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Research Article

Global Stability of Traveling Waves for a More General Nonlocal Reaction-Diffusion Equation

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The purpose of this paper is to investigate the global stability of traveling front solutions with noncritical and critical speeds for a more general nonlocal reaction-diffusion equation with or without delay. Our analysis relies on the technical weighted energy method and Fourier transform. Moreover, we can get the rates of convergence and the effect of time-delay on the decay rates of the solutions. Furthermore, according to the stability results, the uniqueness of the traveling front solutions can be proved. Our results generalize and improve the existing results.

1. Introduction

In the study of biology and other subject fields, the reaction-diffusion equations with delays are usually utilized to depict the population distribution and physical evolution process and so forth, for instance, [1–6]. In this paper, we will study a more general reaction-diffusion equation

$$\begin{aligned} u_t(t,x) &= D\Delta u(t,x) \\ &\quad + F\left(u(t,x), \int_{-\infty}^{+\infty} J(x-y) f(u(t-r,y))\,\mathrm{d}y\right), \\ &\qquad t>0,\ x\in\mathbb{R} \end{aligned} \tag{1}$$

with the initial data

$$u(s,x) = u_0(s,x), \quad s\in[-r,0],\ x\in\mathbb{R}, \tag{2}$$

where Δ is the Laplacian operator on $\mathbb{R}$, $D>0$, $r\geq 0$. Some mathematical models in the literature can be depicted by (1) with proper selections of F, J, and f. For instance, setting

$$\begin{aligned} F(x,y) &= -\delta x + \epsilon y, \\ J(y) &= \frac{1}{\sqrt{4\pi\alpha}} e^{-y^2/4\alpha}, \quad \alpha>0, \end{aligned} \tag{3}$$

(1) turns to the following famous nonlocal Nicholson's blowflies population model

$$\begin{aligned} &u_t(t,x) - D\Delta u(t,x) + \delta u(t,x) \\ &\quad = \epsilon \int_{-\infty}^{+\infty} J(y) f(u(t-r,x-y))\,\mathrm{d}y, \\ &\qquad t>0,\ x\in\mathbb{R}, \end{aligned} \tag{4}$$

where $f(x)$ is often formed as

$$\begin{aligned} f_1(x) &= pxe^{-ax^q}, \\ f_2(x) &= \frac{px}{1+ax^q}, \\ &\qquad p>0,\ q>0,\ a>0 \end{aligned} \tag{5}$$

in (4). Particularly, if $q=1$, $f(x)=f_1(x)$, (4) is Nicholson's birth rate function. Furthermore, setting

$$\begin{aligned} D &= 1, \\ r &= 0, \\ F(x,y) &= -x^2 + y, \end{aligned}$$

$$f(x) = x,$$
$$J(y) = \frac{1}{\sqrt{4\pi\alpha}} e^{-y^2/4\alpha}, \quad \alpha \longrightarrow 0^+, \tag{6}$$

(1) turns to the classical Fisher-KPP equation as follows:

$$u_t(t,x) - \Delta u(t,x) = u(1-u), \quad t > 0, \ x \in \mathbb{R}. \tag{7}$$

Now we impose some assumptions on (1) as follows:

(A_1) $J(y)$ is a continuous nonnegative function with $J(-y) = J(y)$, $y \in \mathbb{R}$ and

$$\int_{-\infty}^{+\infty} J(y)\,\mathrm{d}y = 1,$$
$$\int_{-\infty}^{+\infty} J(y) e^{\lambda y}\mathrm{d}y < \infty, \quad \text{for any } \lambda > 0. \tag{8}$$

(A_2) Let $K > 0$, $F \in C^2([0,K] \times [0, f(K)], \mathbb{R})$, $F(0, f(0)) = F(K, f(K)) = 0$, and $F(x, f(x)) > 0$ for any $x \in (0,K)$. In addition, $F(x,y) \le \partial_1 F(0,0)x + \partial_2 F(0,0)y$, $\partial_2 F(x,y) > 0$, $\partial_{ij} F(x,y) \le 0$ $(i,j = 1,2)$ for any $(x,y) \in [0,K] \times [0, f(K)]$, where $\partial_1 F(x,y) = \partial F(x,y)/\partial x$ and $\partial_2 F(x,y) = \partial F(x,y)/\partial y$.

(A_3) $f \in C^2([0,K), \mathbb{R}_+)$, $f(0) = 0$, and $f'(x) \ge 0$, $f''(x) \le 0$, for any $x \in [0,K]$.

(A_4) $\partial_1 F(K, f(K)) + \partial_2 F(K, f(K)) f'(K) < 0$.

From (A_2), it is easy to see that $u_- = 0$, $u_+ = K$ are two constant equilibria $u_\pm$. Throughout this paper, a traveling front solution of (1) connecting $u_\pm$ is a nondecreasing solution with the form $u(t,x) = \phi(\xi)$ $(\xi = x + ct)$; that is, it satisfies the following ordinary differential equation:

$$\begin{aligned} c\phi'(\xi) &= D\phi''(\xi) \\ &\quad + F\left(\phi(\xi), \int_{-\infty}^{+\infty} J(y) f(\phi(\xi - y - cr))\,\mathrm{d}y\right), \end{aligned} \tag{9}$$

where c is the traveling wave speed.

In the past few years, the study on traveling waves of the reaction-diffusion equations has drawn wide attention. One of the important and difficult problems is the stability of traveling waves. For example, the authors in [7, 8] and the references therein proved the stability of traveling waves of reaction-diffusion equations without time-delay. In fact, for the time-delayed reaction-diffusion equations, Schaaf [1] firstly studied the stability of the traveling waves by applying a spectral analysis. Later, there are many great contributions on this issue on both time-delayed and nonlocal reaction-diffusion equations. For instance, in [9] the authors investigated that the traveling front solutions with noncritical speeds were globally asymptotically stable by using the super- and subsolution method. Though the study of the stability of traveling waves in monostable condition is difficult, Mei in [10] firstly showed nonlinear stability of the traveling front solutions of a time-delayed diffusive Nicholson blowflies equation by employing a technical weighted energy method. Then Mei and coauthors in [11–15] further obtained global stability using both the weighted method and the comparison principle. Among them, the authors in [11] developed and improved the wave stability results showed in [10]. By using the above methods, Wu et al. in [16] showed the exponential stability of traveling wavefronts in monostable reaction-advection-diffusion equations with nonlocal delay, which improved some previous works. In a word, there are three commonly used methods for proving the stability of traveling waves, which we mentioned above.

The most challenging problem, however, is the stability of the critical traveling wave solutions to local or nonlocal time-delayed equations. It is also very important because the critical wave speed is the spreading speed. The methods mentioned above can not be used to solve this problem. As a matter of fact, as early as in 1978, by using the maximum principle method, Uchiyama [17] gave the local stability of the traveling waves including the critical waves (no convergence rate). Immediately, Moet [18] proved that the critical waves of the KPP equation were algebraically stable by using the Green function method. Later, Kirchgässner [19] and Gallay [20] showed the stability of the critical waves by using the spectral method and the renormalization group method for parabolic equations, respectively. Recently, for some nonlocal time-delayed reaction-diffusion equations, Mei, Ou and Zhao [21] and Wang [22] proved the globally exponential stability of traveling front solutions with noncritical speeds and globally algebraical stability of traveling front solutions with critical speed by using the weighted energy method and Green's function method. Particularly, Mei and Wang [23] considered a class of nonlocal time-delayed Fisher-KPP type reaction-diffusion equations in n-dimensional space. They obtained the exponential stability of all noncritical planar wavefronts and the algebraic stability of the critical planar wavefronts by using the weighted energy method coupled with Fourier transform. Furthermore, the convergence rates were obtained in the sense with L^1-initial perturbation. Very recently, Chern et al. [24] studied the stability of critical traveling waves for a kind of nonmonotone time-delayed reaction-diffusion equations by using the technical weighted energy method with some new developments.

The main purpose of this paper is to investigate the stability of the traveling front solutions of (1) including the traveling waves with critical speed. First, let us review the works of the existence of the traveling front solutions of (1). In [25], Wang showed the existence of traveling front solutions with speed $c > c^*$ for (1) with nonmonotone nonlinearity by constructing a closed and convex subset in a suitable Banach space and using the fixed point theorem, where c^* is the minimal wave speed. Recently, for the reaction-diffusion equations with nonlocal delays, Tian [26] proved the existence of the traveling waves with $c \ge c^*$ by using the finite time-delay approximation method coupled with the monotone semiflows theorem. Then in this paper, by using the technical weighted energy method and Fourier

transform, we obtained the exponential stability of traveling front solutions with noncritical speeds and the algebraic stability of the traveling front solutions with critical speed of (1). Furthermore, the convergence rates and the effect of time-delay on the decay rates of the solutions were showed. At last, motivated by Lin, Lin, and Mei [27], we show the uniqueness of traveling front solutions for (1).

