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Research Article

Existence and Nonexistence of Solutions for Fourth-Order Nonlinear Difference Boundary Value Problems via Variational Methods

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This paper is concerned with boundary value problems for a fourth-order nonlinear difference equation. Via variational methods and critical point theory, sufficient conditions are obtained for the existence of at least two nontrivial solutions, the existence of n distinct pairs of nontrivial solutions, and nonexistence of solutions. Some examples are provided to show the effectiveness of the main results.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ the sets of all natural numbers, integers, and real numbers, respectively. Let the symbol $*$ denote the transpose of a vector. For any integers c and d with $c \leq d$, $[c,d]_{\mathbb{Z}}$ is defined by the discrete interval $\{c, c+1, \dots, d\}$.

Now, we are concerned with the existence and nonexistence solutions to the fourth-order nonlinear difference equation

$$\Delta^2 \left(p(t-2)\Delta^2 x(t-2)\right) + q(t)x(t) = f(t, x(t)), \quad t \in [1,T]_{\mathbb{Z}}, \tag{1}$$

satisfying the boundary value conditions

$$\Delta^k x(-1) = \Delta^k x(T-1), \quad k = 0,1,2,3, \tag{2}$$

where $1 \leq T \in \mathbb{Z}$, Δ is the forward difference operator defined by $\Delta x(t) = x(t+1) - x(t)$, $\Delta^k x(t) = \Delta(\Delta^{k-1}x(t))$ $(2 \leq k \leq 4)$, $\Delta^0 x(t) = x(t)$, $p(t) \in C([-1,T]_{\mathbb{Z}}, \mathbb{R})$ with $p(-1) = p(T-1)$, $p(0) = p(T)$, $q(t) \in C([1,T]_{\mathbb{Z}}, \mathbb{R})$, $f(t,x) \in C([1,T]_{\mathbb{Z}} \times \mathbb{R}, \mathbb{R})$.

As usual, a solution of (1), (2), in other words, a function $x : [-1, T+2]_{\mathbb{Z}} \longrightarrow \mathbb{R}$, satisfies both (1) and (2).

We may think of boundary value problem (BVP) (1), (2) as being a discrete analogue of the following fourth-order nonlinear differential equation:

$$\left[p(s)x''(s)\right]'' - q(s)x(s) = f(t, x(s)), \quad t \in (0,1), \tag{3}$$

with boundary value conditions

$$x^{(k)}(0) = x^{(k)}(1), \quad k = 0,1,2,3. \tag{4}$$

(3) includes the following differential equation:

$$x^{(4)}(s) = f(s, x(s)), \quad s \in \mathbb{R}, \tag{5}$$

which is used to describe the bending of an elastic beam [1]. Equations similar in structure to (3) have been studied by many researchers using a variety of methods; see, for example, [2–12].

It is well-known that the study of nonlinear difference equations [13–29] has long been an important one as a result of the fact that they arise in numerical solutions of both ordinary and partial differential equations as well as in applications to different areas of applied mathematics and physics.

Domshlak and Matakaev [17] in 2001 investigated the oscillation properties of the delay difference equation

$$x(n+1) - x(n) + b(n)x(n-k) = 0, \quad b(n) > 0,\ n \geq 1, \tag{6}$$

for $k = 2$ and $k = 3$ near the 2-periodic critical states with respect to its oscillation properties. By making use of "the Sturmian comparison method: discrete version", they obtained some conditions for the existence and for the nonexistence of eventually positive solution.

Using the critical point theory, Yang [27] studied the following higher order nonlinear difference equation:

$$\begin{aligned} &\sum_{i=0}^{n} r(i)(x(k-i) + x(k+i)) \\ &\quad = f(k, x(k+M), \ldots, x(k), \ldots, x(k-M)), \\ &\qquad n \in \mathbb{N},\ k \in \mathbb{Z}(1,T), \end{aligned} \tag{7}$$

with boundary value conditions

$$\begin{aligned} x(1-m) &= x(2-m) = \cdots = x(0) = 0, \\ x(T+1) &= x(T+2) = \cdots = x(T+m) = 0. \end{aligned} \tag{8}$$

Some sufficient conditions for the existence of the solution to the boundary value problem (7), (8) are obtained.

