

Research Article

Spatial Dynamics for a Generalized Solow Growth Model

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The existence of nontrivial equilibrium and poverty traps for a generalized Solow growth model with concave and nonconcave production functions is investigated. The explicit solutions of the growth model, which is expressed by a differential equation with corresponding boundary conditions, are employed to illustrate the spatial dynamics of the model in different economic regions. Numerical method is used to justify the validity of the theoretical analysis.

1. Introduction

The distribution over space of spatial economic activities has been investigated in many literatures, in which economic geographers study how and why people make their location choices, consider the reasons of production agglomeration, and find the formation of cities and migration flows. Early regional economic growth models focus on capital, labor, pollution flows, individuals' welfare, and the policymaker both in discrete and continuous cases. In recent years, the new economic geography emerges in economic analysis, where economic geographers employ a refined specification of whole market structures and several precise assumptions on the mobility of production factors. Fujita and Thisse [1] and Krugman [2] use a general equilibrium framework to explain production, consumption, and price formation in local and global economy. Mossay [3] studies that continuous space is not incompatible with regional divergence in different migration schemes which are decided by idiosyncrasies in location taste. Krugman [4] states that the economy always displays regional convergence and migration follows utility-level differentials.

Since the assumption of a continuous space structure in economic model fits better modern economies, several continuous space extensions of economic models have been discussed. Brito [5] has investigated spatial capital accumulation and capital mobility in the spatial Ramsey growth framework. In [5], production and capital accumulation are distributed in continuous space and capital differentials drive

the spatial capital dynamics. La Torre et al. [6] investigate the optimal dynamics of capital and pollution in a spatial economic growth model. Camacho and Barahona [7] analyze spatial optimal land use and environmental degradation. Boucekkine et al. [8] and Camacho et al. [9] illustrate how to choose optimal trajectories of capital and consumption by maximizing an objective function in a spatial Ramsey model with continuous space. Industrial dynamics and economic geography over time and space are considered in Capasso et al. [10]. The relationship between the growth process and stable spatially nonhomogenous distribution for per capital and income is described by Xepapadeas and Yannacopoulos [11]. Boucekkine et al. [12] prove the characterization of the optimal dynamics in a spatial AK model across as a circle and Fabbri [13] investigates the generalized model with AK production function in a generic geographic structure. For other studies of spatial economic growth model, the reader is referred to [14–18] and the references therein.

The classical Solow model describes the evolution in time of gross output which is based upon the input factors, labor, capital, and technology (see [19]). In the Solow model, there exist multiple steady states when production function is not concave for any level of capital. Kamihigashi and Soy [20] establish an optimal discrete time problem when an s-shaped production function is assumed. Ferrara et al. [21] discuss the stability of the classical Solow model with delay and nonconvex technology. Brianzoni et al. [22] investigate the dynamics of a discrete growth model with nonconcave production function in local and global economy. Camacho

and Zou [23] illustrate the long-run structure for the spatial distribution of the capital and the spatially homogeneous steady state when capital movement is modeled by local diffusion and spatial dynamics. Neto and Claeysen [24] consider the stability of a spatial Solow model with labor mobility and prove that capital induced labor migration is a necessary condition for the spatiotemporal dynamics of the model. Capasso et al. [25] prove the steady state of the classical Solow model with a nonconcave production function and analyze the convergence of a spatial Solow model with technological diffusion. In our paper, we generalize the classical Solow model in [25] and obtain the steady states of the generalized Solow model in a geographic structure. In open regions, we prove the existence and uniqueness of solution for a partial differential equation with corresponding boundary conditions, which is different from convergence analysis in [25].

Precisely, the aim of this paper is to investigate the steady states of a generalized Solow model with both concave and nonconcave production functions and obtain the asymptotic properties of solutions for the model in continuous space and bounded time. Using the monotonic property of functions, we choose a function to show the steady state of the generalized Solow model with a concave production function. When production function is not concave, we introduce a nonconcave production function into the generalized Solow model to get the existence of nontrivial equilibrium and poverty traps. Our main contribution is to obtain the explicit solution of an ordinary differential equation which expresses the generalized Solow model in close regions and prove the existence and uniqueness of solution for a partial differential equation in open economy. The obtained result in our work shows the asymptotic capital distribution across space. This is different from those in [23], where solution of a spatial Solow model is given in global economy. On the proof of existence and uniqueness of solution for the generalized Solow model in open economy, we apply an approach used in Boucekkine et al. [8], in which a generalized Ramsey growth model is investigated in global economy.

