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Research Article

Boundedness and Asymptotic Stability for the Solution of Homogeneous Volterra Discrete Equations

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We consider homogeneous linear Volterra Discrete Equations and we study the asymptotic behaviour of their solutions under hypothesis on the sign of the coefficients and of the first- and second-order differences. The results are then used to analyse the numerical stability of some classes of Volterra integrodifferential equations.

1. Introduction

Linear Volterra Discrete Equations (VDEs) are usually represented according to two types of formulae (see, e.g., [1] and references therein, [2, XIII-10], [3, Chap. 7]):

$$x_{n+1} - x_n = g_{n+1} + a_{n+1}x_{n+1} + \sum_{j=0}^{n+1} b_{n+1,j}x_j, \qquad n \geq 0, \ x_0 \text{ given}, \tag{1}$$

$$x_n = f_n + \sum_{j=0}^{n} c_{n,j}x_j, \quad n \geq 1, \ x_0 \text{ given}. \tag{2}$$

Even if each of the equations above can be easily transformed into the other, we read in the literature (see, e.g., [4]) that (1) is the discrete analogue of a Volterra Integrodifferential Equation (VIDE), whereas (2) is seen as the discrete version of a second kind Volterra Integral Equation (VIE). This is due to the fact that the simple position

$$\begin{aligned} c_{n+1,n+1} &= a_{n+1}, \\ c_{n+1,n} &= 1 + b_{n+1,n}, \\ c_{n+1,j} &= b_{n+1,j}, \quad j = 0, \dots, n-2, \end{aligned} \tag{3}$$

which transforms (2) into (1) is not meaningful when we are dealing with numerical analysis of Volterra equations.

To be more specific, a simple numerical method for the VIDE, $y'(t) = g(t) + A(t)y(t) + \int_0^t K(t,s)y(s)ds$, has the form

$$\begin{aligned} y_{n+1} - y_n = {} & hg(t_{n+1}) + hA(t_{n+1})\, y_{n+1} \\ & + h^2 \sum_{j=0}^{n+1} w_{n+1,j} K(t_{n+1}, t_j)\, y_j, \end{aligned} \tag{4}$$

where h is the stepsize, $t_n = nh$, $w_{n,j}$ are given weights, and $y_n \approx y(t_n)$.

Using (3), namely, $c_{n+1,n+1} = hA(t_{n+1})$, $c_{n+1,n} = h^2 w_{n+1,n}K(t_{n+1}, t_n) - 1$, and $c_{n+1,j} = h^2 w_{n+1,j}K(t_{n+1}, t_j)$, (4) turns into the form of (2), the analysis of which would be complicated by the fact that the coefficients do not have the same dependence on h. For such a reason, in this paper we focus on the following homogeneous VDE:

$$x_{n+1} - x_n = a_{n+1}x_{n+1} + \sum_{j=0}^{n+1} b_{n+1,j}x_j, \quad n \geq 0, \tag{5}$$

where x_0 is given and $b_{n,j} = 0$ for $j > n$, and we study its asymptotic properties exploiting its particular form.

For the sake of completeness, there is also another type of VDE widely used in literature (see, e.g., [5, 6])

$$x_{n+1} = A_n x_n + \sum_{j=0}^{n} B_{n,j} x_j, \quad n \geq 0, \ x_0 \text{ given.} \tag{6}$$

This is an explicit equation which can be recasted in the form (5) with $b_{n+1,n+1} = a_{n+1} = 0$, by imposing $A_n = 1$.

Asymptotic analysis of difference equations of the form (5) or its explicit version often appeared in literature in the last decades. Some of them deal with the convolution case ($b_{n,j} = b_{n-j}$); see, for instance, [6] and the references therein and [7–12]. Most of the known results for the nonconvolution case are based on the hypothesis of double summability of the coefficients ($\sum_{n=0}^{+\infty} \sum_{j=0}^{n} |b_{n,j}| < +\infty$); see [1, 4, 13–19]. Another interesting approach, resembling the study of continuous VIDE (see, e.g., [20, 21]), basically requires that the coefficient a_{n+1} of (5), assumed to be negative, in some sense “prevails” on the summation of the remaining coefficients b_{nj}. Here we would like to add another piece to the framework regarding the analysis of VDE behaviour, by considering hypotheses based on the sign of the coefficients and of their first and second differences.

Since (5) is homogeneous, it has always the trivial solution. Therefore, all the results that follow are valid automatically and no assumptions are necessary when $x_0 = 0$. From now on we assume that the given datum x_0 is different from zero and we want to analyse the behaviour of the corresponding solution with respect to the trivial one. In Section 2 we report our main results on the asymptotic behaviour of the nontrivial solution to (5) which are then used, in Section 3, to prove the boundedness of the solution and the convergence to zero in some cases of interest.

In the whole paper it is assumed the empty sum convention $\sum_{j=m}^{M} v_j = 0$, if $M < m$.

