u, v, w)$ and corresponding sequences of states P^j, L^j, S^j, Q^{tj}, and Q^{pj}. Since there is a finite number of uniformly bounded sequences, there exist $(u^*, v^*, w^*) \in U_{ad}$ and P^*, L^*, S^*, Q^{t*}, and $Q^{p*} \in IR^{T+1}$ such that, on a subsequence, $(u^j, v^j, w^j) \longrightarrow (u^*, v^*, w^*)$, $P^j \longrightarrow P^*$, $L^j \longrightarrow L^*$, $S^j \longrightarrow S^*$, $Q^{tj} \longrightarrow Q^{t*}$, and $Q^{pj} \longrightarrow Q^{p*}$. Finally, due to the finite dimensional structure of system (2) and the objective function $J(u; v; w)$, $(u^*; v^*; w^*)$ is an optimal control with corresponding states P^*, L^*, S^*, Q^{t*}, and Q^{p*}. Therefore $\inf_{(u,v,w)\in U_{ad}} J(u, v, w)$ is achieved. □

In order to derive the necessary condition for optimal control, the pontryagins maximum principle in discrete time given in [10, 11, 14–16] was used. This principle converts into a problem of minimizing a Hamiltonian H_k at time step k defined by

$$H_k = A_k S_k + B_k L_k + C_k Q_k^t + \frac{D_k}{2} c_1 u_k^2 + \frac{E_k}{2} c_2 v_k^2 + \frac{F_k}{2} c_3 w_k^2 + \sum_{j=1}^{5} \lambda_{j,k+1} f_{j,k+1}, \tag{8}$$

where $f_{j,k+1}$ is the right side of the system of difference equations (2) of the j^{th} state variable at time step $k+1$.

Theorem 2. *Given an optimal control $(u_k^*, v_k^*, w_k^*) \in U_{ad}$ and the solutions P_k^*, L_k^*, S_k^*, Q_k^{t*}, and Q_k^{p*} of the corresponding state system (2), there exist adjoint functions $\lambda_{1,k}$, $\lambda_{2,k}$, $\lambda_{3,k}$, $\lambda_{4,k}$, and $\lambda_{5,k}$ satisfying*

$$\begin{aligned}
\lambda_{1,k} &= \beta_1\rho\left(\lambda_{3,k+1}-\lambda_{1,k+1}\right)\frac{S_k}{1+m_2S_k} \\
&\quad + \beta_1\left(1-\rho\right)\left(\lambda_{2,k+1}-\lambda_{1,k+1}\right)\frac{L_k}{1+m_1L_k} \\
&\quad + \lambda_{1,k+1}\left(1-\mu\right). \\
\lambda_{2,k} &= B_k + \beta_1\left(1-\rho\right)\left(\lambda_{2,k+1}-\lambda_{1,k+1}\right)\frac{P_k}{\left(1+m_1L_k\right)^2} \\
&\quad + \beta_2\left(\lambda_{3,k+1}-\lambda_{2,k+1}\right)\frac{S_k}{1+m_3S_k} \\
&\quad + \lambda_{2,k+1}\left(1-\mu\right) \\
&\quad + \left(\lambda_{5,k+1}-\lambda_{2,k+1}\right)\left(c_1u_k+\beta_3\right). \\
\lambda_{3,k} &= A_k + \beta_1\rho\left(\lambda_{3,k+1}-\lambda_{1,k+1}\right)\frac{P_k}{\left(1+m_2S_k\right)^2} \\
&\quad + \beta_2\left(\lambda_{3,k+1}-\lambda_{2,k+1}\right)\frac{L_k}{\left(1+m_3S_k\right)^2} \\
&\quad + \lambda_{3,k+1}\left(1-\mu-\lambda-\gamma\right) + \gamma\lambda_{4,k+1}\left(1-\sigma\right) \\
&\quad + \left(\lambda_{5,k+1}-\lambda_{3,k+1}\right)c_2v_k. \\
\lambda_{4,k} &= C_k + \alpha\lambda_{3,k+1} + \lambda_{4,k+1}\left(1-\mu\right) \\
&\quad + \left(\lambda_{5,k+1}-\lambda_{4,k+1}\right)c_3w_k. \\
\lambda_{5,k} &= \lambda_{5,k+1}\left(1-\mu\right).
\end{aligned} \tag{9}$$

With the transversality conditions at time T, $\lambda_{1,T}=\lambda_{5,T}=0$, $\lambda_{2,T}=B_T$, $\lambda_{3,T}=A_T$, and $\lambda_{4,T}=C_T$.