This paper is organized as follows. Section 2 states a generalized Solow model with concave and nonconcave production functions. We prove the existence of nontrivial equilibrium and poverty traps when several conditions are satisfied and get the time path of capital per effective worker in close economic regions. Section 3 presents the generalized Solow model in open economy and we obtain the existence and uniqueness of the solution for the model. In Section 4, we use numerical method to analyze the solution of the corresponding partial differential equation with boundary conditions by discretizing the space variable. Our conclusions are presented in Section 5.

2. A Generalized Solow Model in Autarkic Regions

2.1. Steady States of the Model with Concave and Nonconcave Production Functions. Assume that there is only one final good signed to be consumed or invested in one economy market. The generalized Solow model describes the evolution

in time for gross output $Y(x, t)$ which depends upon the input factors, labor $L(x, t)$, capital $K(x, t)$, and technology $A(x, t)$ at work in location $x \in \Omega$. The typical mathematical expression linking the input variable is

$$Y(x, t) = A(x, t) F(K(x, t), L(x, t)), \quad (1)$$

where F is a production function which is continuous and twice differentiable. The usual production function satisfies constant returns to scale or linear homogeneity, which means that the marginal products depend only on the ratio K/L . All inputs satisfy

$$F(0, L) = F(K, 0) = 0. \quad (2)$$

The evolution of the physical capital over time is described by the following differential equation:

$$\frac{\partial}{\partial t} K(x, t) = Y(x, t) - C(x, t) - \eta K(x, t), \quad (3)$$

where the change of capital stock $(\partial/\partial t)K(x, t)$ is related to the saved quantity after consumption $C(x, t)$ and depreciation $\eta K(x, t)$. If a traction of the output is saved, we have

$$Y(x, t) - C(x, t) = sY(x, t). \quad (4)$$

We rewrite (3) in the form

$$\frac{\partial}{\partial t} K(x, t) = sA(x, t) F(K(x, t), L(x, t)) - \eta K(x, t). \quad (5)$$

Using a new variable $k(x, t) = K(x, t)/L(x, t)$, we obtain

$$\frac{1}{L(x, t)} F(K(x, t), L(x, t)) = F(k(x, t), 1) \quad (6)$$

and

$$\begin{aligned} \frac{\partial}{\partial t} k(x, t) &= \frac{\partial}{\partial t} \left(\frac{K(x, t)}{L(x, t)} \right) \\ &= \frac{L(x, t) (\partial/\partial t) K(x, t) - K(x, t) (\partial/\partial t) L(x, t)}{L^2(x, t)} \\ &= \frac{1}{L(x, t)} \frac{\partial}{\partial t} K(x, t) - n(x, t) \frac{K(x, t)}{L(x, t)}, \end{aligned} \quad (7)$$

where $n(x, t) = (1/L(x, t))(\partial/\partial t)L(x, t)$ denotes the constant growth rate of the labor input. From (5) and (7), we have the following generalized Solow model:

$$\frac{\partial}{\partial t} k(x, t) = sA(x, t) f(k(x, t)) - (\eta + n)k(x, t), \quad (8)$$

where $f(k(x, t)) \equiv F(k(x, t), 1)$, $n(x, t) = n$ and $k(x, t)$ denotes the capital stock held by the representative household located at x and $t, x \in \Omega \subset R, t \geq 0$.

The hypothesis of a concave production function has played a crucial role in many economic growth models based on intertemporal allocation. It describes the maximum output for all possible combinations of input factors and determines the way that the economic model evolves in time. Usually, a production function $f(k)$ fulfills the so-called Inada conditions (see [26]), if $f(k)$ satisfies the following assumptions:

- (A₁) $f(k)$ is nonnegative, increasing, and concave;
 (A₂) $\lim_{k \rightarrow 0} f'(k) = +\infty$, $\lim_{k \rightarrow \infty} f'(k) = 0$, $f(0) = 0$.