2. Main Results

Let $v_{n,j}$ be a double-indexed sequence and define $\Delta_1 v_{n,j} = v_{n+1,j} - v_{n,j}$, $\Delta_2 v_{n,j} = v_{n,j+1} - v_{n,j}$, and $\Delta_{12} v_{n,j} = \Delta_1(\Delta_2 v_{n,j})$. Our main result gives sufficient conditions for (5) to have a solution x_n vanishing at infinity.

Theorem 1. *Consider (5) and assume that*

(i) $\exists \overline{n} > 0$ *such that* $2a_n + b_{n,n} \leq 0, \ n \geq \overline{n}$,

(ii) $b_{n,0} \leq 0, \ n \geq 2, \ \Delta_1 b_{n,0} \geq 0, \ n \geq \overline{n}$,

(iii) $\Delta_2 b_{n,j} \leq 0, \ n \geq 2, \ j \leq n-2$,

(iv) $\Delta_{12} b_{n,j} \geq 0, \ n \geq \overline{n} \geq 1, \ j \leq n-1$.

Then, for any $x_0 \in \mathbb{R}$, there exists x^* such that $|x_n| \leq x^*$, $\forall n > 0$. If, in addition,

(v) $2a_n + b_{n,n} \leq -a^* < 0, \ n > \overline{n} \geq 1$,

or

(v_{bis}) $\Delta_{12} b_{n,j} \geq b^* > 0, \ n \geq \overline{n} \geq 1, \ j \leq n-1$,

then, for any $x_0 \in \mathbb{R}$, $\lim_{n\to+\infty} x_n = 0$.

Proof. Set $\alpha_n = a_{n+1}x_{n+1} + \sum_{j=0}^{n+1} b_{n+1,j} x_j$, then

$$x_{n+1}^2 - x_n^2 = 2x_{n+1}\alpha_n - \alpha_n^2, \tag{7}$$

and hence

$$2x_{n+1}\alpha_n = 2x_{n+1}^2 \left(a_{n+1} + b_{n+1,n+1}\right) + 2x_{n+1} \sum_{j=0}^{n} b_{n+1,j} x_j. \tag{8}$$

The second addendum in the right-hand side of (8) can be written as

$$2x_{n+1} \sum_{j=0}^{n+1} b_{n+1,j} x_j = 2x_{n+1} \sum_{j=0}^{n} b_{n+1,j} \Delta X_j, \tag{9}$$

where

$$X_{n+1} = \sum_{j=0}^{n} x_j, \quad n \geq 0, \qquad X_0 = 0. \tag{10}$$

Applying the summation by parts rule, we have

$$2x_{n+1} \sum_{j=0}^{n} b_{n+1,j} x_j = 2x_{n+1} \left(b_{n+1,n+1} X_{n+1} - \sum_{j=0}^{n} X_{j+1} \Delta_2 b_{n+1,j} \right). \tag{11}$$

By adding and subtracting $X_{n+1} \sum_{j=0}^{n} \Delta_2 b_{n+1,j}$ in the right-hand side and by setting

$$V_{n,j} = X_{n+1} - X_{j+1} = \sum_{i=j+1}^{n} x_i, \qquad j \leq n-1, \ V_{n,j} = 0 \text{ for } j \geq n, \tag{12}$$

we get

$$\begin{aligned} 2x_{n+1} \sum_{j=0}^{n} b_{n+1,j} x_j &= 2x_{n+1} \left(b_{n+1,n+1} X_{n+1} + \sum_{j=0}^{n} V_{n,j} \Delta_2 b_{n+1,j} - X_{n+1} \sum_{j=0}^{n} \Delta_2 b_{n+1,j} \right) \\ &= 2x_{n+1} \left(\sum_{j=0}^{n} V_{n,j} \Delta_2 b_{n+1,j} + X_{n+1} b_{n+1,0} \right). \end{aligned} \tag{13}$$

Now, taking into account the fact that

$$2x_{n+1} \sum_{j=m}^{n} x_j = \left(\sum_{j=m}^{n+1} x_j \right)^2 - \left(\sum_{j=m}^{n} x_j \right)^2 - x_{n+1}^2, \qquad 0 \leq m \leq n, \tag{14}$$

we have

$$2x_{n+1}\sum_{j=0}^{n}b_{n+1,j}x_j = \sum_{j=0}^{n}\Delta_2 b_{n+1,j}\left(\Delta_1 V_{n,j}^2 - x_{n+1}^2\right) + b_{n+1,0}\left(\Delta X_{n+1}^2 - x_{n+1}^2\right) = \sum_{j=0}^{n}\Delta_2 b_{n+1,j}\Delta_1 V_{n,j}^2 - x_{n+1}^2 b_{n+1,n+1} + b_{n+1,0}\Delta X_{n+1}^2. \quad (15)$$

By (8) and (15), (7) becomes

$$x_{n+1}^2 - x_n^2 = x_{n+1}^2\left(2a_{n+1} + b_{n+1,n+1}\right) + \sum_{j=0}^{n}\Delta_2 b_{n+1,j}\Delta_1 V_{n,j}^2 + b_{n+1,0}\Delta X_{n+1}^2 - \alpha_n^2. \quad (16)$$