In 2010, He, Yang and Yang [18] considered the following second-order three-point discrete boundary value problem:

$$\begin{aligned} \Delta^2 x(t-1) + f(t, x(t)) = 0, \quad t \in [1,n]_{\mathbb{Z}}, \\ x(0) = 0, \\ x(n+1) = \alpha x(m). \end{aligned} \tag{9}$$

By using the topological degree theory and the fixed point index theory, they provided sufficient conditions for the existence of sign-changing solutions, positive solutions, and negative solutions.

Investigating the high order difference equation

$$\begin{aligned} &\Delta^n \left(r(t-n)\phi_c\left(\Delta^n x(t-n)\right)\right) \\ &\quad = (-1)^n f(t, x(t+1), x(k), x(t-1)), \quad t \in \mathbb{Z}, \end{aligned} \tag{10}$$

Leng [22] established some new criteria for the existence and multiplicity of periodic and subharmonic solutions of (10) based on the linking theorem in combination with variational technique.

Leszczyński [23] considered the difference equation with both advance and retardation,

$$\begin{aligned} &\Delta^2 \left(\gamma(t-1)\phi_{p(t)}\left(\Delta^2 x(t-2)\right)\right) \\ &\quad = f(t, x(t+1), x(t), x(t-1)), \quad t \in \mathbb{Z}[1,k], \end{aligned} \tag{11}$$

with boundary value conditions

$$\begin{aligned} \Delta x(-1) &= \Delta x(0) = 0, \\ x(k+1) &= x(k+2) = 0. \end{aligned} \tag{12}$$

They applied the direct method of the calculus of variations and the mountain pass technique to prove the existence of at least one and at least two solutions. Nonexistence of nontrivial solutions is also undertaken.

In this paper, we shall study the boundary value problem for a fourth-order nonlinear difference equation (1), (2). Via variational methods and critical point theory, sufficient conditions are obtained for the existence of at least two nontrivial solutions, the existence of n distinct pairs of nontrivial solutions, and nonexistence of solutions. The motivation for the present work stems from the recent papers [3, 5].

Throughout the whole paper, we suppose that there exists a function $G(t,x)$ such that

$$G(t,x) = \int_0^x f(t,v)\,dv, \tag{13}$$

for any $(t,x) \in [1,T]_{\mathbb{Z}} \times \mathbb{R}$.

Let

$$\underline{q} = \min\{q(t) \mid t = 1, 2, \ldots, T\}, \tag{14}$$

and

$$\overline{q} = \max\{q(t) \mid t = 1, 2, \ldots, T\}. \tag{15}$$

The remainder of this article is organized as follows. In Section 2, we shall give some preliminary lemmas and establish the variational structure of BVP (1), (2). In Section 3, we shall give sufficient conditions to the existence and nonexistence solutions. In Section 4, we shall complete the proofs of the main results. Some examples illustrating our main results are given in Section 5.

For the basic knowledge of variational methods, the reader is referred to [30–32].

2. Preliminary Lemmas

Assume that X is a real Banach space and $I \in C^1(X, \mathbb{R})$ is a continuously Fréchet differentiable functional defined on X. As usual, I is said to satisfy the Palais-Smale condition if any sequence $\{x_k\}_{k=1}^{\infty} \subset X$ for which $\{I(x_k)\}_{k=1}^{\infty}$ is bounded and $I'(x_k) \longrightarrow 0$ as $k \longrightarrow \infty$ possesses a convergent subsequence. Here, the sequence $\{x_k\}_{k=1}^{\infty}$ is called a Palais-Smale sequence.

Let X be a real Banach space. We denote by the symbol B_r the open ball in X about 0 of radius r, ∂B_r its boundary, and $\overline{B}_r$ its closure.

In the present article, we define a vector space X by

$$\begin{aligned} X := \big\{x : [-1, T+2]_{\mathbb{Z}} \longrightarrow \mathbb{R} \mid \Delta^k x(-1) \\ = \Delta^k x(T-1),\ k = 0, 1, 2, 3\big\}, \end{aligned} \tag{16}$$

and for any $x \in X$, define

$$(x, y) := \sum_{t=1}^{T} x(t)\, y(t), \quad \forall x, y \in X, \tag{17}$$

and

$$\|x\| := \left(\sum_{t=1}^{T} x^2(t)\right)^{1/2}, \quad \forall x \in X. \tag{18}$$

Remark 1. For any $x \in X$, it is easy to see that

$$\begin{aligned} x(-1) &= x(T-1), \\ x(0) &= x(T), \\ x(1) &= x(T+1), \\ x(2) &= x(T+2). \end{aligned} \tag{19}$$

As the case stands, X is isomorphic to $\mathbb{R}^T$. In the following and in the sequel, when we write $x = (x(1), x(2), \ldots, x(T)) \in \mathbb{R}^T$, we always imply that x can be extended to a vector in X so that (19) is satisfied.