Condition (A₂) implies that the marginal productivity of capital is only related to the distribution of the per capital stock of capital. It is possible to get high returns when one invests only a small amount of money. Obviously, this is not realistic. Even if a basic structure is established for production, one might still get small returns when the returns increase to the point where the law of diminishing returns takes effect. This fact is known as poverty traps (see [27]).

By the change of capital per worker, or to be more precise, one conclusion of (8) shows that the capital per worker can reach its steady state described in the following proposition.

Proposition 1. *Assume that $A(x, t) = 1$ and $f(k)$ is a concave production function. At location $x_0 \in \Omega$, if $\eta + n \in (0, 1)$ and $s \in (0, 1)$, (8) has a unique steady state \bar{k} which is given by*

$$\bar{k} = \varphi^{-1} \left(\frac{\eta + n}{s} \right), \quad (9)$$

where $\varphi(k) = f(k)/k$.

Proof. Equation (8) reaches its steady state if and only if

$$0 = sf(k) - (\eta + n)k. \quad (10)$$

Let

$$\varphi(k) = \frac{f(k)}{k} = \frac{\eta + n}{s}. \quad (11)$$

The function φ gives the output-to-capital ratio in the economy. Since f is continuous, decreasing, and satisfies the Inada conditions, we have

$$\varphi'(k) = \frac{f'(k)k - f(k)}{k^2} = -\frac{F_L}{k^2} \quad (12)$$

and

$$\varphi(0) = f'(0) = +\infty, \quad (13)$$

$$\varphi(\infty) = f'(\infty) = 0.$$

Since $\varphi'(k) < 0$, the function $\varphi(k) \equiv f(k)/k$ is a decreasing function, which means that (11) has a solution and the solution is unique. Therefore, the steady state of the economy is unique and given by

$$\bar{k} = \varphi^{-1} \left(\frac{\eta + n}{s} \right). \quad (14)$$

□

Remark 2. The steady state of the economy in Proposition 1 can be described in other words. That is to say, if there is no technological progress and exponential labor growth, the rate of capital per worker is determined by three variables: saving rate s , depreciation rate η , and growth rate of labor input n . Moreover, the capital per worker increases whenever $sf(k) > (\eta + n)k$ and decreases otherwise.

If $f(\cdot)$ is a Cobb-Douglas production function, that is, $f(k) = k^\alpha$, the economy always converges to the so-called steady state where the capital per capita is constant. Namely, there is just enough money saved to cover the population growth and the amount of capital lost through depreciation. Thus, in the short term, the economy grows faster and both the output per capita and the capital per capita increase. In the long term, the economy tends to a new steady state and the economic growth rate eventually is n again. Moreover, the saving rate has no effect on the growth rate of the economy. Therefore, the main consequence of the spatial Solow model is that the only way to obtain enduring economic growth is to assume a nonconstant technological progress.

When $f(\cdot)$ is not concave for any level of capital, the economy is history-dependent and the poverty traps exist in the long-run structure. Using the similar method in [25], we consider a nonconcave production function

$$f(k) = \frac{k^p}{\gamma_1 + \gamma_2 k^p}, \quad (15)$$

where all involved parameters are nonnegative and $p \geq 2$.

We focus on the existence of a poverty trap for (8) at location $x_0 \in \Omega$. For simplicity, we assume $A = 1$, $s \in (0, 1)$, and $\eta + n = \delta \in (0, 1)$. If $sf(k^*) - \delta k^* = 0$, we have a nontrivial equilibrium k^* , with $k^* \neq 0$. That is,

$$\frac{s(k^*)^p}{\gamma_1 + \gamma_2 (k^*)^p} - \delta k^* = 0. \quad (16)$$

The following proposition states the existence of a poverty trap if

$$\delta < \bar{\delta} = \frac{s[\gamma_1(p-1)/\gamma_2]^{(p-1)/p}}{\gamma_1 p}. \quad (17)$$