The rest of the paper is organized as follows. In Section 2, we introduce some preliminaries used later. In Section 3, we will prove the stability of the traveling front solutions for (1) by using the weighted energy method and Fourier transform. According to the stability results, in Section 4 the uniqueness of the traveling front solutions can be proved. In the last section, we apply our results to some models.

2. Preliminaries

In this section, we introduce some notations as follows. We assume that $C > 0$ represents a general constant and $C_i > 0$ denotes a concrete constant. Set I to be an interval, ordinarily $I = \mathbb{R}$. Take

$$\begin{aligned} L^p(I) &= \left\{ f(x) : \int_I f^p(x)\,\mathrm{d}x < \infty,\ x \in I \right\}, \\ W^{k,p}(I) &= \left\{ f(x) \in L^p(I) : \frac{\mathrm{d}^i f}{\mathrm{d}x^i} \in L^p(I),\ i \right. \\ &\quad \left. = 0, 1, \ldots, k,\ x \in I \right\} \quad (k \geq 0,\ p \geq 1), \\ L^p_\omega(I) &= \left\{ f(x) : \int_I \omega(x) f^p(x)\,\mathrm{d}x < \infty,\ x \in I \right\}, \\ &\qquad p \geq 1,\ \omega(x) \geq 0 \end{aligned} \tag{10}$$

with the norm defined by

$$\|f\|_{L^p_\omega} = \left(\int_I \omega(x) f^p(x)\,\mathrm{d}x \right)^{1/p}. \tag{11}$$

Moreover, $W^{k,p}_\omega(I)$ is the Sobolev space with the norm given by

$$\|f\|_{W^{k,p}_\omega(I)} = \left(\sum_{i=0}^{k} \int_I \omega(x) \left| \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x) \right|^p \mathrm{d}x \right)^{1/p}. \tag{12}$$

In addition, we assume that T denotes a positive number and $\mathscr{B}$ represents a Banach space. Also we let $C([0,T];\mathscr{B})$ be the space of the $\mathscr{B}$-valued continuous functions on$[0,T]$. Similarly, we can define the corresponding spaces of $\mathscr{B}$-valued functions on $[0,\infty)$.

Next we present some previous results which will be needed in the proofs of our results later.

Lemma 1 (see [28]). *Set $z(t)$ to be the solution to the following linear time-delayed ODE with time-delay $r > 0$*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} z(t) + c_1 z(t) &= c_2 z(t-r), \\ z(s) &= z_0(s), \quad s \in [-r, 0]. \end{aligned} \tag{13}$$

Thus

$$\begin{aligned} z(t) &= e^{-c_1(t+r)} e_r^{\bar{c}_2 t} z_0(-r) \\ &\quad + \int_{-r}^{0} e^{-c_1(t-s)} e_r^{\bar{c}_2(t-r-s)} \left[z_0'(s) + c_1 z_0(s) \right] \mathrm{d}s, \end{aligned} \tag{14}$$

where

$$\bar{c}_2 := c_2 e^{c_1 r}, \tag{15}$$

and $e_r^{\bar{c}_2 t}$ is the delayed exponential function defined by

$$e_r^{\bar{c}_2 t} = \begin{cases} 0, & -\infty < t < -r, \\ 1, & -r \leq t < 0, \\ 1 + \dfrac{\bar{c}_2 t}{1!}, & 0 \leq t < r, \\ 1 + \dfrac{\bar{c}_2 t}{1!} + \dfrac{\bar{c}_2^2 (t-r)^2}{2!}, & r \leq t < 2r, \\ \vdots & \vdots \\ 1 + \dfrac{\bar{c}_2 t}{1!} + \dfrac{\bar{c}_2^2 (t-r)^2}{2!} + \cdots + \dfrac{\bar{c}_2^m [t-(m-1)r]^m}{m!}, & (m-1)r \leq t < mr, \\ \vdots & \vdots \end{cases} \tag{16}$$

and $e_r^{\bar{c}_2 t}$ is the fundamental solution as

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \bar{c}_2 z(t-r), \qquad z(s) \equiv 1, \quad s \in [-r, 0]. \tag{17}$$

Lemma 2 (see [23]). *Set $c_1 \geq 0$ and $c_2 \geq 0$. Thus the solution $z(t)$ of (13) satisfies*

$$|z(t)| \leq C_0 e^{-c_1 t} e_r^{\bar{c}_2 t}, \tag{18}$$

where

$$C_0 := e^{-c_1 r} |z_0(-r)| + \int_{-r}^{0} e^{c_1 s} \left| z_0'(s) + c_1 z_0(s) \right| \mathrm{d}s, \tag{19}$$

and the fundamental solution $e_r^{\bar{c}_2 t}$ with $\bar{c}_2 t > 0$ of (17) satisfies

$$e_r^{\bar{c}_2 t} \leq C(1+t)^{-\gamma} e^{\bar{c}_2 t}, \quad t > 0 \tag{20}$$

for arbitrary constant $\gamma > 0$.

Moreover, if $c_1 \geq c_2 \geq 0$, there is a number $0 < \varepsilon_1(r) \leq 1$ such that

$$e^{-c_1 t} e_r^{\bar{c}_2 t} \leq C e^{-\varepsilon_1 (c_1 - c_2) t}, \quad t > 0 \tag{21}$$

and the solution of (13) satisfies

$$|z(t)| \leq C e^{-\varepsilon_1 (c_1 - c_2) t}, \quad t > 0, \tag{22}$$

where ε_1 is uniquely determined by

$$(1 - \varepsilon_1) c_1 + \varepsilon_1 c_2 - c_2 e^{\varepsilon_1 (c_1 - c_2) r} = 0. \tag{23}$$

Moreover, clearly, conditions $(A_1) - (A_3)$ guarantees the existence of the traveling front solutions of (1) which was showed in [25, 26, 29]. So we have the following result.

Theorem 3 (existence of traveling waves). *Suppose that $(A_1) - (A_3)$ hold, there is a pair (c^*, λ^*) determined by*

$$\mathscr{P}(\lambda^*, c^*) = 0, \qquad \frac{\partial}{\partial \lambda} \mathscr{P}(\lambda^*, c^*) = 0, \tag{24}$$

where

$$\mathscr{P}(\lambda, c) = D\lambda^2 - c\lambda + \partial_1 F(0,0) + \partial_2 F(0,0) f'(0) \int_{-\infty}^{+\infty} J(y) e^{-\lambda(y+cr)} \mathrm{d}y. \tag{25}$$

For any $c \geq c^*$, (1) has a nondecreasing traveling front solution $\phi(x+ct)$ such that $\phi(-\infty) = 0$ and $\phi(+\infty) = K$, while for any $0 < c < c^*$, (1) has no traveling front solution $\phi(x+ct)$ connecting 0 and K. Moreover, when $c > c^*$, the traveling front solutions with noncritical speeds satisfy $\lim_{\xi \to -\infty} \phi(\xi) e^{-\lambda_1 \xi} = \mu$ with some positive constant μ.

Furthermore, when $c > c^*$, $\mathscr{P}(\lambda, c) = 0$ has two distinct positive real roots $\lambda_1(c)$ and $\lambda_2(c)$ with $0 < \lambda_1(c) < \lambda_2(c)$ such that

$$\mathscr{P}(\lambda, c) < 0 \quad \text{for } \lambda_1 < \lambda < \lambda_2. \tag{26}$$

When $c = c^*$, it holds that

$$\mathscr{P}(\lambda^*, c^*) = 0 \quad \text{for } \lambda_1 = \lambda^* = \lambda_2. \tag{27}$$

Now we define a weight function with a number $\lambda > 0$,

$$\omega(\xi) = \begin{cases} e^{-\lambda(\xi - \xi_0)}, & \xi \leq \xi_0, \\ 1, & \xi > \xi_0 \end{cases} \tag{28}$$

with a sufficiently large number $\xi_0 \gg 1$, where $\lambda \in (\lambda_1(c), \lambda^*)$ when $c > c^*$, but $\lambda = \lambda^*$ when $c = c^*$. Obviously $\omega(\xi) \geq 1$ for all $\xi \in \mathbb{R}$ and $\omega(\xi) \to +\infty$ as $\xi \to -\infty$.