Summing up over n, for all $N > \overline{n} \geq 0$, we have

$$x_{N+1}^2 - x_0^2 = \sum_{n=0}^{N}x_{n+1}^2\left(2a_{n+1} + b_{n+1,n+1}\right) + \sum_{n=0}^{N}\sum_{j=0}^{n}\Delta_2 b_{n+1,j}\Delta_1 V_{n,j}^2 + \sum_{n=0}^{N}b_{n+1,0}\Delta X_{n+1}^2 - \sum_{n=0}^{N}\alpha_n^2. \quad (17)$$

Now, let us consider the double summation at the right-hand side. By inverting the summation order, applying the summation by part rule and recalling that $V_{j,j} = 0$, it becomes

$$\sum_{n=0}^{N}\sum_{j=0}^{n}\Delta_2 b_{n+1,j}\Delta_1 V_{n,j}^2 = \sum_{j=0}^{N}\Delta_2 b_{N+2,j}V_{N+1,j}^2 - \sum_{n=0}^{N}\sum_{j=0}^{n}\Delta_{12} b_{n+1,j}V_{n+1,j}^2. \quad (18)$$

Taking account of this and applying the summation by part rule also to the third addendum in (17), we get

$$x_{N+1}^2 - x_0^2 = \sum_{n=0}^{N}x_{n+1}^2\left(2a_{n+1} + b_{n+1,n+1}\right) + \sum_{j=0}^{N}\Delta_2 b_{N+2,j}V_{N+1,j}^2 - \sum_{n=0}^{N}\sum_{j=0}^{n}\Delta_{12} b_{n+1,j}V_{n+1,j}^2 + b_{N+2,0}X_{N+2}^2 - b_{1,0}X_2^2 - \sum_{n=0}^{N}\Delta_1 b_{n+1,0}X_{n+2}^2 - \sum_{n=0}^{N}\alpha_n^2. \quad (19)$$

In view of the first group of hypotheses (i)–(iv), this implies

$$x_{N+1}^2 \leq x_0^2 + \sum_{n=0}^{\overline{n}-2}x_{n+1}^2\left(2a_{n+1} + b_{n+1,n+1}\right) - \sum_{n=0}^{\overline{n}-2}\sum_{j=0}^{n}\Delta_{12} b_{n+1,j}V_{n+1,j}^2 - b_{1,0}\left(x_0 + x_1\right)^2 - \sum_{n=0}^{\overline{n}-1}\Delta_1 b_{n+1,0}X_{n+2}^2. \quad (20)$$

As the whole right-hand side does not depend on N, (20) assures the boundedness of $|x_n|$ and the first part of the theorem is proved.

In order to prove the second part of our result, let us proceed by contradiction. Assume that

$$\lim_{N\to+\infty}\sum_{n=\overline{n}}^{N}x_n^2 = +\infty. \quad (21)$$

From (19) and (20) we have $x_{N+1}^2 \leq C + \sum_{n=\overline{n}}^{N+1}(2a_n + b_{n,n})x_n^2$ and in view of (v) $x_{N+1}^2 \leq C - a^*\sum_{n=\overline{n}}^{N+1}x_n^2$, which leads to an absurd because of (21). So the series in (21) converges and $\lim_{n\to+\infty}x_n^2 = 0$.

Now consider hypothesis ($\mathrm{v_{bis}}$) and once again proceed by contradiction. Assume that the series $\sum_{n=0}^{+\infty}x_n$ does not converge, therefore $\exists\epsilon > 0$ and two increasing sequences of integers $\{n_i\}$ and $\{l_i\}$ with $n_i > l_i$, $l_0 > \overline{n}$ such that $|\sum_{j=l_i+1}^{n_i+1}x_j| \geq \epsilon$, and hence

$$V_{n_i+1,l_i}^2 \geq \epsilon^2, \quad \forall i \geq 0. \quad (22)$$

Again, using (19) and (20), we write

$$x_{N+1}^2 \leq C - \sum_{n=\overline{n}}^{N}\sum_{j=0}^{n}\Delta_{12} b_{n+1,j}V_{n+1,j}^2. \quad (23)$$

Because of (iv) we can write

$$\sum_{n=\overline{n}}^{N}\sum_{j=0}^{n}\Delta_{12} b_{n+1,j}V_{n+1,j}^2 \geq \sum_{i=0}^{m(N)}\Delta_{12} b_{n_i+1,l_i}V_{n_i+1,l_i}^2, \quad (24)$$

with $m(N)$ such that $n_{m(N)} < N < n_{m(N)} + 1$. This together with (22) and ($\mathrm{v_{bis}}$) leads to $x_{N+1}^2 \leq C - \epsilon^2 b^* m(N)$. Since $m(N) \to +\infty$, as N increases, this is absurd. Hence, the series $\sum_{n=0}^{+\infty}x_n$ converges and the desired result follows. □