For any $x \in X$, let the functional I be denoted by

$$\begin{aligned} I(x) := {} & \frac{1}{2}\sum_{t=1}^{T} p(t-2)\left(\Delta^2 x(t-2)\right)^2 \\ & + \frac{1}{2}\sum_{t=1}^{T} q(t)(x(t))^2 - \sum_{t=1}^{T} G(t, x(t)). \end{aligned} \tag{20}$$

Then $I \in C^1(X, \mathbb{R})$ and

$$\begin{aligned} \frac{\partial I}{\partial x(t)} = {} & \Delta^2\left(p(t-2)\,\Delta^2 x(t-2)\right) + q(t)\,x(t) \\ & - f(t, x(t)), \quad t \in [1, T]_{\mathbb{Z}}. \end{aligned} \tag{21}$$

Thus, $I'(x) = 0$ if and only if

$$\begin{aligned} \Delta^2\left(p(t-2)\,\Delta^2 x(t-2)\right) + q(t)\,x(t) = f(t, x(t)), \\ t \in [1, T]_{\mathbb{Z}}. \end{aligned} \tag{22}$$

Thereupon a function $x \in X$ is a critical point of the functional I on X if and only if x is a solution of BVP (1), (2).

Let P be the $T \times T$ matrix. If $T \geq 5$, let

$$P = \begin{pmatrix} b(1) & a(1) & p(1) & 0 & \cdots & 0 & p(T-1) & a(T) \\ a(1) & b(2) & a(2) & p(2) & \cdots & 0 & 0 & p(T) \\ p(1) & a(2) & b(3) & a(3) & \cdots & 0 & 0 & 0 \\ 0 & p(2) & a(3) & b(4) & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a(T-3) & p(T-3) & 0 \\ 0 & 0 & 0 & 0 & \cdots & b(T-2) & a(T-2) & p(T-2) \\ p(T-1) & 0 & 0 & 0 & \cdots & a(T-2) & b(T-1) & a(T-1) \\ a(T) & p(T) & 0 & 0 & \cdots & p(T-2) & a(T-1) & b(T) \end{pmatrix}, \tag{23}$$

where $b(k) = p(k) + 4p(k-1) + p(k-2)$, $a(k) = -2(p(k-1) + p(k))$, $k = 1, 2, \ldots, T$.

If $T = 4$, let

$$P = \begin{pmatrix} p(-1) + 4p(0) + p(1) & -2\left(p(0) + p(1)\right) & p(1) + p(3) & -2\left(p(3) + p(4)\right) \\ -2\left(p(0) + p(1)\right) & p(0) + 4p(1) + p(2) & -2\left(p(1) + p(2)\right) & p(2) + p(4) \\ p(1) + p(3) & -2\left(p(1) + p(2)\right) & p(1) + 4p(2) + p(3) & -2\left(p(2) + p(3)\right) \\ -2\left(p(3) + p(4)\right) & p(2) + p(4) & -2\left(p(2) + p(3)\right) & p(2) + 4p(3) + p(4) \end{pmatrix}. \tag{24}$$

If $T = 3$, let

$$P = \begin{pmatrix} p(-1) + 4p(0) + p(1) & p(2) - 2(p(0) + p(1)) & p(1) - 2(p(2) + p(3)) \\ p(2) - 2(p(0) + p(1)) & p(0) + 4p(1) + p(2) & p(3) - 2(p(1) + p(2)) \\ p(1) - 2(p(2) + p(3)) & p(3) - 2(p(1) + p(2)) & p(1) + 4p(2) + p(3) \end{pmatrix}. \quad (25)$$

If $T = 2$, let

$$P = \begin{pmatrix} p(-1) + 4p(0) + 3p(1) & -2p(0) - 4p(1) - 2p(2) \\ -2p(0) - 4p(1) - 2p(2) & p(0) + 4p(1) + 3p(2) \end{pmatrix}. \quad (26)$$

If $T = 1$, let $P = (0)$.