3. The Stability of Traveling Front Solutions

In this section, we will prove the stability of all traveling front solutions with time-delay or not. Firstly, we show the following boundedness and establish the comparison principle for (1). Here we omit the proofs of these results since it is essentially the same as that of [14].

Lemma 4 (boundedness). *Assume $(A_1) - (A_4)$ hold and the initial data satisfy*

$$0 = u_- \leq u_0(s,x) \leq u_+ = K, \quad \text{for } (s,x) \in [-r, 0] \times \mathbb{R}, \tag{29}$$

then the solution $u(t,x)$ of the Cauchy problem (1) and (2) exists uniquely and satisfies

$$0 = u_- \leq u(t,x) \leq u_+ = K, \quad \text{for } (t,x) \in [0, +\infty) \times \mathbb{R}. \tag{30}$$

Lemma 5 (comparison principle). *Set $\overline{u}(t,x)$ and $\underline{u}(t,x)$ to be the solution of (1) and (2) with the initial data $\overline{u}_0(s,x)$ and $\underline{u}_0(s,x)$, respectively. If*

$$0 = u_- \leq \underline{u}_0(s,x) \leq \overline{u}_0(s,x) \leq u_+ = K, \quad \text{for } (s,x) \in [-r, 0] \times \mathbb{R}; \tag{31}$$

thus

$$0 = u_- \leq \underline{u}(t,x) \leq \overline{u}(t,x) \leq u_+ = K, \quad \text{for } (t,x) \in [0, +\infty) \times \mathbb{R}. \tag{32}$$

For the given $u_0(s,x)$, if

$$0 = u_- \leq u_0(s,x) \leq u_+ = K, \quad \text{for } (s,x) \in [-r, 0] \times \mathbb{R}, \tag{33}$$

set

$$\begin{aligned} U_0^+(s,x) &= \max\{u_0(s,x), \phi(x+cs)\}, \quad \text{for } (s,x) \in [-r, 0] \times \mathbb{R}, \\ U_0^-(s,x) &= \min\{u_0(s,x), \phi(x+cs)\}, \quad \text{for } (s,x) \in [-r, 0] \times \mathbb{R}; \end{aligned} \tag{34}$$

thus

$$u_- \le U_0^-(s,x) \le u_0(s,x) \le U_0^+(s,x) \le u_+, \quad \text{for } (s,x) \in [-r,0]\times\mathbb{R}, \tag{35}$$

$$u_- \le U_0^-(s,x) \le \phi(x+cs) \le U_0^+(s,x) \le u_+, \quad \text{for } (s,x) \in [-r,0]\times\mathbb{R}. \tag{36}$$

Let $U^+(t,x)$ and $U^-(t,x)$ be the corresponding solutions of (1) and (2) with initial data $U_0^+(s,x)$ and $U_0^-(s,x)$ defined in (34) respectively; that is,

$$\begin{aligned} &U_t^\pm(t,x) = D\Delta U^\pm(t,x) + F\Big(U^\pm(t,x), \\ &\quad \int_{-\infty}^{+\infty} J(y) f(U^\pm(t-r,x-y))\,dy\Big), \quad t>0,\ x\in\mathbb{R}, \\ &U^\pm(s,x) = U_0^\pm(s,x), \quad \text{for } (s,x)\in[-r,0]\times\mathbb{R}. \end{aligned} \tag{37}$$

By Lemma 5, we can get

$$u_- \le U^-(t,x) \le u(t,x) \le U^+(t,x) \le u_+, \quad \text{for } (t,x)\in(0,+\infty)\times\mathbb{R}, \tag{38}$$

$$u_- \le U^-(t,x) \le \phi(x+ct) \le U^+(t,x) \le u_+, \quad \text{for } (t,x)\in(0,+\infty)\times\mathbb{R}. \tag{39}$$

Now we need the following three steps.

Firstly, we will prove the convergence of $U^+(t,x)$ to $\phi(x+ct)$.

For any $c \ge c^*$, set $\xi = x+ct$ and

$$\begin{aligned} v(t,\xi) &= U^+(t,x) - \phi(x+ct), \\ v_0(s,\xi) &= U_0^+(s,x) - \phi(x+cs). \end{aligned} \tag{40}$$

From (36) and (39) we get

$$\begin{aligned} v(t,\xi) &\ge 0, \\ v_0(s,\xi) &\ge 0. \end{aligned} \tag{41}$$

Then it can be verified that $v(t,\xi)$ defined in (40) satisfies

$$\begin{aligned} v_t + cv_\xi - Dv_{\xi\xi} - \partial_1 F(\xi) v &= \partial_2 F(\xi) \int_{-\infty}^{+\infty} J(y) \\ &\quad \cdot f'(\phi(\xi-y-cr)) v(t-r,\xi-y-cr)\,dy + Q \end{aligned} \tag{42}$$

with the initial data

$$v(s,\xi) = v_0(s,\xi), \quad (s,\xi)\in[-r,0]\times\mathbb{R}, \tag{43}$$

where

$$\begin{aligned} \widehat{\phi}(\xi) &= \int_{-\infty}^{+\infty} J(y) f(\phi(\xi-y-cr))\,dy, \\ F(\xi) &= F\big(\phi(\xi), \widehat{\phi}(\xi)\big) \end{aligned} \tag{44}$$

and

$$\begin{aligned} Q &= F\Big(v(t,\xi)+\phi(\xi), \int_{-\infty}^{+\infty} J(y) \\ &\quad \cdot f(v(t-r,\xi-y-cr)+\phi(\xi-y-cr))\,dy\Big) \\ &\quad - F(\xi) - \partial_1 F(\xi) v(t,\xi) - \partial_2 F(\xi) \int_{-\infty}^{+\infty} J(y) \\ &\quad \cdot f'(\phi(\xi-y-cr)) v(t-r,\xi-y-cr)\,dy. \end{aligned} \tag{45}$$

By using the mollification, Zorns lemma, and energy method, we obtain the global existence for the solution of (42)-(43).

For the nonlinearity Q, applying Taylor's formula to (45) and noting $(A_2)-(A_3)$, we have

$$\begin{aligned} Q &\le \frac{1}{2}\partial_{11}F(\overline{\phi},\widetilde{\phi}) v^2 + \partial_{12}F(\overline{\phi},\widetilde{\phi}) \\ &\quad \cdot \Big[\int_{-\infty}^{+\infty} J(y)[f(v+\phi)-f(\phi)]\,dy\Big] v + \frac{1}{2} \\ &\quad \cdot \partial_{22}F(\overline{\phi},\widetilde{\phi}) \Big[\int_{-\infty}^{+\infty} J(y)[f(v+\phi)-f(\phi)]\,dy\Big]^2 \\ &\le 0, \end{aligned} \tag{46}$$

where $\overline{\phi}$ is some function between ϕ and $\phi+v$; $\widetilde{\phi}$ is some function between $\widehat{\phi}$ and $\int_{-\infty}^{+\infty} J(y) f(v+\phi)\,dy$. Then (42) becomes

$$\begin{aligned} v_t + cv_\xi - Dv_{\xi\xi} - \partial_1 F(\xi) v - \partial_2 F(\xi) \int_{-\infty}^{+\infty} J(y) \\ \cdot f'(\phi(\xi-y-cr)) v(t-r,\xi-y-cr)\,dy \le 0. \end{aligned} \tag{47}$$

Set $\widetilde{v}(t,\xi)$ to be the solution of the following equation with the original data $v_0(s,\xi)$:

$$\begin{aligned} &\widetilde{v}_t + c\widetilde{v}_\xi - D\widetilde{v}_{\xi\xi} - \partial_1 F(0,0)\widetilde{v} - \partial_2 F(0,0) f'(0) \\ &\quad \cdot \int_{-\infty}^{+\infty} J(y)\widetilde{v}(t-r,\xi-y-cr)\,dy = 0, \\ &\qquad t>0,\ \xi\in\mathbb{R}, \\ &\widetilde{v}(s,\xi) = v_0(s,\xi) = \widetilde{v}_0(s,\xi), \quad (s,\xi)\in[-r,0]\times\mathbb{R}. \end{aligned} \tag{48}$$

By the assumptions $\partial_{ij}F(x,y) \le 0$, $f'(x) \le f'(0)$ and the comparison principle, we see that

$$0 \le v(t,\xi) \le \widetilde{v}(t,\xi), \quad t>0,\ \xi\in\mathbb{R}. \tag{49}$$