Proposition 3. *At location $x_0 \in \Omega$, if $\delta < s[\gamma_1(p-1)/\gamma_2]^{(p-1)/p}/\gamma_1 p$, (8) shows two nontrivial equilibria k_1^* and k_2^* , with $k_1^* < k_2^*$. The economy is unstable at k_1^* while it is stable at k_2^* .*

Proof. From (16), we have

$$\delta = \frac{s(k^*)^{p-1}}{\gamma_1 + \gamma_2 (k^*)^p}. \quad (18)$$

Let $\varphi(k) = s(k)^{p-1}/(\gamma_1 + \gamma_2(k)^p)$. Using the basic notions of calculus, when $\varphi'(\bar{k}^*) = 0$, we obtain the maximum of the function $\varphi(k)$ at point $\bar{k}^* = [\gamma_1(p-1)/\gamma_2]^{1/p}$. Then the optimal value is

$$\bar{\delta} = \varphi(\bar{k}^*) = \frac{s[\gamma_1(p-1)/\gamma_2]^{(p-1)/p}}{\gamma_1 p}. \quad (19)$$

If $\delta < \bar{\delta}$, there exist two equilibria k_1^* and k_2^* , with $k_1^* < [\gamma_1(p-1)/\gamma_2]^{1/p} < k_2^*$. We come back to (16). Let

$$g(k) := \frac{sk^p}{\gamma_1 + \gamma_2 k^p} - \delta k. \quad (20)$$

We have the first derivative of g , which is given by

$$g'(k) = \frac{\gamma_1 p s k^{p-1}}{(\gamma_1 + \gamma_2 k^p)^2} - \delta. \quad (21)$$

We verify that $g'(k_1^*) > 0$ and $g'(k_2^*) < 0$, which means that the economy is unstable at k_1^* while it is stable at k_2^* . \square

Remark 4. Proposition 3 shows that when the level of physical capital surpasses the threshold k_1^* , it converges certainly to a nontrivial steady state k_2^* , which is similar to those of the classical Solow model in [25].

2.2. Asymptotic Analysis. Assume that all regions are closed and there is are capital flows among the regions, which means that real transfers of goods between regions cannot be financed and there is no trade between regions. Furthermore, a mathematical representation of the assumption is that all locations have access to goods in modern economy (see [1]). Since there are no arbitrage opportunities in an autarkic region (see [5]), we write the regional balance equation as

$$\int_{\Omega} \left\{ \frac{\partial k(x, t)}{\partial t} - sA(x, t) f(k(x, t)) + \eta k(x, t) \right\} dx = 0, \quad \forall (x, t) \in \Omega \times [0, T], \quad (22)$$

where we assume $L(x, t) = L$ for all t , which implies $n = 0$ and the capital stock depreciates at a fixed rate η . We set $s = 1$ for simplicity. The production function and the technology affect the economic growth. According to the regional balance equation, the instantaneous budget constraint of household at $x \in \Omega$ is written in the form

$$\frac{\partial k(x, t)}{\partial t} = A(x, t) f(k(x, t)) - \eta k(x, t), \quad \forall (x, t) \in \Omega \times [0, T]. \quad (23)$$

In addition to (23), we assume that the initial capital distribution $k(x, 0)$ is known and that there is no capital flow through the boundary $\partial\Omega$. Since there is no capital flow if the location is far away from the origin, we have

$$\lim_{x \rightarrow +\infty} \frac{\partial k(x, t)}{\partial x} = 0. \quad (24)$$

We write the initial problem of (23) in the form

$$\frac{\partial k(x, t)}{\partial t} = A(x, t) f(k(x, t)) - \eta k(x, t), \quad (x, t) \in \Omega \times [0, T], \quad (25)$$

$$k(x, 0) = k_0(x), \quad x \in \Omega.$$

In order to present the solution of problem (25), we denote $k(t)(x) \equiv k(x, t)$. Then, we have

$$k_t(t) = A(x, t) f(k(t)) - \eta k(t), \quad (x, t) \in \Omega \times [0, T], \quad (26)$$

$$k(0) = k_0, \quad x \in \Omega,$$

where $k_t(t) = dk(t)/dt$. Therefore, problem (26) is equivalent to the standard one-dimensional Solow model.