Then the functional $I(x)$ can be rewritten as

$$I(x) = \frac{1}{2}x^* Px + \frac{1}{2}\sum_{t=1}^{T} q(t)(x(t))^2 - \sum_{t=1}^{T} G(t, x(t)). \quad (27)$$

Lemma 2 (linking theorem [30]). *Let X be a real Banach space, $X = X_1 \oplus X_2$, where X_1 is finite dimensional. Assume that $I \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition and the following:*

(I_1) *There are positive constants r and ρ such that $I|_{\partial B_\rho \cap X_2} \geq r$.*

(I_2) *There is a $\nu \in \partial B_1 \cap X_2$ and a positive constant $\tilde{c} \geq \rho$ such that $I|_{\partial\Phi} \leq 0$, where $\Phi = (\overline{B}_{\tilde{c}} \cap X_1) \oplus \{s\nu \mid 0 < s < \tilde{c}\}$.*

Then I possesses a critical value $c \geq r$, where

$$c = \inf_{\varphi \in \Pi} \sup_{x \in \Pi} I(\varphi(x)), \quad (28)$$

and $\Pi = \{\varphi \in C(\overline{\Phi}, X) \mid \varphi|_{\partial\Omega} = id\}$, where id denotes the identity operator.

Lemma 3 (Clark theorem [30]). *Let X be a real Banach space, $I \in C^1(X, \mathbb{R})$, with I being even, bounded from below, and satisfying Palais-Smale condition. Suppose $I(0) = 0$, there is a set $\Omega \subset X$ such that Ω is homeomorphic to S^{T-1} (T-1 dimension unit sphere) by an odd map, and $\sup_\Omega I < 0$. Then I has at least T distinct pairs of nonzero critical points.*

3. Main Results

We now state our main theorems in this paper.

Theorem 4. *Suppose that the function $G(t, x) \geq 0$ and the following conditions are satisfied:*

(p) *For any $t \in [-1, T]_{\mathbb{Z}}$, $p(t) > 0$.*

(q) *For any $t \in [1, T]_{\mathbb{Z}}$, $q(t) \leq 0$, $q(0) = 0$ and $\underline{\lambda} + \underline{q} > 0$.*

(W) *$b(1) + a(1) + p(1) + p(T-1) + a(T) = 0$, $b(2) + a(2) + p(2) + p(T) + a(1) = 0$, $b(k) + a(k) + p(k) + p(k-2) + a(k-1) = 0$, $k = 3, 4, \ldots, T$.*

(G_1) *There are constants $\rho_1 > 0$ and $c_1 \in (0, (1/2)(\underline{\lambda} + \underline{q}))$ such that*

$$G(t, x) \leq c_1 x^2, \quad \forall t \in [1, T]_{\mathbb{Z}}, \ |x| \leq \rho_1. \quad (29)$$

(G_2) *There are constants $c_2 \in ((1/2)(\overline{\lambda} + \overline{q}), +\infty)$ and $c_3 > 0$ such that*

$$G(t, x) \geq c_2 x^2 - c_3, \quad \forall t \in [1, T]_{\mathbb{Z}}, \ |x| \in \mathbb{R}. \quad (30)$$

Here $\underline{\lambda}$ and $\overline{\lambda}$ are constants which can be referred to (32) and (33).

Then BVP (1), (2) has at least three solutions.

Corollary 5. *Suppose that the function $G(t, x) \geq 0$ and the conditions (p), (q), (W), (G_1), and (G_2) are satisfied. Then BVP (1), (2) has at least two nontrivial solutions.*

Theorem 6. *Suppose that the function $G(t, x) \geq 0$, (p), (q), (W) and the following conditions are satisfied:*

(G_3) *For any $(t, x) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}$, $\lim_{|x| \to 0}(G(t, x)/x^2) = 0$.*

(G_4) *For any $(t, x) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}$, there are constants $c_4 > 0$, $\alpha > 2$ and $c_5 > 0$ such that*

$$G(t, x) \geq c_4 |x|^\alpha - c_5. \quad (31)$$

Then BVP (1), (2) has at least three solutions.