Take

$$\overline{v}(t,\xi) = e^{-\lambda(\xi-\xi_0)}\widetilde{v}(t,\xi), \quad \lambda\in(\lambda_1(c),\lambda^*], \tag{50}$$

then $\overline{v}(t,\xi)$ satisfies the following equation:

$$\begin{aligned}&\overline{v}_t + l_1\overline{v}_\xi - D\overline{v}_{\xi\xi} + l_2\overline{v} = \partial_2 F(0,0) f'(0)\\&\quad\cdot\int_{-\infty}^{+\infty} J(y) e^{-\lambda(y+cr)}\overline{v}(t-r,\xi-y-cr)\,\mathrm{d}y,\\&\qquad t>0,\ \xi\in\mathbb{R},\\&\overline{v}(s,\xi) = e^{-\lambda(\xi-\xi_0)}\widetilde{v}_0(s,\xi) = \overline{v}_0(s,\xi),\\&\qquad (s,\xi)\in[-r,0]\times\mathbb{R},\end{aligned}\tag{51}$$

where

$$\begin{aligned}l_1 &= c - 2D\lambda,\\ l_2 &= c\lambda - D\lambda^2 - \partial_1 F(0,0) > 0.\end{aligned}\tag{52}$$

When $r>0$, by taking Fourier transform

$$\begin{aligned}\widehat{v}(t,\eta) = \mathscr{F}[\overline{v}] &:= \int_{-\infty}^{+\infty}\overline{v}(t,\xi)e^{-\mathrm{i}\xi\eta}\mathrm{d}\xi,\\ \widehat{v}_0(t,\eta) &= \mathscr{F}[\overline{v}_0],\ \mathrm{i}=\sqrt{-1}\end{aligned}\tag{53}$$

to (51), we have

$$\begin{aligned}&\frac{\mathrm{d}\widehat{v}}{\mathrm{d}t} + A(\eta)\widehat{v} = B(\eta)\widehat{v}(t-r,\eta),\\&\widehat{v}(s,\eta) = \widehat{v}_0(s,\eta),\\&\qquad s\in[-r,0],\end{aligned}\tag{54}$$

where

$$A(\eta) = D\eta^2 + l_2 + \mathrm{i}l_1\eta,\tag{55}$$

$$\begin{aligned}&B(\eta)\\&\quad = \partial_2 F(0,0) f'(0)\int_{-\infty}^{+\infty} J(y) e^{-\lambda(y+cr)}e^{-\mathrm{i}\eta(y+cr)}\mathrm{d}y.\end{aligned}\tag{56}$$

According to Lemma 1, we get the solution of (54) as

$$\begin{aligned}\widehat{v}(t,\eta) &= e^{-A(\eta)(t+r)}e_r^{\mathscr{B}(\eta)t}\widehat{v}_0(-r,\eta)\\&\quad+\int_{-r}^{0} e^{-A(\eta)(t-s)}e_r^{\mathscr{B}(\eta)(t-r-s)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\widehat{v}_0(s,\eta)\right.\\&\quad\left.+A(\eta)\widehat{v}_0(s,\eta)\right)\mathrm{d}s,\end{aligned}\tag{57}$$

where

$$\mathscr{B}(\eta) := B(\eta)e^{A(\eta)r}.\tag{58}$$

Next by taking the inverse Fourier transform to (57), we get

$$\begin{aligned}\overline{v}(t,\xi) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\mathrm{i}\xi\eta}e^{-A(\eta)(t+r)}e_r^{\mathscr{B}(\eta)t}\widehat{v}_0(-r,\eta)\,\mathrm{d}\eta\\&\quad+\int_{-r}^{0}\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\mathrm{i}\xi\eta}e^{-A(\eta)(t-s)}e_r^{\mathscr{B}(\eta)(t-r-s)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right.\\&\quad\left.\cdot\widehat{v}_0(s,\eta)+A(\eta)\widehat{v}_0(s,\eta)\right)\mathrm{d}\eta\,\mathrm{d}s,\end{aligned}\tag{59}$$

where the inverse Fourier transform is given by

$$\begin{aligned}\overline{v}(t,\xi) = \mathscr{F}^{-1}[\widehat{v}] &:= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\mathrm{i}\xi\eta}\widehat{v}(t,\eta)\,\mathrm{d}\eta,\\&\qquad \mathrm{i}=\sqrt{-1}.\end{aligned}\tag{60}$$

Now we will prove the asymptotic behavior of $\overline{v}(t,\xi)$.

Lemma 6 (decay rates for $r>0$). *Suppose $v_0\in C([-r,0];W_\omega^{2,1}(\mathbb{R}))$ and $v_{0s}\in C([-r,0];L_\omega^1(\mathbb{R}))$. Thus, when $k_1\ge k_2$, there is a number $0<\varepsilon\le 1$ determined in Lemma 2 such that the solution of (51) satisfies*

$$\|\overline{v}(t)\|_{L^\infty(\mathbb{R})} \le Ct^{-1/2}e^{-\varepsilon(k_1-k_2)t},\quad t>0.\tag{61}$$

Proof. Set

$$I_1(t,\eta) := e^{-A(\eta)(t+r)}e_r^{\mathscr{B}(\eta)t}\widehat{v}_0(-r,\eta),\tag{62}$$

$$\begin{aligned}I_2(t-s,\eta) &:= e^{-A(\eta)(t-s)}e_r^{\mathscr{B}(\eta)(t-r-s)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\widehat{v}_0(s,\eta)\right.\\&\quad\left.+A(\eta)\widehat{v}_0(s,\eta)\right)\end{aligned}\tag{63}$$

and

$$k_0 := D\eta^2 + l_2 > 0.\tag{64}$$

It follows from (55) and (56) that

$$\left|e^{-A(\eta)(t+r)}\right| = e^{-(D\eta^2+l_2)(t+r)} = e^{-k_0(t+r)},\tag{65}$$

$$\begin{aligned}&|B(\eta)| \le \partial_2 F(0,0) f'(0)\int_{-\infty}^{+\infty} J(y)e^{-\lambda(y+cr)}\mathrm{d}y\\&\quad =: k_2,\\&|\mathscr{B}(\eta)| := \left|B(\eta)e^{A(\eta)r}\right| \le k_2 e^{(D\eta^2+l_2)r} = k_2 e^{k_0 r}\\&\quad =: \overline{k}_2\end{aligned}\tag{66}$$

and

$$\left|e_r^{\mathscr{B}(\eta)t}\right| \le e_r^{\overline{k}_2 t}.\tag{67}$$

Because of $\widehat{v}(t,\eta) = \mathscr{F}[\overline{v}]$, we can get

$$\begin{aligned}\sup_{\eta\in\mathbb{R}}\left|\widehat{v}_0(-r,\eta)\right| &\le \int_{-\infty}^{\infty}\left|\overline{v}_0(-r,\xi)\right|\mathrm{d}\xi\\&= \|v_0(-r)\|_{L_\omega^1(\mathbb{R})}.\end{aligned}\tag{68}$$

By (62) and combining with (65)-(68), we have

$$\left|I_1(t,\eta)\right| \le \left|e^{-k_0(t+r)}e_r^{\overline{k}_2 t}\widehat{v}_0(-r,\eta)\right|.\tag{69}$$

Let $k_1 = l_2$. Noticing $k_1\ge k_2$ for $c\ge c^*$ and $\overline{k}_2 = k_2 e^{k_0 r}$, from Lemma 2, it holds that

$$\begin{aligned}e^{-k_0 t}e_r^{\overline{k}_2 t} &\le Ce^{-\varepsilon(k_0-k_2)t} = Ce^{-\varepsilon(D\eta^2+l_2-k_2)t}\\&= Ce^{-\varepsilon Dt\eta^2}e^{-\varepsilon(k_1-k_2)t},\end{aligned}\tag{70}$$

where $0 < \varepsilon < 1$. Then,

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\xi\eta} I_1(t,\eta)\,\mathrm{d}\eta\right| &\le C\left\|v_0(-r)\right\|_{L^1_\omega(\mathbb{R})}\int_{-\infty}^{+\infty} e^{-\varepsilon Dt\eta^2} e^{-\varepsilon(k_1-k_2)t}\mathrm{d}\eta \\ &\le C\left\|v_0(-r)\right\|_{L^1_\omega(\mathbb{R})} t^{-1/2} e^{-\varepsilon(k_1-k_2)t}.\end{aligned} \tag{71}$$