Theorem 5. *Suppose that A is a positive constant. If $f(k(t)) = k^\alpha(t)$ is a Cobb-Douglas production function, there exists the time path of capital per effective worker to (26).*

Proof. By a change of variables $w(t) = k^{1-\alpha}(t)$, we transform (26) into the following linear differential equation:

$$w_t(t) + (1 - \alpha)\eta w(t) = (1 - \alpha)A. \quad (27)$$

Using the knowledge of ordinary differential equation, we get the mild solution (see [12]) of (27), which is given by

$$\begin{aligned} w(t) &= e^{(\alpha-1)\eta t} w(0) + \int_0^t e^{(t-s)(\alpha-1)\eta} (1 - \alpha) A ds \\ &= e^{(\alpha-1)\eta t} w(0) + (1 - \alpha) A e^{(\alpha-1)\eta t} \int_0^t e^{(1-\alpha)\eta s} ds \\ &= e^{(\alpha-1)\eta t} w(0) + \frac{A}{\eta} e^{(\alpha-1)\eta t} [e^{(1-\alpha)\eta t} - 1] \\ &= e^{(\alpha-1)\eta t} \left(w(0) - \frac{A}{\eta} \right) + \frac{A}{\eta}. \end{aligned} \quad (28)$$

Combining the initialization $w(0) = k^{1-\alpha}(0)$ into (28), we obtain

$$k(t) = \left[\left(k^{1-\alpha}(0) - \frac{A}{\eta} \right) e^{(\alpha-1)\eta t} + \frac{A}{\eta} \right]^{1/(1-\alpha)}, \quad (29)$$

which shows the time path of capital per effective worker in autarkic regions. \square

Remark 6. Assume that r is an exogenous interest rate with $\lambda = r + \eta$. We get the lifetime budget constraint expressed in present discounted value form for agent if (23) is multiplied by e^{-rt} . Assume that all debts are required to be paid off. Integrating over t from $t = 0$ to ∞ , we obtain a reformulation of the instantaneous budget constraint in the static form

$$\int_0^\infty e^{-rt} \{k_0 + A(x, t) f(k(x, t)) - \lambda k(x, t)\} dt = 0. \quad (30)$$

Remark 7. Equation (30) shows an investment optimization problem which is required to characterize the spatial structure of the capital stock. When there exists quadratic adjustment cost denoted by $(\alpha/2)[\partial k(x, t)/\partial t]^2$ and $A(x, t) = 1$, Brock et al. [28] investigates the following investment optimization problem:

$$\begin{aligned} \max_{k'(x, \cdot)} \int_0^\infty e^{-rt} \left\{ f(k(x, t), k^e(x, t)) - \lambda k(x, t) \right. \\ \left. - \frac{\alpha}{2} \left[\frac{\partial k(x, t)}{\partial t} \right]^2 \right\} dt, \quad \forall x \in \Omega, \end{aligned} \quad (31)$$

where $k^e(x, t)$ denotes the geographical spillovers.

3. A Generalized Solow Growth Model in Open Regions

When capital and goods flow among open regions and there are no (intertemporal) adjustment costs, the aggregate balance equation for region Ω becomes

$$\int_{\Omega} \left\{ \frac{\partial k(x, t)}{\partial t} - A(x, t) f(k(x, t)) + \eta k(x, t) + i(x, t) \right\} dx = 0, \quad (32)$$

where $i(x, t) \neq 0$ is the household's net trade balance of household x at time t . The trade is matched by reallocations of capital among regions in a centralized economy. When a decentralized equilibrium is set, we assume that there exists an interspatial capital market where stocks are traded. From (32), for a region Ω , the budget constraint follows

$$\frac{\partial k(x, t)}{\partial t} = A(x, t) f(k(x, t)) - \eta k(x, t) - i(x, t), \quad (33)$$