Furthermore, for $k = 0,1,2\ldots T-1$ and $c_1 = c_2 = c_3 = 1$, the optimal controls u_k^*, v_k^*, and w_k^* are given by

$$\begin{aligned}
u_k^* &= \min\left[b;\max\left(a,\frac{1}{D_k}\left[\left(\lambda_{2,k+1}-\lambda_{5,k+1}\right)L_k\right]\right)\right] \\
v_k^* &= \min\left[d;\max\left(c,\frac{1}{E_k}\left[\left(\lambda_{3,k+1}-\lambda_{5,k+1}\right)S_k\right]\right)\right] \\
w_k^* &= \min\left[f;\max\left(e,\frac{1}{F_k}\left[\left(\lambda_{4,k+1}-\lambda_{5,k+1}\right)Q_k^t\right]\right)\right]
\end{aligned} \tag{10}$$

Proof. The Hamiltonian at time step k is given by

$$\begin{aligned}
H_k &= A_kS_k + B_kL_k + C_kQ_k^t + \frac{D_k}{2}c_1u_k^2 + \frac{E_k}{2}c_2v_k^2 + \frac{F_k}{2} \\
&\quad \cdot c_3w_k^2 + \lambda_{1,k+1}f_{1,k+1} + \lambda_{2,k+1}f_{2,k+1} + \lambda_{3,k+1}f_{3,k+1} \\
&\quad + \lambda_{4,k+1}f_{4,k+1} + \lambda_{5,k+1}f_{5,k+1} = A_kS_k + B_kL_k \\
&\quad + C_kQ_k^t + \frac{D_k}{2}c_1u_k^2 + \frac{E_k}{2}c_2v_k^2 + \frac{F_k}{2}c_3w_k^2 + \lambda_{1,k+1}\Bigg[\Lambda \\
&\quad + \left(1-\mu\right)P_k - \beta_1\left(1-\rho\right)\frac{P_kL_k}{1+m_1L_k} \\
&\quad - \beta_1\rho\frac{P_kS_k}{1+m_2S_k}\Bigg] + \lambda_{2,k+1}\Bigg[\left(1-\mu\right)L_k \\
&\quad + \beta_1\left(1-\rho\right)\frac{P_kL_k}{1+m_1L_k} - \beta_2\frac{L_kS_k}{1+m_3S_k} - c_1u_kL_k \\
&\quad - \beta_3L_k\Bigg] + \lambda_{3,k+1}\Bigg[\left(1-\mu-\lambda-\gamma\right)S_k \\
&\quad + \beta_1\rho\frac{P_kS_k}{1+m_2S_k} + \beta_2\frac{L_kS_k}{1+m_3S_k} + \alpha Q_k^t - c_2v_kS_k\Bigg] \\
&\quad + \lambda_{4,k+1}\left[\left(1-\mu-\alpha\right)Q_k^t + \gamma\left(1-\sigma\right)S_k - w_kQ_k^t\right] \\
&\quad + \lambda_{5,k+1}\left[\left(1-\mu\right)Q_k^p + \gamma\sigma S_k + c_1u_kL_k + c_2v_kS_k\right. \\
&\quad \left. + c_3w_kQ_k^t + \beta_3L_k\right]
\end{aligned} \tag{11}$$

For $k = 0,1\ldots T-1$ the optimal controls $u_k, v_k,, w_k$ can be solved from the optimality condition,

$$\begin{aligned}
\frac{\partial H_k}{\partial u_k} &= 0, \\
\frac{\partial H_k}{\partial v_k} &= 0, \\
\frac{\partial H_k}{\partial w_k} &= 0
\end{aligned} \tag{12}$$

that are

$$\frac{\partial H_k}{\partial u_k} = D_k c_1 u_k + (\lambda_{5,k+1} - \lambda_{2,k+1}) c_1 L_k = 0$$
$$\frac{\partial H_k}{\partial v_k} = E_k c_2 v_k + (\lambda_{5,k+1} - \lambda_{3,k+1}) c_2 S_k = 0 \qquad (13)$$
$$\frac{\partial H_k}{\partial w_k} = F_k c_3 w_k + (\lambda_{5,k+1} - \lambda_{4,k+1}) c_3 Q_k^t = 0$$