Corollary 7. *Suppose that the function $G(t, x) \geq 0$, (p), (q), (W) and the conditions (G_3) and (G_4) are satisfied. Then BVP (1), (2) has at least two nontrivial solutions.*

Theorem 8. *Suppose that the function $G(t, x) \geq 0$, (p), (q), (W), (G_1), (G_2) and the following condition are satisfied:*

(ψ) *$f(t, -x) = -f(t, x)$, $\forall (t, x) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}$.*

Then BVP (1), (2) has at least n distinct pairs of nontrivial solutions, where n is the dimension of X_2 which can be referred to (34).

Theorem 9. *Suppose that the following conditions are satisfied:*

(p') *For any $t \in [-1, T]_{\mathbb{Z}}$, $p(t) \leq 0$.*

(q') *For any $t \in [1, T]_{\mathbb{Z}}$, $q(t) \leq 0$.*

(ϕ) *For any $x \notin 0$, $t \in [1, T]_{\mathbb{Z}}$, $xf(t, x) > 0$.*

Then BVP (1), (2) has no nontrivial solutions.

4. Proofs of the Main Results

In this section, we shall finish proofs of the main results via variational methods.

Proof of Theorem 4. The matrix P satisfies that P is positive semidefinite. In fact, from (W), it is obvious that 0 is an eigenvalue of P with an eigenvector $(1,1,\ldots,1)^*$. Define the eigenvalues of P by $\lambda_1,\lambda_2,\ldots,\lambda_T$.

Denote

$$\underline{\lambda}=\min\left\{\lambda_k \mid \lambda_k\neq 0,\ k=1,2,\ldots,T\right\}, \tag{32}$$

and

$$\overline{\lambda}=\max\left\{\lambda_k \mid \lambda_k\neq 0,\ k=1,2,\ldots,T\right\}. \tag{33}$$

Let $X_1=\{(d,d,\ldots,d)^*\in X \mid d\in\mathbb{R}\}$. Clearly, X_1 is an invariant subspace of X. The space X_2 is defined by

$$X=X_1\oplus X_2. \tag{34}$$

Let $\{x_k\}_{k\in\mathbb{N}}\subset X$ be such that $\{I(x_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(x_k)\longrightarrow 0$ as $k\longrightarrow\infty$. Thus, for any $k\in\mathbb{N}$, there exists a positive constant C such that

$$-C\leq I\left(x_k\right)\leq C. \tag{35}$$

For any $\{x_k\}_{k\in\mathbb{N}}\subset X$, it comes from (27) and (G_2) that

$$\begin{aligned}
-C&\leq I\left(x_k\right)\\
&=\frac{1}{2}\left(x_k\right)^* Px_k+\frac{1}{2}\sum_{t=1}^{T}q\left(t\right)\left(x_k\left(t\right)\right)^2\\
&\quad-\sum_{t=1}^{T}G\left(t,x_k\left(t\right)\right)\\
&\leq\frac{1}{2}\overline{\lambda}\left\|x_k\right\|^2+\frac{1}{2}\overline{q}\left\|x_k\right\|^2-\sum_{t=1}^{T}\left[c_2\left(x_k\left(t\right)\right)^2-c_3\right]\\
&=\left[\frac{1}{2}\left(\overline{\lambda}+\overline{q}\right)-c_2\right]\left\|x_k\right\|^2+c_3T.
\end{aligned} \tag{36}$$

Therefore,

$$\left[c_2-\frac{1}{2}\left(\overline{\lambda}+\overline{q}\right)\right]\left\|x_k\right\|^2\leq C+c_3T. \tag{37}$$

From (G_2), we have $c_2>(1/2)(\overline{\lambda}+\overline{q})$. (37) implies that $\{x_k\}_{k\in\mathbb{N}}$ is a bounded in X. Since the dimension of X is finite, $\{x_k\}_{k\in\mathbb{N}}$ possesses a convergent subsequence. Consequently, the functional $I(x)$ satisfies the Palais-Smale condition. Therefore, it suffices to prove that $I(x)$ satisfies the conditions (I_1) and (I_2) of Lemma 2.

For any $t\in\mathbb{Z}[1,T]$, it follows from (q), (G_1), and (G_2) that $G(t,0)=0$ and $f(t,0)=0$. Hence, $x=0$ is a trivial solution of BVP (1), (2).