$$(x, t) \in \Omega \times [0, T].$$

In order to eliminate interregional arbitrage opportunities, we consider that capital always flows from regions with low marginal productivity of capital to regions with high marginal productivity of capital. Since $i(x, t)$ is symmetry of capital account balance, when there are no institutional barriers to capital flows and the regions are internally homogeneous, the current account balance $i(x, t)$ is measured by the symmetry of the difference of capital intensities with the adjacent regions

$$\int_{\Omega} i(x, t) dx = - \left[\frac{\partial k(x_i + \Delta x_i, t)}{\partial x} - \frac{\partial k(x_i, t)}{\partial x} \right] \quad (34)$$

$$= - \int_{x_i}^{x_i + \Delta x_i} \frac{d}{dx} \left(\frac{\partial k}{\partial x} \right) dx$$

See [8], that is,

$$i(x, t) = - \frac{d}{dx} \left(\frac{\partial k}{\partial x} \right) \quad (35)$$

$$= - \lim_{\Delta x \rightarrow 0} \frac{k_x(x + \Delta x, t) - k_x(x, t)}{\Delta x}$$

$$= - \frac{\partial^2 k(x, t)}{\partial x^2}.$$

Substituting (35) into (32), we have

$$\int_{\Omega} \left\{ \frac{\partial k(x, t)}{\partial t} - \frac{\partial^2 k(x, t)}{\partial x^2} - A(x, t) f(k(x, t)) + \eta k(x, t) \right\} dx = 0 \quad (36)$$

for $\forall t \geq 0$.

If we assume that the space interval is sufficiently large and there is no capital flow, we have the following Neumann boundary condition:

$$\frac{\partial k}{\partial x}(x, t) = 0, \quad \partial\Omega \times [0, T]. \quad (37)$$

Then, the budget constraint (36) is written in the form

$$\frac{\partial k(x, t)}{\partial t} - \frac{\partial^2 k(x, t)}{\partial x^2} = A(x, t) f(k(x, t)) - \eta k(x, t), \quad \Omega \times [0, T], \quad (38)$$

$$\frac{\partial k}{\partial x} = 0, \quad \partial\Omega \times [0, T],$$

$$k(x, 0) = k_0(x) > 0, \quad x \in \Omega.$$

System (38) describes the distribution of capital for the generalized Solow model in open economic regions. In recent economic analysis, Boucekkine et al. [8] consider the solutions of a generalized Ramsey growth model in global economy. Following the method of [8], we investigate the existence and uniqueness of solution for system (38) in the next theorem.

Theorem 8. *Suppose (A_1) and (A_2) hold. For any given finite T , if there exist positive constants a and b with $0 < k_0(x) < ae^{b|x^2|}$, then, problem (38) has a unique solution $k(x, t) \in C^{2,1}(\Omega \times [0, T])$, which is given by*

$$k(x, t) = \int_{\Omega} k_0(\xi) G(x, \xi, t) d\xi \quad (39)$$

$$+ \int_0^t \int_{\Omega} (Af(k(\xi, \tau)) - \eta k(\xi, \tau)) \cdot G(x, \xi, t - \tau) d\xi d\tau,$$

where

$$G(x, \xi, t) = \begin{cases} \frac{1}{2\sqrt{\pi t}} \exp\left[-\frac{(x - \xi)^2}{4t}\right], & t > 0, \\ 0, & t \leq 0. \end{cases} \quad (40)$$

Proof. Defining a sequence $\{k^{(n)}\}, n \geq 1$, we have the following iteration process:

$$\Delta k^{(n)} = k_t^{(n)} - k_{xx}^{(n)} \quad (41)$$

$$= Af(k^{(n-1)}(x, t)) - \eta k^{(n-1)}(x, t),$$

$$(x, t) \in \Omega \times [0, T],$$

$$k^{(n)}(x, 0) = k_0(x), \quad x \in \Omega,$$

and the vectors

$$\begin{aligned} U_h(t) &= (u_j(t)), \\ F_h(U_h(t)) &= f(u_j(t)), \\ j &= 0, \dots, N, \end{aligned} \quad (56)$$

we rewrite (52) in the form

$$\frac{dU_h}{dt}(t) - \frac{1}{\eta L^2} B U_h(t) = \frac{A(t)}{\eta} F_h(U_h(t)) - U_h(t). \quad (57)$$