It is well known that one of the most used tools in the stability analysis of VDEs is the Lyapunov approach [22–28]. As already mentioned in the introduction, among the results that can be obtained by Lyapunov techniques, the most popular are based on the hypothesis that the coefficients are summable (e.g., the result in [28, Th. 2] applied to (5) requires, among other hypotheses, that $\sum_{i=0}^{n-1}\sum_{j=n+1}^{+\infty}|b_{j,i}/(1-a_j-b_{j,j})| < +\infty$). Our attempt to construct new functionals for the form (5) leads inevitably to this type of hypothesis. Very few are the cases where no summability requirements are made. One of these can be found in [26, Th. 2.2], where a Lyapunov functional is constructed which allows the stability analysis of an explicit equation, provided that some conditions on the sign of the coefficients and of their Δ's are satisfied. In order to compare the technique developed in this paper with the Lyapunov one, we refer precisely to this theorem and consider (5) with $b_{n,n} = 0$ and $a_n = a < 0$. In this situation the hypotheses of Theorem 1 proved above guarantee that the solution vanishes also in few cases not covered by Theorem 2.2 in [26] (just to mention one example, the coefficient of x_n, which should be negative in [26], is allowed to assume whatever sign here).

Remark 2. It is easy to see that if hypothesis (iii) in Theorem 1 holds also for $j = n - 1$, then ($\mathrm{v}_{\mathrm{bis}}$) assures $b_{nn} \le -b^*$, for all $n > \bar{n}$. Therefore, if we assume $a_n \le 0, n \ge \bar{n}$, hypothesis ($\mathrm{v}_{\mathrm{bis}}$) becomes sufficient for (v). So ($\mathrm{v}_{\mathrm{bis}}$) does no more represent an alternative with respect to (v) and can be dropped out. In this case Theorem 1 can be stated as follows.

Corollary 3. *Consider (5) and assume that $a_n \le 0$, for $n \ge \bar{n}$, $\Delta_2 b_{n,n-1} \le 0$, and that (ii)–(v) hold. Then, for any $x_0 \in \mathbb{R}$, $\lim_{n\to+\infty} x_n = 0$.*

Furthermore, we point out that checking assumption ($\mathrm{v}_{\mathrm{bis}}$) of Theorem 1 may be difficult; hence the following result can be useful.

Corollary 4. *Consider (5) and assume that (i)–(iv) hold and*

$$\Delta_{12} b_{n,n-1} \ge b^*, \quad n \ge \bar{n}, \tag{25}$$

with $b^* > 0$. Then, for any $x_0 \in \mathbb{R}$, $\lim_{n\to+\infty} x_n = 0$.

Proof. From (iv) and the definition of $V_{n,j}$ in (12),

$$\begin{aligned}\sum_{n=\bar{n}}^{N} \sum_{j=0}^{n} \Delta_{12} b_{n+1,j} V_{n+1,j}^2 &\ge \sum_{n=\bar{n}}^{N} \Delta_{12} b_{n+1,n} V_{n+1,n}^2 \\ &= \sum_{n=\bar{n}}^{N} \Delta_{12} b_{n+1,n} x_{n+1}^2.\end{aligned} \tag{26}$$

The desired result is readily obtained by using (23) and (25). □

We want to underline that Theorem 1 is strongly inspired by [29] where the asymptotic behaviour of a nonlinear VIDE is studied and that "in some sense" our result can be viewed as its discrete analogue. This will be illustrated in the following section.

Remark 5. Observe that, when $\Delta_{12} b_{n,j}$ is of convolution type, hypothesis (iv) in Theorem 1 becomes $\Delta^2 b_n \le 0$, so that the advantage of using hypothesis (iv), which allowed $\Delta_{12} b_{n,j}$ to have a constant sign only definitely with respect to n, is completely lost. This drawback can be overcome if we know that the sign of x_n is definitely constant, as it is shown in the following theorem.

Theorem 6. *Consider (5) and assume that*

(a) *$\exists \bar{n} > 0$ such that $x_n \ge 0$ (<0), for $n \ge \bar{n}$,*
(b) *$\exists n^* > \bar{n}$ such that $\beta_n = 2a_n + 2b_{n,n} - b_{n,\bar{n}} \le 0$, for $n \ge n^* \ge \bar{n}$,*
(c) *$b_{n,0} \le 0,\ n \ge 0,\ \Delta_1 b_{n,0} \ge 0,\ n \ge n^* \ge \bar{n}$,*
(d) *$\Delta_2 b_{n,j} \le 0,\ n \ge 1,\ j < n$,*
(e) *$\exists p \ge 0$ such that $\Delta_{12} b_{n,j} \ge 0,\ n \ge \bar{n} + p,\ 0 \le j \le \bar{n} - 2$,*
(f) *$\beta_n \le -a^* < 0,\ n \ge n^* \ge \bar{n}$,*

or

($\mathrm{f}_{\mathrm{bis}}$) *$\Delta_{12} b_{n,j} \ge b^* > 0,\ n \ge \bar{n} + p,\ 0 \le j \le \bar{n} - 2$.*

Then, for any $x_0 \in \mathbb{R}$, $\lim_{n\to+\infty} x_n = 0$.