On the other hand, by applying the characters of Fourier transform, there holds

$$\begin{aligned}\left|\mathrm{i}\eta\widehat{v}_0(s,\eta)\right| &= \left|\int_{-\infty}^{\infty} \mathrm{i}\eta\overline{v}_0(s,\xi)\, e^{-\mathrm{i}\xi\eta}\mathrm{d}\xi\right| = \left|\widehat{\overline{v}_{0\xi}}(s,\xi)\right| \\ &\le \int_{-\infty}^{\infty}\overline{v}_{0\xi}(s,\xi)\,\mathrm{d}\xi = \left\|v_{0\xi}(s)\right\|_{L^1_\omega(\mathbb{R})},\end{aligned} \tag{72}$$

and

$$\begin{aligned}\sup_{\eta\in\mathbb{R}}\left|A(\eta)\widehat{v}_0(s,\eta)\right| &= \sup_{\eta\in\mathbb{R}}\left|\left(D\eta^2 + l_2 + il_1\eta\right)\widehat{v}_0(s,\eta)\right| \\ &\le C\left\|v_0(s)\right\|_{W^{2,1}_\omega(\mathbb{R})}, \quad s\in[-r,0].\end{aligned} \tag{73}$$

Then, by a direct calculation, we obtain

$$\begin{aligned}\left|I_2(t-s,\eta)\right| &= \left|e^{-A(\eta)(t-s)} e_r^{\mathscr{B}(\eta)(t-r-s)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\widehat{v}_0(s,\eta) + A(\eta)\widehat{v}_0(s,\eta)\right)\right| \le e^{-(D\eta^2+l_2)(t-s)} e_r^{\overline{k}_2(t-r-s)}\sup_{\eta\in\mathbb{R}}\left|\frac{\mathrm{d}}{\mathrm{d}s}\right. \\ &\quad \left.\cdot\,\widehat{v}_0(s,\eta) + A(\eta)\widehat{v}_0(s,\eta)\right| \\ &\le Ce^{-\varepsilon D\eta^2(t-s)} e^{-\varepsilon(k_1-k_2)(t-s)}\left(\left\|v_{0s}(s)\right\|_{L^1_\omega(\mathbb{R})} + \left\|v_0(s)\right\|_{W^{2,1}_\omega(\mathbb{R})}\right)\end{aligned} \tag{74}$$

and

$$\begin{aligned}&\left|\int_{-r}^{0}\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\mathrm{i}\xi\eta} e^{-A(\eta)(t-s)} e_r^{\mathscr{B}(\eta)(t-r-s)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\widehat{v}_0(s,\eta) + A(\eta)\widehat{v}_0(s,\eta)\right)\mathrm{d}\eta\,\mathrm{d}s\right| \le C\int_{-r}^{0} e^{-\varepsilon(k_1-k_2)(t-s)}\left(\left\|v_{0s}(s)\right\|_{L^1_\omega(\mathbb{R})}\right. \\ &\quad \left. + \left\|v_0(s)\right\|_{W^{2,1}_\omega(\mathbb{R})}\right)\int_{-\infty}^{+\infty} e^{-\varepsilon D\eta^2(t-s)}\mathrm{d}\eta\,\mathrm{d}s \le Ct^{-1/2}e^{-\varepsilon(k_1-k_2)t}.\end{aligned} \tag{75}$$

From (71) and (75), we can get the result. □

Then, for traveling front solutions with noncritical speeds, from Lemma 6 we get the following exponential decay:

$$\left\|\overline{v}(t)\right\|_{L^\infty(\mathbb{R})} \le Ct^{-1/2}e^{-\epsilon(k_1-k_2)t}. \tag{76}$$

But if $c = c^*$, from (25) and (27), we see that

$$\begin{aligned}k_1 &= c^*\lambda^* - D(\lambda^*)^2 - \partial_1 F(0,0) \\ &= \partial_2 F(0,0)\, f'(0)\int_{-\infty}^{+\infty} J(y)\, e^{-\lambda^*(y+c^*r)}\mathrm{d}y = k_2.\end{aligned} \tag{77}$$

Then, for traveling front solutions with critical speeds, from Lemma 6 we get the following algebraic decay:

$$\left\|\overline{v}(t)\right\|_{L^\infty(\mathbb{R})} \le Ct^{-1/2}. \tag{78}$$

Since $v(t,\xi) \le \widetilde{v}(t,\xi) = \overline{v}(t,\xi)e^{\lambda(\xi-\xi_0)} \le \overline{v}(t,\xi)$ for $\xi \le \xi_0$, we can directly get the following result for $v(t,\xi)$.

Lemma 7. *There holds that*

$$\left\|v(t)\right\|_{L^\infty(I_1)} \le Ct^{-1/2}e^{-\epsilon(k_1-k_2)t}, \quad \textit{for } c > c^*,\ t > 0, \tag{79}$$

and

$$\left\|v(t)\right\|_{L^\infty(I_1)} \le Ct^{-1/2}, \quad \textit{for } c = c^*,\ t > 0, \tag{80}$$

where $I_1 = (-\infty, \xi_0]$.

Moreover, we need to prove the decay rate of $v(t,\xi)$ for $\xi \ge \xi_0$.

Lemma 8. *There holds that*

$$\left\|v(t)\right\|_{L^\infty(I_2)} \le Ct^{-1/2}e^{-\nu t}, \quad \textit{for } c > c^*,\ t > 0, \tag{81}$$

with some constant ν *which satisfies*

$$0 < \nu < \min\{\nu_1, \nu_2\} \tag{82}$$

and

$$\left\|v(t)\right\|_{L^\infty(I_2)} \le Ct^{-1/2}, \quad \textit{for } c = c^*,\ t > 0, \tag{83}$$

where $I_2 = [\xi_0, +\infty)$, $\nu_1 = \varepsilon(k_1 - k_2)$, *and* $\nu_2 > 0$ *is the root of the equation* $-\partial_1 F(K, f(K)) - \nu - \partial_2 F(K, f(K)) f'(K)e^{\nu r} = 0$.

Proof. Now we consider the following equations, for $c > c^*$:

$$\begin{aligned}&v_t + cv_\xi - Dv_{\xi\xi} - \partial_1 F(\xi)\, v \le \partial_2 F(\xi)\int_{-\infty}^{+\infty} J(y) \\ &\quad \cdot f'(\phi(\xi - y - cr))\, v(t-r, \xi - y - cr)\,\mathrm{d}y, \\ &\qquad\qquad \text{for } t > 0,\ \xi\in I_2, \\ &v|_{\xi=\xi_0} \le C_1(1+t)^{-1/2}e^{-\epsilon(k_1-k_2)t}, \quad \text{for } t > 0, \\ &v(s,\xi) = v_0(s,\xi), \quad \text{for } s\in[-r,0],\ \xi\in I_2,\end{aligned} \tag{84}$$

and for $c = c^*$,

$$\begin{aligned} &v_t + cv_\xi - Dv_{\xi\xi} - \partial_1 F(\xi) v \le \partial_2 F(\xi) \int_{-\infty}^{+\infty} J(y) \\ &\quad \cdot f'(\phi(\xi - y - cr)) v(t - r, \xi - y - cr) \, dy, \\ &\qquad \text{for } t > 0,\ \xi \in I_2, \\ &v|_{\xi=\xi_0} \le C_2 (1+t)^{-1/2}, \quad \text{for } t > 0, \\ &v(s,\xi) = v_0(s,\xi), \quad \text{for } s \in [-r, 0],\ \xi \in I_2, \end{aligned} \tag{85}$$

with two constants $C_1 > 0$ and $C_2 > 0$.