So, for $c_1 = c_2 = c_3 = 1$, we have

$$u_k = \frac{1}{D_k}(\lambda_{2,k+1} - \lambda_{5,k+1}) L_k$$
$$v_k = \frac{1}{E_k}(\lambda_{3,k+1} - \lambda_{5,k+1}) S_k \qquad (14)$$
$$w_k = \frac{1}{F_k}(\lambda_{4,k+1} - \lambda_{5,k+1}) Q_k^t$$

However, if $c_i = 0$ for $i = 1, 2, 3$, the control attached to this case will be eliminated and removed.

By the bounds in U_{ad} of the controls, it is easy to obtain u_k^*, v_k^*, and w_k^* in the form of (10). □

4. Simulation

4.1. Algorithm. In this section, we present the results obtained by solving numerically the optimality system. This system consists of the state system, adjoint system, initial and final time conditions, and the controls characterization. So, the optimality system is given by the following.

Step 1. $P_0 = p_0$, $L_0 = l_0$, $S_0 = s_0$, $Q_0^t = q_0^t$, $Q_0^p = q_0^p$, $\lambda_{1,T} = \lambda_{5,T} = 0$, $\lambda_{2,T} = B_T$, $\lambda_{3,T} = A_T$, $\lambda_{4,T} = C_T$, and given $u_{k,0}^*$, $v_{k,0}^*$, and $w_{k,0}^*$

Step 2. For $k = 0; 1; \ldots; T-1$ do:

$$P_{k+1} = \Lambda + (1-\mu) P_k - \beta_1 (1-\rho) \frac{P_k L_k}{1 + m_1 L_k} - \beta_1 \rho \frac{P_k S_k}{1 + m_2 S_k}$$
$$L_{k+1} = (1-\mu) L_k + \beta_1 (1-\rho) \frac{P_k L_k}{1 + m_1 L_k} - \beta_2 \frac{L_k S_k}{1 + m_3 S_k} - \beta_3 L_k - c_1 u_k L_k$$
$$S_{k+1} = (1-\mu-\lambda-\gamma) S_k + \beta_1 \rho \frac{P_k S_k}{1 + m_2 S_k} + \beta_2 \frac{L_k S_k}{1 + m_3 S_k} + \alpha Q_k^t - c_2 v_k S_k$$
$$Q_{k+1}^t = (1-\mu-\alpha) Q_k^t + \gamma (1-\sigma) S_k - c_3 w_k Q_k^t$$
$$Q_{k+1}^p = (1-\mu) Q_k^p + \gamma\sigma S_k + \beta_3 L_k + c_1 u_k L_k + c_2 v_k S_k + c_3 w_k Q_k^t$$
$$\vdots$$
$$\vdots$$
$$\lambda_{1,T-k} = \beta_1 \rho (\lambda_{3,T-k+1} - \lambda_{1,T-k+1}) \frac{S_k}{1 + m_2 S_k} + \beta_1 (1-\rho)(\lambda_{2,T-k+1} - \lambda_{1,T-k+1}) \frac{L_k}{1 + m_1 L_k} + \lambda_{1,n-k+1}(1-\mu)$$
$$\lambda_{2,T-k} = B_k + \beta_1 (1-\rho)(\lambda_{2,T-k+1} - \lambda_{1,T-k+1}) \frac{P_k}{(1 + m_1 L_k)^2} + \beta_2 (\lambda_{3,T-k+1} - \lambda_{2,T-k+1}) \frac{S_k}{1 + m_3 S_k} + \lambda_{2,T-k+1}(1-\mu) + (\lambda_{5,T-k+1} - \lambda_{2,T-k+1})(c_1 u_k + \beta_3)$$
$$\lambda_{3,T-k} = A_k + \beta_1 \rho (\lambda_{3,T-k+1} - \lambda_{1,T-k+1}) \frac{P_k}{(1 + m_2 S_k)^2} + \beta_2 (\lambda_{3,T-k+1} - \lambda_{2,T-k+1}) \frac{L_k}{(1 + m_3 S_k)^2} + \lambda_{3,T-k+1}(1-\mu-\lambda-\gamma) + \gamma \lambda_{4,T-k+1}(1-\sigma) + (\lambda_{5,T-k+1} - \lambda_{3,T-k+1}) c_2 v_k$$
$$\lambda_{4,T-k} = C_k + \alpha \lambda_{3,T-k+1} + \lambda_{4,T-k+1}(1-\mu) + (\lambda_{5,T-k+1} - \lambda_{4,T-k+1}) c_3 w_k$$
$$\lambda_{5,T-k} = \lambda_{5,T-k+1}(1-\mu)$$
$$u_{k+1} = \min\left[b; \max\left(a, \frac{1}{D_k}\left[(\lambda_{2,T-k+1} - \lambda_{5,T-k+1}) L_k\right]\right)\right]$$
$$v_{k+1} = \min\left[d; \max\left(c, \frac{1}{E_k}\left[(\lambda_{3,T-k+1} - \lambda_{5,T-k+1}) S_k\right]\right)\right]$$
$$w_{k+1} = \min\left[f; \max\left(e, \frac{1}{F_k}\left[(\lambda_{4,T-k+1} - \lambda_{5,T-k+1}) Q_k^t\right]\right)\right] \qquad (15)$$