Proof. First of all observe that (a) assures $2x_{n+1} V_{n,j} \ge 0,\ n \ge j \ge \bar{n} - 1$. From here and (13) we derive

$$\begin{aligned}2x_{n+1} \sum_{j=0}^{n} b_{n+1,j} x_j \le \sum_{j=0}^{\bar{n}-2} \Delta_2 b_{n+1,j} 2x_{n+1} V_{n,j} \\ + 2x_{n+1} X_{n+1} b_{n+1,0}.\end{aligned} \tag{27}$$

Now, proceeding as in the proof of Theorem 1, we arrive to

$$\begin{aligned}x_{N+1}^2 - x_0^2 \le \sum_{n=0}^{N} \beta_{n+1} x_{n+1}^2 + \sum_{j=0}^{\bar{n}-2} \Delta_2 b_{N+2,j} V_{N+1,j}^2 \\ - \sum_{n=0}^{N} \sum_{j=0}^{\bar{n}-2} \Delta_{12} b_{n+1,j} V_{n+1,j}^2 + b_{N+2,0} X_{N+2}^2 \\ - b_{1,0} X_1^2 - \sum_{n=0}^{N} \Delta_1 b_{n+1,0} X_{n+2}^2 - \sum_{n=0}^{N} \alpha_n^2,\end{aligned} \tag{28}$$

which, taking into account (b), (c), and (e), assures

$$x_{N+1}^2 \le C + \sum_{n=n^*-1}^{N} \beta_{n+1} x_{n+1}^2, \tag{29}$$

or

$$x_{N+1}^2 \le C - \sum_{n=\bar{n}+p-1}^{N} \sum_{j=0}^{\bar{n}-2} \Delta_{12} b_{n+1,j} V_{n+1,j}^2, \tag{30}$$

which corresponds to (14) and (23) of Theorem 1, respectively. The desired result follows as in the proof of Theorem 1. □

As a consequence of this result, the following can be easily proved.

Corollary 7. *Under assumptions (b)–(f) of Theorem 6 a sequence x_n, obtained by (5) with $x_0 \in \mathbb{R}$, cannot diverge and if it is convergent then its limit is zero.*

Remark 8. If, in Theorem 6, $\bar{n} = 1$, then hypothesis (e) can be removed, and the theorem assumes a simplified form.

3. Examples of Applications

Consider the following theoretical examples of application of Corollaries 3 and 4.

It can be easily seen that (5), with the choices for a_n and $b_{n,j}$

$$a_n = 0, \quad b_{nj} = -e^{-(n-j)} - \frac{1}{(n+1)^2}; \tag{31}$$

$$a_n = \frac{1}{2}\left(\frac{n}{n+1} + \frac{1}{(n+1)^2}\right), \quad b_{nj} = -e^{-(n-j)} - \frac{1}{(n+1)^2}; \tag{32}$$

$$a_n = -1, \quad b_{nj} = -e^{-(2n-j)} j (n-1)^2; \tag{33}$$

$$a_n = -1, \quad b_{nj} = -e^{-(2n-j)} j (n-1)^2 - 1, \tag{34}$$

satisfies the assumptions of the corollaries in the previous section.

To be more specific (31) fulfills both corollaries with $\overline{n} = 0$. Equation (33) satisfies only Corollary 3 with $\overline{n} = 3$, whereas (32) satisfies Corollary 4 but not 2.1, because $b_{nn} + a_n \to 0$. Equation (34) is only a slight modification of (33) and, like (33), it fulfills all the hypotheses of Corollary 3; furthermore it can be easily seen that $\sum_{j=0}^{n} |b_{nj}|$ is an unbounded sequence. So (5) with coefficients as in (34) is an example of VDE with vanishing solution and nonsummable coefficients.

As a counterexample, consider (5) with coefficients given by

$$a_n = 1 + \frac{1}{2(n+1)}\left(1 + 2n - \frac{1}{(n+1)^2}\right), \quad b_{nj} = -e^{-(n-j)} - \frac{1}{(n+1)^2}. \tag{35}$$

Here condition (i) for the coefficients a_n and $b_{n,n}$ in Theorem 1 is violated and the boundedness of the solution of (5) is not guaranteed any more. In fact, this is clear in Figure 1, which shows the actual behaviour of x_n.