In order to obtain the time discretization, we choose a grid on the interval $[0, T]$ by $t_k = \tau k$, where $\tau = T/m$ and $k = 0, \dots, m$, and denote the values by approximating $U_h(t)$ at the discrete point of time. Using a backward difference quotient, we approximate the time derivative in the form

$$\frac{dU_{h,\tau}}{dt}(t_k) \approx \frac{1}{\tau} (U_{h,\tau}(t_k) - U_{h,\tau}(t_{k-1})). \quad (58)$$

Then, we obtain

$$\begin{aligned} [U_{h,\tau}(t_k) - U_{h,\tau}(t_{k-1})] - \frac{\tau}{\eta L^2} B U_{h,\tau}(t_k) \\ = \frac{\tau A_j(t_k)}{\eta} F_h(U_{h,\tau}(t_k)) - \tau U_{h,\tau}(t_k), \end{aligned} \quad (59)$$

that is,

$$\begin{aligned} U_{h,\tau}(t_k) + \tau U_{h,\tau}(t_k) - \frac{\tau}{\eta L^2} B U_{h,\tau}(t_k) \\ = \frac{\tau A_j(t_k)}{\eta} F_h(U_{h,\tau}(t_k)) + U_{h,\tau}(t_{k-1}). \end{aligned} \quad (60)$$

Defining

$$F'_h(U_{h,\tau}(t_k)) = \text{diag} \left(\frac{df}{du}(u_j(t_k)) \right), \quad (61)$$

where

$$\begin{aligned} f(u_j(t_k)) &\approx f(u_j(t_{k-1})) \\ &+ \frac{df}{du}(u_j(t_{k-1})) (u_j(t_k) - u_j(t_{k-1})) \end{aligned} \quad (62)$$

for all $j = 0, \dots, N$ and $k = 1, \dots, m$, we write (60) in the form

$$\begin{aligned} \left[I + \tau I - \frac{\tau}{\eta L^2} B - \frac{\tau A_j(t_k)}{\eta} F'_h(U_{h,\tau}(t_{k-1})) \right] U_{h,\tau}(t_k) \\ = \frac{\tau A_j(t_k)}{\eta} [F_h(U_{h,\tau}(t_{k-1})) \\ - F'_h(U_{h,\tau}(t_{k-1})) U_{h,\tau}(t_{k-1})] + U_{h,\tau}(t_{k-1}), \end{aligned} \quad (63)$$

where $I \in R^{N+1 \times N+1}$ is the identity matrix. From (63), we obtain

$$U_{h,\tau}(t_k) = \bar{B}^{-1} D, \quad (64)$$

where

$$\bar{B} = I + \tau I - \frac{\tau}{\eta L^2} B - \frac{\tau A_j(t_k)}{\eta} F'_h(U_{h,\tau}(t_{k-1})) \quad (65)$$

and

$$\begin{aligned} D = \frac{\tau A_j(t_k)}{\eta} [F_h(U_{h,\tau}(t_{k-1})) \\ - F'_h(U_{h,\tau}(t_{k-1})) U_{h,\tau}(t_{k-1})] + U_{h,\tau}(t_{k-1}). \end{aligned} \quad (66)$$

5. Conclusions

In this paper, we have investigated a generalized Solow model with continuous space and bounded time. Introducing concave and nonconcave production functions into the generalized Solow model in close regions, we get the steady states of the model when several conditions are satisfied. The asymptotic properties of solutions for the generalized Solow model are proved. We obtain the explicit time path of capital per effective worker by solving an ordinary differential equation in close regions and prove the existence and uniqueness of the solution for the generalized Solow model in open regions. The obtained results show the asymptotic capital distribution across space. Discretizing the space variable, we employ numerical method for the system of partial differential equation to justify the validity of the theoretical analysis.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The article is a joint work of two authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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