end for

Step 3. For $k = 0; 1; \ldots; T$; write:

$$P_k^* = P_k,$$
$$L_k^* = L_k,$$

TABLE 2: The description of parameters used for the definition of discrete time systems (1). We used just arbitrary academic data.

P_0	L_0	S_0	Q_0^t	Q_0^p	σ	ρ	β_3	α
5.10^3	2.10^3	1.10^3	2.10^3	1.10^3	0.5	0.1	0.01	0.7
Λ	μ	β_1	m_1	m_2	m_3	λ	γ	β_2
8.10^2	0.04	0.6	1	1	1	0.05	0.7	0.05

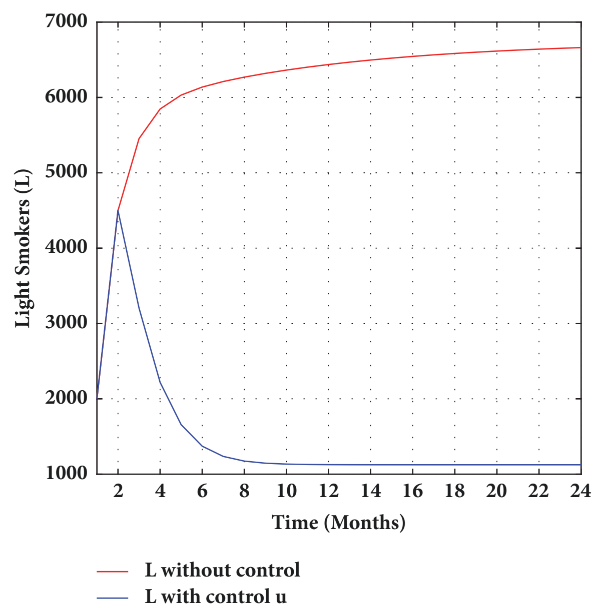


FIGURE 2: The evolution of the light smokers with and without controls.