Theorem 6 can be applied to the following example:

$$a_n = -20, \quad x_0 = -10, \quad b_{n,j} = \begin{cases} -\frac{1}{2}\left((n+1)e^{-n} + \frac{1}{(n+1)^2}\right), & j = 0, \\ -\left((n+1-j)e^{-(n-j)} + \frac{1}{(n+1)^2}\right), & 1 \le j \le n. \end{cases} \tag{36}$$

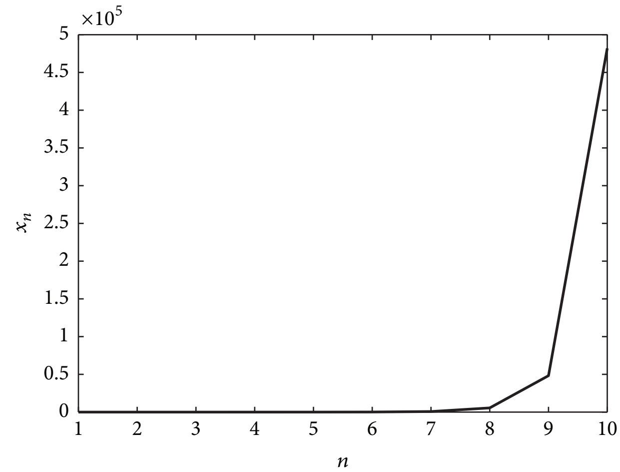

Figure 1: Problem (35): unbounded x_n.

First of all we need to show that (a) holds with $\overline{n} = 2$. Starting from the initial condition given in (36), by simple computation, we have $x_1 = -0.003 < 0$, $0 < x_2 < 1$. Our aim is to prove that $0 < x_n \le 1$, for $n \ge 2$. Let us proceed by induction on n. Assume $0 \le x_j \le 1$, $j = 2, \dots, n-1$ and verify that the same is true for x_n given by

$$\left(1 + 20 - b_{n,n}\right) x_n = b_{n,0} x_0 + b_{n,1} x_1 + \sum_{j=2}^{n-2} b_{n,j} x_j + \left(1 + b_{n,n-1}\right) x_{n-1}. \tag{37}$$

From the definition of $b_{n,j}$ in (36), it easily follows that $-4 < 1 + \sum_{j=0}^{n-1} b_{n,j} < 0$. Then taking into account the induction hypothesis and that $b_{n,n-1} \le -1$, we obtain

$$-4 < \sum_{j=0}^{n-2} b_{n,j} x_j + \left(1 + b_{n,n-1}\right) x_{n-1} < 0. \tag{38}$$

As $x_0 = -10$ and $x_1 < 0$, it turns out that the right-hand side of (37) is positive, then $x_n > 0$. On the other hand (38) implies $x_n < (b_{n,0} x_0 + b_{n,1} x_1)/21$, $n \ge 2$, with $b_{n,0} \le -1/2$, $b_{n,1} \le -1$, which assures $0 < x_n < 1$. We conclude that hypothesis (a) of Theorem 6 is satisfied. Since (d) is true and (c), (f) are obvious with $n^* = 0$, it remains to prove (e). In our case (e) corresponds to $\exists p \ge 0$ such that $\Delta_{12} b_{n,j} \ge 0$, for $n \ge 2 + p$ and $j = 0$. Observe that $\Delta_{12} b_{n,0}$ can be written as $\Delta_{12} b_{n,0} = b_{n+1,1} - b_{n,1} - 2(b_{n+1,0} - b_{n,0}) + \Delta_1 b_{n,0}$. So $\Delta_{12} b_{n,0} \ge 0$, $n \ge 2$, since (c) holds and $b_{n+1,1} - b_{n,1} - 2(b_{n+1,0} - b_{n,0}) = (e-1)e^{-(n+1)}(n(e-1) - 2)$. In conclusion, all the hypotheses of Theorem 6 are fulfilled and $x_n \to 0$, as can be seen in Figure 2.

Remark 9. We want explicitly to mention that Theorem 1 cannot be applied to (36) because $\Delta_{12} b_{n,n-1} = (e^2 - 4e + 3)/e^2 < 0$, $\forall n \ge 2$, and hypothesis (iv) is not satisfied.

Finally, we observe that, if in (36) we choose $b_{n,0}$ according to the remaining coefficients for $j \ne 0$, that is, $b_{n,0} = -((n+1)e^{-n} + 1/(n+1)^2)$, then Theorem 6 is still valid with $\overline{n} = 1$.

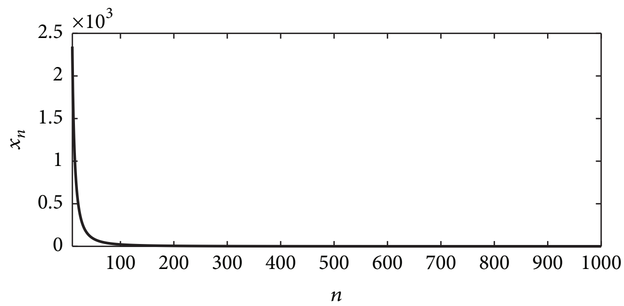

Figure 2: Problem (36): vanishing x_n.

So we are in the case of Remark 8 and the assumption (e) of Theorem 6 can be ignored.