From the proof of (36), we have that $I(x)$ is bounded from above in X. Denote

$$c_0=\sup_{x\in X}I\left(x\right). \tag{38}$$

As a consequence, on the one hand, there is a sequence $\{x_i\}$ in X such that

$$c_0=\lim_{i\longrightarrow\infty}I\left(x_i\right). \tag{39}$$

By (27), on the other hand, the functional $I(x)$ satisfies

$$I\left(x\right)\leq\left[\frac{1}{2}\left(\overline{\lambda}+\overline{q}\right)-c_2\right]\|x\|^2+c_3T,\quad\forall x\in X. \tag{40}$$

(40) means that $\lim_{\|x\|\longrightarrow+\infty}I(x)=-\infty$ which implies that $\{x_i\}$ is bounded. As a result, $\{x_i\}$ has a convergent subsequence in X denoted by $\{x_{i_k}\}$. Let

$$x_0=\lim_{k\longrightarrow+\infty}x_{i_k}. \tag{41}$$

By the reason of the continuity of $I(x)$ in x, it is easy to see that $I(x_0)=c_0$. That is, $x_0\in X$ is a critical point of $I(x)$.

For any $x\in X_2$, $\|x\|\leq\rho_1$, via (27) and (G_1), the functional $I(x)$ satisfies

$$\begin{aligned}
I\left(x\right)&=\frac{1}{2}x^*Px+\frac{1}{2}\sum_{t=1}^{T}q\left(t\right)\left(x\left(t\right)\right)^2-\sum_{t=1}^{T}G\left(t,x\left(t\right)\right)\\
&\geq\frac{1}{2}\underline{\lambda}\|x\|^2+\frac{1}{2}\underline{q}\|x\|^2-c_1\sum_{t=1}^{k}\left(x\left(t\right)\right)^2\\
&\geq\left[\frac{1}{2}\left(\underline{\lambda}+\underline{q}\right)-c_1\right]\|x\|^2.
\end{aligned} \tag{42}$$

Take

$$r=\left[\frac{1}{2}\left(\underline{\lambda}+\underline{q}\right)-c_1\right]\rho_1^2. \tag{43}$$

Then

$$I\left(x\right)\geq r,\quad\forall x\in X_2\cap\partial B_{\rho_1}. \tag{44}$$

In other words, there exist two positive constants r and ρ_1 such that $I|_{\partial B_{\rho_1}\cap X_2}\geq r$. Hence, $I(x)$ satisfies (I_1) of Lemma 2.

For any $x\in X_1$, $Px=0$. Combining with (q), we have

$$\begin{aligned}
I\left(x\right)&=\frac{1}{2}x^*Px+\frac{1}{2}\sum_{t=1}^{T}q\left(t\right)\left(x\left(t\right)\right)^2-\sum_{t=1}^{T}G\left(t,x\left(t\right)\right)\\
&\leq-\sum_{t=1}^{T}G\left(t,x\left(t\right)\right)\leq 0.
\end{aligned} \tag{45}$$

Therefore, $x_0\notin X_1$ and the critical point x_0 of $I(x)$ corresponding to the critical value c_0 is a nontrivial solution of BVP (1), (2).

Next, we shall prove (I_2) of Lemma 2.

Take $y \in \partial B_1 \cap X_2$; for any $z \in X_1$ and $\xi \in \mathbb{R}$, let $x = \xi y + z$. By (F_2), we have

$$\begin{aligned} I(x) &= \frac{1}{2}(\xi y + z)^* P(\xi y + z) \\ &\quad + \frac{1}{2}\sum_{t=1}^{T} q(t)(\xi y(t) + z(t))^2 \\ &\quad - \sum_{t=1}^{T} G(t, \xi y(t) + z(t)) \\ &\leq \frac{1}{2}(\xi y)^* P(\xi y) + \frac{1}{2}\sum_{t=1}^{T} q(t)(\xi y(t) + z(t))^2 \\ &\quad - \sum_{n=1}^{k}\left[c_2(\xi y(t) + z(t))^2 - c_3\right] \\ &\leq \frac{1}{2}\overline{\lambda}\xi^2 + \frac{1}{2}\overline{q}\sum_{t=1}^{T}(\xi y(t) + z(t))^2 \\ &\quad - c_2\sum_{t=1}^{T}(\xi y(t) + z(t))^2 + c_3 T \\ &= \left[\frac{1}{2}(\overline{\lambda} + \overline{q}) - c_2\right]\xi^2 + \left(\frac{1}{2}\overline{q} - c_2\right)\|z\|^2 + c_3 T \\ &\leq \left(\frac{1}{2}\overline{q} - c_2\right)\|z\|^2 + c_3 T. \end{aligned} \tag{46}$$