When $c > c^*$, take

$$V(t,\xi) = C_3 (1 + t + r)^{-1/2} e^{-\nu t}, \quad t > 0, \tag{86}$$

where $C_3 > v_0(s,\xi) \ge 0$ is a large number. Now we can choose $t_0 \gg 1$ and $0 < \nu < \min\{\nu_1, \nu_2\}$ such that for $t > t_0, \xi \ge \xi_0 \gg 1$,

$$\begin{aligned} &-\frac{1}{2(1+t+r)} - \nu - \partial_1 F(\xi) - \partial_2 F(\xi) \\ &\quad \cdot e^{\nu r} \left(\frac{1+t+r}{1+t}\right)^{1/2} \int_{-\infty}^{+\infty} J(y) \\ &\quad \cdot f'(\phi(\xi - y - cr)) \, dy = -\left[\partial_1 F(K, f(K)) \right. \\ &\quad \left. + \partial_2 F(K, f(K)) f'(K)\right] - \nu + \left[\partial_1 F(K, f(K)) \right. \\ &\quad \left. - \partial_1 F(\xi)\right] - \int_{-\infty}^{+\infty} J(y) \left[\partial_2 F(\xi) f'(\phi(\xi - y - cr)) \right. \\ &\quad \left. - \partial_2 F(K, f(K)) f'(K)\right] dy - \frac{1}{2(1+t+r)} \\ &\quad - \partial_2 F(\xi) \left[e^{\nu r} \left(\frac{1+t+r}{1+t}\right)^{1/2} - 1\right] \int_{-\infty}^{+\infty} J(y) \\ &\quad \cdot f'(\phi(\xi - y - cr)) \, dy \ge 0. \end{aligned} \tag{87}$$

Thus, we can verify that

$$\begin{aligned} &V_t + cV_\xi - DV_{\xi\xi} - \partial_1 F(\xi) V \ge \partial_2 F(\xi) \int_{-\infty}^{+\infty} J(y) \\ &\quad \cdot f'(\phi(\xi - y - cr)) V(t - r, \xi - y - cr) \, dy, \\ &\qquad \text{for } t > t_0,\ \xi \in I_2, \\ &V|_{\xi=\xi_0} \ge C_1 (1+t)^{-1/2} e^{-\epsilon(k_1 - k_2)t}, \quad \text{for } t > t_0, \\ &V|_{t=s} \ge v_0(s,\xi), \quad \text{for } s \in [-r, 0],\ \xi \in I_2. \end{aligned} \tag{88}$$

On the other hand, for the interval $[0, t_0]$, by choosing $C_3 > 0$ large enough, we can obtain $v(t,\xi) \le V(t,\xi)$; that is, $V(t,\xi)$ is an upper solution of (84). Therefore, for $c > c^*$, we obtain

$$\begin{aligned} &0 \le v(t,\xi) \le V(t,\xi) = C_3 (1+t+r)^{-1/2} e^{-\nu t}, \\ &\qquad t > 0,\ \xi \in I_2. \end{aligned} \tag{89}$$

When $c = c^*$, set

$$W(t,\xi) = C_4 (1+t+r)^{-1/2}, \quad t > 0, \tag{90}$$

where $C_4 > v_0(s,\xi) \ge 0$ is a large number. Similarly, we can obtain

$$\begin{aligned} &W_t + cW_\xi - DW_{\xi\xi} - \partial_1 F(\xi) W \ge \partial_2 F(\xi) \int_{-\infty}^{+\infty} J(y) \\ &\quad \cdot f'(\phi(\xi - y - cr)) W(t - r, \xi - y - cr) \, dy, \\ &\qquad \text{for } t > t_0,\ \xi \in I_2, \\ &W|_{\xi=\xi_0} \ge C_4 (1+t)^{-1/2}, \quad \text{for } t > t_0, \\ &W|_{t=s} \ge v_0(s,\xi), \quad \text{for } s \in [-r, 0],\ \xi \in I_2. \end{aligned} \tag{91}$$

Therefore, for $c = c^*$, we obtain

$$\begin{aligned} &0 \le v(t,\xi) \le W(t,\xi) = C_4 (1+t+r)^{-1/2}, \\ &\qquad t > 0,\ \xi \in I_2. \end{aligned} \tag{92}$$

This completes the proof. □

From Lemmas 7 and 8, we prove the L^∞– convergence directly as follows.

Lemma 9. *There holds that*

(1) if $c > c^*$, one gets

$$\sup_{x\in\mathbb{R}} \left|U^+(t,x) - \phi(x+ct)\right| \le Ct^{-1/2} e^{-\nu t}, \quad t > 0, \tag{93}$$

with some sufficiently small constant $0 < \nu < \min\{\nu_1, \nu_2\}$;

(2) if $c = c^*$, one gets

$$\sup_{x\in\mathbb{R}} \left|U^+(t,x) - \phi(x + c^* t)\right| \le Ct^{-1/2}, \quad t > 0. \tag{94}$$

Secondly, we will prove the convergence of $U^-(t,x)$ to $\phi(x+ct)$.

As shown in the above process, we can similarly prove the following convergence of $U^-(t,x)$ to $\phi(x+ct)$.

Lemma 10. *There holds that*

(1) if $c > c^*$, one can get

$$\sup_{x\in\mathbb{R}} \left|U^-(t,x) - \phi(x+ct)\right| \le Ct^{-1/2} e^{-\nu t}, \quad t > 0, \tag{95}$$

with some sufficiently small constant $0 < \nu < \min\{\nu_1, \nu_2\}$;

(2) if $c = c^*$, one can get

$$\sup_{x\in\mathbb{R}} \left|U^-(t,x) - \phi(x + c^* t)\right| \le Ct^{-1/2}, \quad t > 0. \tag{96}$$

Lastly, we will prove the convergence of $u(t,x)$ to $\phi(x+ct)$.

Lemma 11. *There holds that:*

(1) if $c > c^*$, one can get

$$\sup_{x\in\mathbb{R}} \left|u(t,x) - \phi(x+ct)\right| \le Ct^{-1/2} e^{-\nu t}, \quad t > 0, \tag{97}$$

with some sufficiently small constant $0 < \nu < \min\{\nu_1, \nu_2\}$;

(2) if $c = c^*$, one can get

$$\sup_{x\in\mathbb{R}} \left|u(t,x) - \phi(x + c^* t)\right| \le Ct^{-1/2}, \quad t > 0. \tag{98}$$

Combining with the above lemmas, we can get the following stability results of the traveling front solutions for (1).

Theorem 12 (stability of traveling waves with time-delay). *Suppose $(A_1)-(A_4)$ hold. When $c \ge c^*$, for the given traveling front solution $\phi(\xi)$ of (1) and $\phi(\pm\infty) = u_\pm$, if the initial data holds*

$$0 = u_- \le u_0(s,x) \le u_+ = K, \quad \text{for } (s,x) \in [-r,0]\times\mathbb{R} \tag{99}$$

and the initial perturbation is $u_0(s,x) - \phi(x+cs) \in C([-r,0]; W^{2,1}_\omega(\mathbb{R})$ and $\partial_s(u_0(s,x) - \phi(x+cs)) \in C([-r,0]; L^1_\omega(\mathbb{R}))$, then the solution of (1) and (2) satisfies the following:

(1) if $c > c^*$, the solution $u(t,x)$ converges to the traveling wave $\phi(x+ct)$ exponentially

$$\sup_{x\in\mathbb{R}} \left|u(t,x) - \phi(x+ct)\right| \le Ct^{-1/2}e^{-\nu t}, \quad t > 0, \tag{100}$$

for some constant ν which satisfies

$$0 < \nu < \min\{\nu_1, \nu_2\}, \tag{101}$$

where

$$\begin{aligned} k_1 &= c\lambda - D\lambda^2 - \partial_1 F(0,0), \\ k_2 &= \partial_2 F(0,0) f'(0) \int_{-\infty}^{+\infty} J(y) e^{-\lambda(y+cr)} \mathrm{d}y, \\ \nu_1 &= \epsilon(r)(k_1 - k_2), \end{aligned} \tag{102}$$

and $\nu_2 > 0$ is the only root of the equation $-\partial_1 F(K, f(K)) - \nu - \partial_2 F(K, f(K)) f'(K) e^{\nu r} = 0$. In addition, $\epsilon(r) \in (0,1)$ is decreasing when $r > 0$ and satisfies $\epsilon(r) \to 1$ as $r \to 0$ and $\epsilon(r) \to 0$ as $r \to \infty$;

(2) if $c = c^*$, the solution $u(t,x)$ converges to the traveling wave $\phi(x + c^* t)$ algebraically

$$\sup_{x\in\mathbb{R}} \left|u(t,x) - \phi(x + c^* t)\right| \le Ct^{-1/2}, \quad t > 0. \tag{103}$$

Remark 13. In Theorem 12, the assumption

$$\begin{aligned} &(\partial_1 F(K, f(K)))^2 \\ &\quad > f'(0) f'(K) \partial_2 F(0,0) \partial_2 F(K,0) \end{aligned} \tag{104}$$

needed in [22] is taken away in this paper. In addition, compared to the stability results obtained in [22], the convergence rates of traveling front solutions with noncritical speeds in this paper are more accurate.