A more practical application of our results is the study of the longtime behaviour of the numerical solution to VIDEs. Let us consider the homogeneous problem

$$y'(t) = A(t)\, y(t) + \int_0^t K(t,s)\, y(s)\, ds, \qquad t \geq 0, \quad y(0) = y_0, \tag{39}$$

and a simple method of family (4), the Backward Euler method (see [30], [12, (3.8)])

$$y_{n+1} - y_n = hA(t_{n+1})\, y_{n+1} + h^2 \sum_{j=1}^{n+1} K(t_{n+1}, t_j)\, y_j, \qquad n \geq 0, \tag{40}$$

where $h > 0$ is the stepsize. With the help of the results of the previous section we can prove the following.

Theorem 10. *Consider (40) and assume that*

(i) $\exists \bar{t}$ *such that* $A(t) \leq 0,\ t > \bar{t}$;

(ii) $K(t,0) \leq 0,\ t \geq 0,\ \partial K(t,0)/\partial t \geq 0,\ t > \bar{t}$;

(iii) $\partial K(t,s)/\partial s \leq 0,\ t > 0$;

(iv) $\partial^2 K(t,s)/\partial t \partial s \geq 0,\ t > \bar{t}$.

Then the solution y_n *of (40) is bounded* $\forall y_0 \in \mathbb{R}$. *If in addition*

(v) $A(t) \leq -A^* < 0,\ t > \bar{t}$,

or

($\mathrm{v_{bis}}$) $\partial^2 K(t,s)/\partial t \partial s \geq K^* > 0,\ t > \bar{t},\ s < \bar{t}$,

then $\lim_{n \to +\infty} y_n = 0$.

Proof. Note that (5) coincides with (40) whenever $a_n = hA(t_n),\ n \geq 1$, and $b_{nj} = h^2 K(t_n, t_j),\ n \geq 1,\ j \leq n$. Now, assumptions (i)–(iv) immediately assure (i)–(iv) of Theorem 1 for any fixed h and $\bar{n}(h)$ such that $\bar{n}(h)h = \bar{t}$. Moreover, (v) implies (v) of Theorem 1 with $a^* = hA^*$, so that Corollary 3 holds. In order to exploit ($\mathrm{v_{bis}}$) note that it is equivalent to $\partial^2 (K(t,s) - K^* ts)/\partial t \partial s > 0,\ t > \bar{t},\ s < \bar{t}$, which in turn implies that, for any fixed h, the function $\Gamma(t,s,h) = K(t+h,s) - K(t,s) - K^* ts$ is increasing with respect to s, so that ($\mathrm{v_{bis}}$) of Theorem 1 is fulfilled with $b^* = h^4 K^*$ and $\bar{n}$ given above. Once again all the hypotheses of Theorem 1 are satisfied and the desired result follows. □

Remark 11. Theorem 10 would be particularly interesting whenever it is known that, under the same hypotheses, also the analytical solution $y(t)$ of (39) goes to zero as t tends to infinity.

As we mentioned in the previous section, the analogue of Theorem 1 in the continuous case can be obtained following the line of the proof of Theorem 1 in [29]. The following is a reformulation of such a theorem suited to our case.

Theorem 12. *Assume that hypotheses (i)–(iv) of Theorem 10 are valid, then the solution* $y(t)$ *of (39) is bounded, for any choice of* $y_0 \in \mathbb{R}$. *Furthermore, if hypothesis (v) or* ($\mathrm{v_{bis}}$) *of Theorem 10 holds and*

$$y \text{ is uniformly continuous}, \tag{41}$$

then $\lim_{t \to +\infty} y(t) = 0$.

Remark 13. From Theorems 1 and 10 it is clear that the asymptotic behaviour of the solution to (39) is preserved both in a generic discretization of the kind (5), where b_{nj} represents the samples $K(t_n, t_j)$, and in the numerical solution obtained by the Backward Euler method (40).

Moreover, it is worth noting that if in (39)

$$\int_0^t |K(t,s)|\, ds < +\infty, \tag{42}$$

then assumption (41) is automatically verified; nevertheless in the discrete case, the summability of the coefficients, which is the analogous of (42), is not required as showed in example (34).

In the literature we sometimes encounter VIDEs with the following structure (see, e.g., [31]):

$$y'(t) = A(t)\, y(t) + \int_0^t f(t-s) \int_s^t y(\tau)\, d\tau\, ds, \qquad t \geq 0, \quad y(0) = y_0. \tag{43}$$

This kind of equation can be easily recast in the form (39), with $K(t,s) = \int_{t-s}^t f(\tau) d\tau$, and the method (40) for it will read

$$y_{n+1} - y_n = hA(t_{n+1})\, y_{n+1} + h^2 \sum_{i=1}^{n+1} \sum_{j=n+1-i}^{n+1} f(t_j)\, y_j, \qquad n \geq 0. \tag{44}$$

Theorem 10 immediately becomes as follows.