Thereby, there exists a constant $\tilde{c} > \rho_1$ such that

$$I(x) \leq 0, \quad \forall x \in \partial\Phi, \tag{47}$$

where $\Phi = (\overline{B}_{\tilde{c}} \cap X_1) \oplus \{s\nu \mid 0 < s < \tilde{c}\}$. According to Lemma 2, $I(x)$ has a critical value $c \geq r > 0$, where

$$c = \inf_{\varphi\in\Pi}\sup_{x\in\Pi} I(\varphi(x)), \tag{48}$$

and $\Pi = \{\varphi \in C(\overline{\Phi}, X) \mid \varphi|_{\partial\Omega} = id\}$.

Similar to the proof of Theorem 1.1 in [22], we can prove that BVP (1), (2) has at least three solutions. For simplicity, its proof is omitted. □

Remark 10. From the course of the proof of Theorem 4, the conclusion of Corollary 5 is evidently correct.

Remark 11. The techniques of the proof of Theorem 6 are just the same as those carried out in the proof of Theorem 4. We do not repeat them here.

Remark 12. According to Theorem 6, it is easy to see that the conclusion of Corollary 7 is true.

Proof of Theorem 8 . For any $t \in [1, T]_{\mathbb{Z}}$, by the continuity of $f(t, x)$ in x, $I(x)$ can be viewed as a continuously differentiable functional defined on X. It comes from (q) and (G_1) that $I(0) = 0$. Owing to the condition (ψ), $I(x)$ is even. From the process of proof of Theorem 4, $I(x)$ is bounded from below and satisfies the Palais-Smale condition. Next, in the light of Clark Theorem, we shall find a set Ω and an odd map such that Ω is homeomorphic to S^{T-1} by an odd map.

Set

$$\Omega = \partial B_{\rho_1} \cap X_2. \tag{49}$$

It is obvious that Ω is homeomorphic to S^{T-1} by an odd map. (44) implies that $\sup_{\Omega}(-I(x)) < 0$. As a result of Clark Theorem, $I(x)$ has at least T distinct pairs of nonzero critical points. As a consequence, BVP (1), (2) has at least T distinct pairs of nontrivial solutions. The desired result is obtained. □

Proof of Theorem 9 . On the contrary, we suppose that BVP (1), (2) has a nontrivial solution. Therefore, $I(x)$ has a nonzero critical point $\tilde{x}$. Since

$$\begin{aligned} \frac{\partial I}{\partial \tilde{x}(t)} &= \Delta^2\left(p(t)\Delta^2\tilde{x}(t-2)\right) + q(t)\tilde{x}(t) \\ &\quad - f(t, \tilde{x}(t)), \quad t \in [1, T]_{\mathbb{Z}}, \end{aligned} \tag{50}$$

we have

$$\begin{aligned} &\sum_{t=1}^{T} f(t, \tilde{x}(t))\tilde{x}(t) \\ &\quad = \sum_{t=1}^{T}\left[\Delta^2\left(p(t)\Delta^2\tilde{x}(t-2)\right) + q(t)\tilde{x}(t)\right]\tilde{x}(t) \\ &\quad = \sum_{t=1}^{T} p(t)\left(\Delta^2\tilde{x}(t)\right)^2 + \sum_{t=1}^{T} q(t)(\tilde{x}(t))^2. \end{aligned} \tag{51}$$

On the one hand, it follows from (p') and (q') that

$$\sum_{t=1}^{T} f(t, \tilde{x}(t))\tilde{x}(t) \leq 0. \tag{52}$$

On the other hand, by (ϕ), we have

$$\sum_{t=1}^{T} f(t, \tilde{x}(t))\tilde{x}(t) > 0, \tag{53}$$

which is a contradiction with (52). □

5. Some Examples

In this section, we shall provide three examples to illustrate our main results.