When $r = 0$, then (51) is reduced to

$$\begin{aligned} &\overline{v}_t + l_1 \overline{v}_\xi - D\overline{v}_{\xi\xi} + l_2 \overline{v} \\ &\quad = \partial_2 F(0,0) f'(0) \int_{-\infty}^{+\infty} J(y) e^{-\lambda y} \overline{v}(t, \xi - y) \mathrm{d}y, \\ &\qquad\qquad t > 0, \ \xi \in \mathbb{R}, \\ &\overline{v}(0,\xi) = e^{-\lambda(\xi-\xi_0)} \widetilde{v}_0(0,\xi) = \overline{v}_0(0,\xi), \quad \xi \in \mathbb{R}. \end{aligned} \tag{105}$$

By taking Fourier transform to (105), we get

$$\begin{aligned} \frac{\mathrm{d}\widehat{v}}{\mathrm{d}t} &= [B(\eta) - A(\eta)]\widehat{v}, \\ \widehat{v}(0,\eta) &= \widehat{v}_0(\eta), \end{aligned} \tag{106}$$

where

$$\begin{aligned} A(\eta) &= D\eta^2 + l_2 + \mathrm{i} l_1 \eta, \\ B(\eta) &= \partial_2 F(0,0) f'(0) \int_{-\infty}^{+\infty} J(y) e^{-\lambda y} e^{-\mathrm{i}\eta y} \mathrm{d}y. \end{aligned} \tag{107}$$

It is clear that

$$\widehat{v}(t,\eta) = \widehat{v}_0(\eta) e^{[B(\eta) - A(\eta)]t}. \tag{108}$$

In the same way, by taking the inverse Fourier transform to (106), we have

$$\overline{v}(t,\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}\xi\eta} \widehat{v}_0(\eta) e^{[B(\eta) - A(\eta)]t} \mathrm{d}\eta. \tag{109}$$

Then by a similar way of Lemma 6, we can get the following decay rates.

Lemma 14 (decay rates for $r = 0$). *Suppose $v_0 \in L^1_\omega(\mathbb{R})$. Thus, when $k_1 \ge k_2$, the solution of (105) satisfies*

$$\|\overline{v}(t)\|_{L^\infty(\mathbb{R})} \le Ct^{-1/2} e^{-(k_1 - k_2)t}, \quad t > 0. \tag{110}$$

Similarly, we can prove the convergence of of $u(t,x)$ to $\phi(x+ct)$ when the time-delay $r = 0$.

Theorem 15 (stability of traveling waves without time-delay). *Suppose $(A_1)-(A_4)$ hold. For the given traveling wave $\phi(\xi)$ of (1) and $\phi(\pm\infty) = u_\pm$ with speed $c \ge c^*$, if the initial data holds*

$$0 = u_- \le u_0(x) \le u_+ = K, \quad \text{for } x \in \mathbb{R} \tag{111}$$

and the initial perturbation $u_0(x) - \phi(x)$ is in $L^1_\omega(\mathbb{R})$, then the solution of (1) and (2) satisfies the following:

(1) if $c > c^*$, the solution $u(t,x)$ converges to the traveling wave $\phi(x+ct)$ exponentially

$$\sup_{x\in\mathbb{R}} \left|u(t,x) - \phi(x+ct)\right| \le Ct^{-1/2}e^{-\nu t}, \quad t > 0, \tag{112}$$

for some constant

$$\begin{aligned} 0 < \nu < \min\{&k_1 - k_2, -\partial_1 F(K, f(K)) \\ &- \partial_2 F(K, f(K)) f'(K)\}, \end{aligned} \tag{113}$$

where

$$k_1 := c\lambda - D\lambda^2 - \partial_1 F(0,0),$$
$$k_2 := \partial_2 F(0,0) f'(0) \int_{-\infty}^{+\infty} J(y) e^{-\lambda y} \mathrm{d}y; \tag{114}$$

(2) if $c = c^*$, the solution $u(t,x)$ converges to the traveling wave $\phi(x + c^* t)$ algebraically

$$\sup_{x\in\mathbb{R}} \left|u(t,x) - \phi(x + c^* t)\right| \le Ct^{-1/2}, \quad t > 0. \tag{115}$$

Remark 16. From Theorems 12 and 15 we conclude that the time-delay affects not only the initial perturbation, but also the convergence rates of the traveling front solutions with noncritical speeds.

4. The Uniqueness of the Traveling Front Solutions

In this part, the uniqueness of the traveling front solutions will be proved on the premise of the stability.

Theorem 17 (uniqueness of traveling front solutions). *Suppose $(A_1) - (A_4)$ hold. Then, for the same $c > c^*$, the traveling front solutions $\phi(x + ct)$ of (1) are unique up to translation.*

Proof. For $r > 0$ and the same $c > c^*$, suppose that $\phi_1(x + ct)$ and $\phi_2(x+ct)$ are two different traveling front solutions. Then it follows from Theorem 3 that

$$\phi_1(\xi) \sim A_1 e^{-\lambda_1|\xi|}, \quad \text{when } \xi \longrightarrow -\infty \tag{116}$$

and

$$\phi_2(\xi) \sim A_2 e^{-\lambda_1|\xi|}, \quad \text{when } \xi \longrightarrow -\infty, \tag{117}$$

where $\xi = x + ct$, A_1, A_2 are two positive constants and λ_1 is defined in Theorem 3. Let us move $\phi_2(x+ct)$ to $\phi_2(x+ct+x_0)$ with some constant x_0. Letting $\xi \to -\infty$, it is evident that $\xi + x_0 < 0$. By choosing

$$x_0 = \frac{1}{\lambda_1} \ln \frac{A_1}{A_2}, \tag{118}$$

if $\xi \to -\infty$, we can get

$$\begin{aligned}\phi_2(\xi + x_0) &\sim A_2 e^{-\lambda_1|\xi+x_0|} = A_2 e^{\lambda_1(\xi+x_0)} \\ &= A_2 e^{\lambda_1 x_0} e^{-\lambda_1|\xi|} = A_1 e^{-\lambda_1|\xi|}.\end{aligned} \tag{119}$$

Then, when $\xi \to -\infty$, there exists $\gamma > \lambda_1$ such that

$$\left|\phi_2(\xi + x_0) - \phi_1(\xi)\right| = O(1) e^{-\gamma|\xi|}, \tag{120}$$

which indicates that

$$\begin{aligned}&\phi_2(\xi + x_0) - \phi_1(\xi) \in W_\omega^{2,1}(\mathbb{R}), \\ &\phi_{2t}(\xi + x_0) - \phi_{1t}(\xi) \in L_\omega^1(\mathbb{R}).\end{aligned} \tag{121}$$

Next, by taking the initial data for (1) as

$$u_0(s,x) = \phi_2(x + cs + x_0), \quad x \in \mathbb{R}, \ s \in [-r, 0], \tag{122}$$

then we get the corresponding solution to (1)

$$u(t,x) = \phi_2(x + ct + x_0). \tag{123}$$

From Theorem 12, for $r > 0$, it holds that

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \left|\phi_2(x + ct + x_0) - \phi_1(x + ct)\right| = 0; \tag{124}$$

that is, $\phi_2(x + ct + x_0) = \phi_1(x + ct)$ for all $x \in \mathbb{R}$ if $t \gg 1$. This completes the proof of the uniqueness of traveling waves up to a constant shift. Similarly, we can obtain the result when the equation is without time-delay. □

5. Applications

In this section, we shall present the following results as a direct consequence of Theorems 12 and 15.

5.1. Consider the following Vector-Disease Model (See [30])

$$\begin{aligned}&u_t(t,x) \\ &\quad = D\Delta u(t,x) - au(t,x) \\ &\qquad + b[1 - u(t,x)] \int_{-\infty}^{+\infty} J(y) u(t-r, x-y) \mathrm{d}y,\end{aligned} \tag{125}$$

where $b > a > 0$, $u(t,x)$ denotes the density of infectious individual at time t and site x, a is the recovery ratio of the infected person, and b is the host-vector contact ratio. D is the diffusion constant.

Clearly, (125) has two constant equilibria $u_- \equiv 0$ and $u_+ \equiv 1 - a/b > 0$. The characteristic equation at $u_- \equiv 0$ is

$$\mathscr{P}(\lambda, c) = D\lambda^2 - c\lambda - a + b \int_{-\infty}^{\infty} J(y) e^{-\lambda(y+cr)} \mathrm{d}y. \tag{126}$$

Set

$$\begin{aligned}&c^* \\ &\quad = \inf\{c > 0 \mid \mathscr{P}(\lambda, c) = 0 \text{ has a positive real root}\}.\end{aligned} \tag{127}$$

There exists a corresponding number $\lambda^* > 0$ such that $\mathscr{P}(\lambda^*, c^*) = 0$. In [30], the authors have justified that c^* is the minimal wave speed of traveling wave solution connecting the two equilibria.