Corollary 14. *Consider (44) and assume that*

(a1) $\exists \bar{t} > 0 : A(t) \le 0,\ t > \bar{t}$,

(a2) $f(t) \le 0,\ t \ge 0$,

(a3) $f'(t) \ge 0,\ t \ge 0$,

(a4) $A(t) \le A^* < 0,\ t \ge \bar{t}$,

or

(a4_{bis}) $f'(t) \ge K^* > 0,\ t > 0$.

Then, for any $y_0 \in \mathbb{R},\ \lim_{n\to+\infty} y_n = 0$.

A comparison to Theorem 12 reveals that, contrarily to (iv), hypothesis (a3) is required to hold in the whole integration range. This is due to the fact that (see Remark 5 in case of (43)) $\partial^2 K(t-s)/\partial t\partial s$ is of convolution type. The application of Theorem 6 leads to the following result.

Theorem 15. *Consider (44) and assume that*

(HA) $\exists \bar{n} \ge 0 : y_n \ge 0\ (<0),\ n > \bar{n}$;

(HB) $\exists \tilde{t} > 0 : A(t) \le 0,\ t > \tilde{t}$;

(HC) $f(t) \le 0,\ t > 0$;

(HD) $f'(t) \ge 0,\ t > \tilde{t}$.

Then, for any $y_0 \in \mathbb{R}$, $\lim_{n\to+\infty} y_n = 0$.

Proof. We want to prove that all the hypotheses of Theorem 6 are fulfilled with $a_n = hA(t_n)$ and $b_{nj} = h^2 \sum_{i=n-j}^{n} f(t_i)$. Let $h > 0$ be fixed and let $\tilde{n} := \tilde{n}(h)$ be such that $\tilde{n}(h)h = \tilde{t}$. Hypothesis (a) of Theorem 6 is obviously true because of (HA). In order to prove (b), observe that (HB) together with (HC) assure that, for $n \ge n^* = \max\{\tilde{n}, \bar{n}\}$,

$$\beta_n = 2hA(t_n) + 2h^2 \sum_{j=0}^{n} f(t_j) - h^2 \sum_{j=n-\tilde{n}}^{n} f(t_j) < h^2 f(\tilde{t}) \le 0. \tag{45}$$

As $b_{n,0} = h^2 f(t_n)$, (HC) implies the first condition in (c), and since $\Delta_1 b_{n,0} = h^2(f(t_{n+1}) - f(t_n))$, (HD) immediately assures the second condition of (c). Furthermore, since $\Delta_2 b_{n,j} = h^2 f(t_{n-j-1})$, (HC) also implies (d) and, taking into account the fact that $\Delta_{1,2} b_{n+1,j} = h^2(f(t_{n+1-j}) - f(t_{n-j}))$, we have that (HD) implies (e) with $p = \max\{0, \tilde{n} - 2\}$. Finally, whenever f is not identically zero, we can always assume, with no loss of generality, that $\tilde{t}$ is such that $f(\tilde{t}) < 0$, so that (45) assures (f) with $A^* = -h^2 f(\tilde{t})$, and the proof is complete. □

As already mentioned above, in [31] equation (3.32) represents the velocity of the centre of mass of a system of N particles in collective motion under alignment and chemotaxis effect. Here $A(t) = 0$ and

$$f(t) = -\frac{c_1 e^{-c_2 t} e^{-c_3/t}}{t^2}, \tag{46}$$

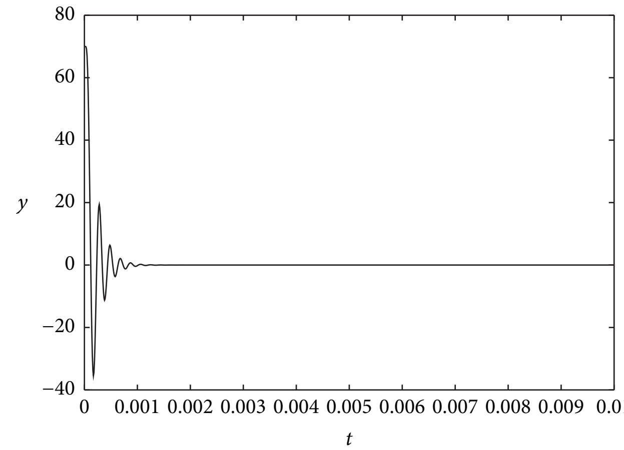

Figure 3: Problem (43)–(46): vanishing y_n.

with c_1, c_2, and c_3 depending on the number of particles, their dimension, and the dynamic of the motion. For significative values of c_1, c_2, and c_3 hypotheses (a1), (a2), (a3), and (a4_{bis}) of Theorem 15 are satisfied. Hence, we expect that in the numerical simulation of (43) obtained by using the Backward Euler method (44), any convergent numerical solution vanishes at infinity. Figure 3 shows exactly this behaviour when integrating (43)–(46) with $c_1 = 10^5,\ c_2 = 10^{-2}$, and $c_3 = 10^{-4}$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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