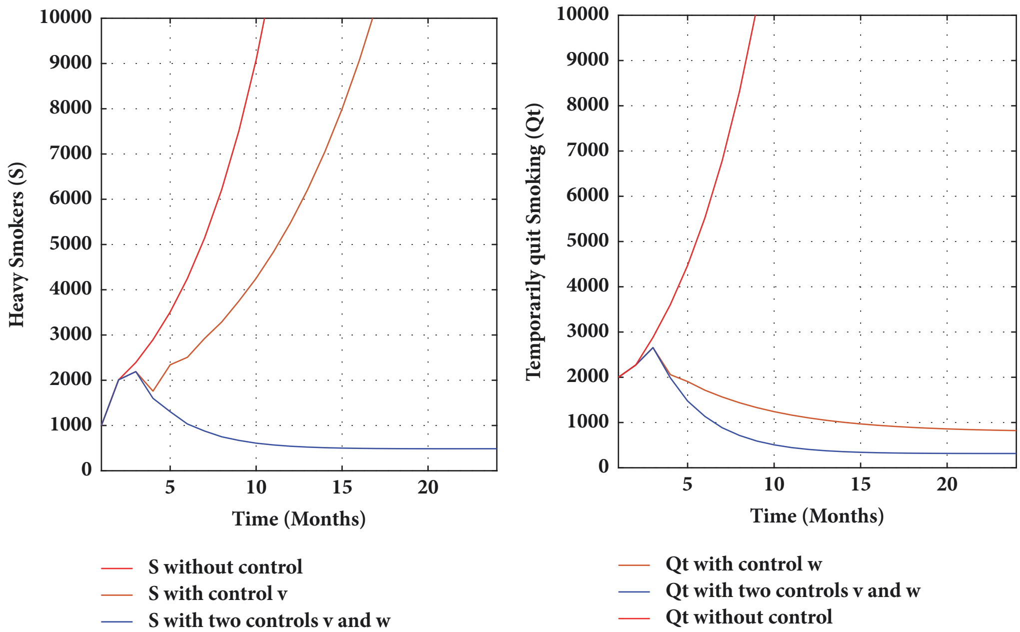
$$
\begin{aligned}
S_k^* &= S_k, \\
Q_k^{t*} &= Q_k^t, \\
Q_k^{p*} &= Q_k^p, \\
u_k^* &= u_k, \\
v_k^* &= v_k, \\
w_k^* &= w_k
\end{aligned}
\tag{16}
$$

end for

In this formulation, there were initial conditions for the state variables and terminal conditions for the adjoints. That is, the optimality system is a two-point boundary value problem with separated boundary conditions at time steps $k = 0$ and $k = T$. We solve the optimality system by an iterative method with forward solving of the state system followed by backward solving of the adjoint system. We start with an initial guess for the controls at the first iteration and then before the next iteration we update the controls by using the characterization. We continue until convergence of successive iterates is achieved.

4.2. Discussion. In this section, we study and analyse numerically the effects of optimal control strategies such as awarness program through media and education, treatment, and psychological support with follow-up for the infected smokers (Table 2).

4.2.1. Strategy A: Control with Awareness Program. Given the importance of the awareness programs in restricting the spreading of smoking, we propose an optimal strategy for this purpose. Hence, we activate the optimal control variable u which represents the awareness program for light smokers. Figure 2 compares the evolution of light



(a) The evolution of the heavy smokers with and without controls

(b) The evolution of the temporary quitters of smoking with and without controls

FIGURE 3

smokers with and without control u in which the effect of the proposed awareness program through media and education is proven to be positive in decreasing the number of light smokers.

4.2.2. Strategy B: Control with Treatment and Psychological Support with Follow-Up. When the number of smokers is so high, it is obligatory to resort to some strategies such as treatment in order to reduce the number of smokers. Therefore, we propose an optimal strategy by using the optimal control v in the beginning. In spite of using the optimal control v, we observe a temporary decrease of the heavy smokers number which is increased again (Figure 3(a)). The reason of this increase is justified by the fact that heavy smokers revert back to smoking after giving up. For improving the effectiveness of this strategy, we add the elements of follow-up and psychocological support which are represented in the proposed strategy by the optimal control variable w (Figure 3(b)). Combining follow-up and psychocological support with treatment results in an obvious decrease in the number of heavy smokers. Also, the proposed strategy has an additional effect in decreasing clearly the number of temporary quitters of smoking.

4.2.3. Strategy C: Control with Awareness Program, Treatment, and Psychological Support with Follow-Up. In this strategy, we combine the two previous strategies to achieve better results. We notice that the numbers of light smokers (Figure 4(a)), heavy smokers (Figure 4(b)), and temporary quitters of smoking (Figure 4(c)) are decreased markedly which leads to satisfactory results.