Set $F(x,y) = -ax + b(1-x)y$ and $f(x) = x$ in (1). Obviously, $\partial_1 F(x,y) = -a - by$, $\partial_2 F(x,y) = b - bx > 0$ for $x \in [0,K]$, $\partial_{ij} F(x,y) \le 0$ for $(x,y) \in [0,K]^2$ and

$$\partial_2 F(K, f(K)) f'(K) + \partial_1 F(K, f(K)) < 0, \tag{128}$$

where $K = 1 - a/b$. Thus we get the following result.

Corollary 18 (stability of traveling waves). *When $c \geq c^*$, for the given traveling wave $\phi(\xi)$ of (125) and $\phi(\pm\infty) = u_\pm$, if the initial data holds*

$$0 = u_- \leq u_0(s,x) \leq u_+ = K, \quad \text{for } (s,x) \in [-r,0] \times \mathbb{R} \tag{129}$$

and the initial perturbation is $u_0(s,x) - \phi(x+cs) \in C([-r,0]; W^{2,1}_\omega(\mathbb{R})$ and $\partial_s(u_0(s,x) - \phi(x+cs)) \in C([-r,0]; L^1_\omega(\mathbb{R}))$, then the solution of (125) and (2) satisfies the following:

(1) if $c > c^*$, the solution $u(t,x)$ converges to the traveling wave $\phi(x+ct)$ exponentially

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+ct)| \leq Ct^{-1/2}e^{-\nu t}, \quad t > 0, \tag{130}$$

for some constant ν which satisfies

$$0 < \nu < \min\{\nu_1, \nu_2\}, \tag{131}$$

where

$$\begin{aligned} k_1 &= c\lambda - D\lambda^2 + a, \\ k_2 &= b\int_{-\infty}^{+\infty} J(y)e^{-\lambda(y+cr)}\mathrm{d}y, \\ \nu_1 &= \epsilon(r)(k_1 - k_2), \end{aligned} \tag{132}$$

and $\nu_2 > 0$ is the only root of the equation $b - \nu - ae^{\nu r} = 0$. In addition, $\epsilon(r) \in (0,1)$ is decreasing when $r > 0$ and satisfies $\epsilon(r) \to 0$ as $r \to \infty$. When $r = 0$, then $\epsilon = 1$ and $\nu_2 = b - a$.

(2) if $c = c^*$, the solution $u(t,x)$ converges to the traveling wave $\phi(x + c^*t)$ algebraically

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+c^*t)| \leq Ct^{-1/2}, \quad t > 0. \tag{133}$$

5.2. Consider the Classical Fisher-KPP Equation (7) with the Initial Data

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}. \tag{134}$$

For (7), it is easy to see that $u_- = 0$, $u_+ = 1$. Obviously, from Theorem 15, we obtain the following result.

Corollary 19 (stability of traveling waves). *For the given traveling wave $\phi(\xi)$ of (7) and $\phi(\pm\infty) = u_\pm$ with speed $c \geq c^*$, if the initial data holds*

$$0 = u_- \leq u_0(x) \leq u_+ = 1, \quad \text{for } x \in \mathbb{R} \tag{135}$$

and the initial perturbation $u_0(x) - \phi(x)$ is in $L^1_\omega(\mathbb{R})$, then the solution of (7) and (134) satisfies the following:

(1) when $c > c^*$, the solution $u(t,x)$ converges to the traveling wave $\phi(x+ct)$ exponentially

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+ct)| \leq Ct^{-1/2}e^{-\nu t}, \quad t > 0, \tag{136}$$

for some constant

$$0 < \nu < \min\{k_1 - k_2, 1\}, \tag{137}$$

where $k_1 := c\lambda - D\lambda^2$, $k_2 := \int_{-\infty}^{+\infty} J(y)e^{-\lambda y}\mathrm{d}y$;

(2) when $c = c^*$, the solution $u(t,x)$ converges to the traveling wave $\phi(x + c^*t)$ algebraically

$$\sup_{x\in R} |u(t,x) - \phi(x+c^*t)| \leq Ct^{-1/2}, \quad t > 0. \tag{138}$$

Remark 20. The authors in [18–20] have showed the stability for (7). Our results generalize and develop the former results. The decay rate of the critical waves of (7) is faster than that of [19], but slower than that in [20].

5.3. Consider the following General Population Model, Which Is Evolved from a Mature Population with an Age Structure of a Single Population (See [6])

$$\begin{aligned} u_t(t,x) &= D\Delta u(t,x) - g(u(t,x)) \\ &\quad + h(u(t,x))\int_{-\infty}^{\infty} J(x-y) f(u(t-r,y))\,\mathrm{d}y, \end{aligned} \tag{139}$$

where $D > 0, r \geq 0$ are constants and f, h and g are Lipschitz continuous functions on any compact interval.

Let $F(x,y) = -g(x) + h(x)y$. Now we assume that

(H_1) let $K > 0$, $f, g, h \in C^2([0,K],\mathbb{R})$, $f(0) = g(0) = 0, g(K) = h(K)f(K), h(x) \times f(x) > g(x), f'(x) \geq 0, f''(x) \leq 0, h(x) > 0, h'(x) \leq 0, h''(x) \leq 0, g''(x) \geq 0$ for $(x,y) \in [0,K] \times [0, f(K)]$.

(H_2) $h(K)f'(K) - g'(K) + h'(K)f(K) < 0$.

Clearly, $u_- = 0$ and $u_+ = K$ are two constant equilibria of (139). And the characteristic equation of (139) at $u_- = 0$ is

$$\begin{aligned} \mathscr{P}(\lambda,c) &= D\lambda^2 - c\lambda - g'(0) \\ &\quad + h(0)f'(0)\int_{-\infty}^{\infty} J(y)e^{-\lambda(y+cr)}\mathrm{d}y. \end{aligned} \tag{140}$$

Set

$$\begin{aligned} c^* &= \inf\{c > 0 \mid \mathscr{P}(\lambda,c) = 0 \text{ has a positive real root}\} \\ &> 0. \end{aligned} \tag{141}$$

It has proved that, for any $c \geq c^*$, (139) admits a traveling front solution $u(t,x) = \phi(x+ct)$ satisfying $\phi(-\infty) = 0$ and $\phi(\infty) = K$. Then we can show the stability of traveling front solutions of (139).

Corollary 21. *Suppose that (A_1) and $(H_1)-(H_2)$ hold. When $c \geq c^*$, for the given traveling wave $\phi(\xi)$ of (139) and $\phi(\pm\infty) = u_\pm$, if the initial data holds*

$$0 = u_- \leq u_0(s,x) \leq u_+ = K, \quad \text{for } (s,x) \in [-r,0] \times \mathbb{R} \tag{142}$$

and the initial perturbation is $u_0(s,x) - \phi(x+cs) \in C([-r,0]; W^{2,1}_\omega(\mathbb{R})$ and $\partial_s(u_0(s,x) - \phi(x+cs)) \in C([-r,0]; L^1_\omega(\mathbb{R}))$, then the solution of (139) and (2) satisfies the following:

(1) if $c > c^*$, the solution $u(t,x)$ converges to the traveling wave $\phi(x+ct)$ exponentially

$$\sup_{x\in\mathbb{R}} \left|u(t,x) - \phi(x+ct)\right| \leq Ct^{-1/2}e^{-\nu t}, \quad t > 0, \tag{143}$$

for some constant ν which satisfies

$$0 < \nu < \min\{\nu_1, \nu_2\}, \tag{144}$$

where

$$\begin{aligned} k_1 &= c\lambda - D\lambda^2 + g'(0), \\ k_2 &= h(0) f'(0) \int_{-\infty}^{+\infty} J(y) e^{-\lambda(y+cr)} \mathrm{d}y, \\ \nu_1 &= \epsilon(r)(k_1 - k_2), \end{aligned} \tag{145}$$

and $\nu_2 > 0$ is the only root of the equation $g'(K) - f(K)h'(K) - \nu - f'(K)h(K)e^{\nu r} = 0$. In addition, $\epsilon(r) \in (0,1)$ is decreasing when $r > 0$ and satisfies $\epsilon(r) \to 0$ as $r \to \infty$. When $r = 0$, then $\epsilon = 1$ and $\nu_2 = g'(K) - f(K)h'(K) - f'(K)h(K)$;

(2) if $c = c^*$, the solution $u(t,x)$ converges to the traveling wave $\phi(x+c^*t)$ algebraically

$$\sup_{x\in\mathbb{R}} \left|u(t,x) - \phi(x+c^*t)\right| \leq Ct^{-1/2}, \quad t > 0. \tag{146}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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