Example 1. For $t \in [1, 3]_{\mathbb{Z}}$, assume that

$$\Delta^2\left((t-2)^2\Delta^2 x(t-2)\right) - t^3 x(t) = \frac{1}{5}(x(t))^4, \tag{54}$$

satisfying the boundary value conditions

$$\begin{aligned} x(-1) &= x(2), \\ \Delta x(-1) &= \Delta x(2), \\ \Delta^2 x(-1) &= \Delta^2 x(2), \\ \Delta^3 x(-1) &= \Delta^3 x(2). \end{aligned} \tag{55}$$

We have

$$\begin{aligned} p(t) &= t^2, \\ q(t) &= -t^3, \\ f(t, u(t)) &= \frac{1}{5}(x(t))^4, \\ & \qquad t \in [1,3]_{\mathbb{Z}}, \end{aligned} \tag{56}$$

with

$$\begin{aligned} p(-1) &= 4, \\ p(0) &= 9, \end{aligned} \tag{57}$$

and

$$G(t, u(t)) = (x(t))^5, \quad t \in [1,3]_{\mathbb{Z}}. \tag{58}$$

Besides,

$$P = \begin{pmatrix} 41 & -16 & -25 \\ -16 & 17 & -1 \\ -25 & -1 & 26 \end{pmatrix}, \tag{59}$$

and the eigenvalues of P are $\lambda_1 = 0$, $\lambda_2 = 21$, and $\lambda_3 = 63$. It is easy to verify that all the suppositions of Theorem 4 are satisfied and then BVP (54), (55) has at least three solutions.

Example 2. For $t \in [1,4]_{\mathbb{Z}}$, assume that

$$\Delta^2 \left((t-2)\Delta^2 x(t-2)\right) - t^6 x(t) = (x(t))^3, \tag{60}$$

satisfying the boundary value conditions

$$\begin{aligned} x(-1) &= x(3), \\ \Delta x(-1) &= \Delta x(3), \\ \Delta^2 x(-1) &= \Delta^2 x(3), \\ \Delta^3 x(-1) &= \Delta^3 x(3). \end{aligned} \tag{61}$$

We have

$$\begin{aligned} p(t) &= t, \\ q(t) &= -t^6, \\ f(t, u(t)) &= (x(t))^3, \\ & \qquad t \in [1,4]_{\mathbb{Z}}, \end{aligned} \tag{62}$$

with

$$\begin{aligned} p(-1) &= 3, \\ p(0) &= 4, \end{aligned} \tag{63}$$

and

$$G(t, u(t)) = \frac{1}{4}(x(t))^4, \quad t \in [1,4]_{\mathbb{Z}}. \tag{64}$$

Besides,

$$P = \begin{pmatrix} 20 & -10 & 4 & -14 \\ -10 & 10 & -6 & 6 \\ 4 & -6 & 12 & -10 \\ -14 & 6 & -10 & 18 \end{pmatrix}, \tag{65}$$

and the eigenvalues of P are $\lambda_1 = 0$, $\lambda_2 \approx 6.8154$, $\lambda_3 \approx 11.1773$, and $\lambda_4 \approx 42.0074$. It is easy to verify that all the suppositions of Theorem 8 are satisfied and then BVP (60), (61) has at least 3 distinct pairs of nontrivial solutions.

Example 3. For $t \in [1,5]_{\mathbb{Z}}$, assume that

$$-\Delta^2 \left((t-2)\Delta^2 x(t-2)\right) - t^6 x(t) = (x(t))^5, \tag{66}$$

satisfying the boundary value conditions

$$\begin{aligned} x(-1) &= x(4), \\ \Delta x(-1) &= \Delta x(4), \\ \Delta^2 x(-1) &= \Delta^2 x(4), \\ \Delta^3 x(-1) &= \Delta^3 x(4). \end{aligned} \tag{67}$$

We have

$$\begin{aligned} p(t) &= -t, \\ q(t) &= -t^6, \\ f(t, u(t)) &= (x(t))^5, \\ & \qquad t \in [1,5]_{\mathbb{Z}}, \end{aligned} \tag{68}$$

with

$$\begin{aligned} p(-1) &= -4, \\ p(0) &= -5, \end{aligned} \tag{69}$$

and

$$G(t, u(t)) = \frac{1}{6}(x(t))^6, \quad t \in [1,5]_{\mathbb{Z}}. \tag{70}$$

It is easy to verify that all the suppositions of Theorem 9 are satisfied and then BVP (66), (67) has no nontrivial solutions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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