5. Conclusion

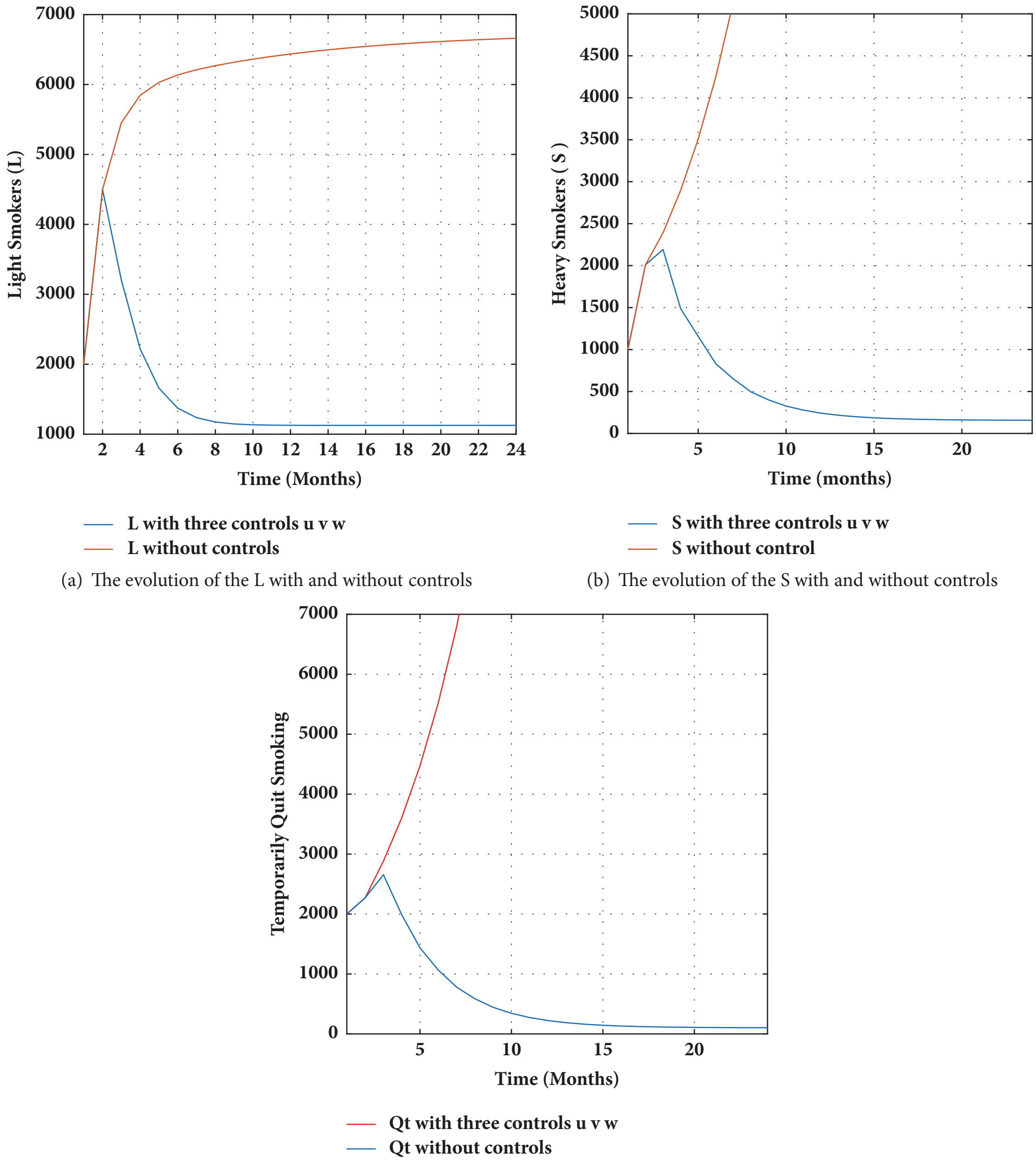
In this paper, we introduced a discrete modeling of smokers in order to minimize the number of light smokers, heavy smokers, and temporary quitters of smoking. We also introduced three controls which, respectively, represent awareness program through education and media, treatment, and psychological support with follow-up. We applied the results of the control theory and we managed to obtain the characterizations of the optimal controls. The numerical simulation of the obtained results showed the effectiveness of the proposed control strategies.

Data Availability

The disciplinary data used to support the findings of this study have been deposited in the Network Repository (http://www.networkrepository.com).

Conflicts of Interest

The authors declare that they have no conflicts of interest.



(a) The evolution of the L with and without controls

(b) The evolution of the S with and without controls

(c) The evolution of the Qt with and without controls

FIGURE 4

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