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Research Article

Pricing Option with Stochastic Interest Rates and Transaction Costs in Fractional Brownian Markets

Lina Song[1] **and Kele Li**[2]

[1]*School of Mathematics, Dongbei University of Finance and Economics, Dalian 116025, China*
[2]*School of Economics, Dongbei University of Finance and Economics, Dalian 116025, China*

Correspondence should be addressed to Lina Song; l_n_song@163.com

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This work deals with European option pricing problem in fractional Brownian markets. Two factors, stochastic interest rates and transaction costs, are taken into account. By the means of the hedging and replicating techniques, the new equations satisfied by zero-coupon bond and the nonlinear equation obeyed by European option are established in succession. Pricing formulas are derived by the variable substitution and the classical solution of the heat conduction equation. By the mathematical software and the parameter estimation methods, the results are reported and compared with the data from the financial market.

1. Introduction

The concept of fractal penetrates into every corner of life and exercises tremendous influence over scientific researches. Fractal theory has existed in many science fields, such as physics, seismology, biology, economics, and finance and even in social science. Fractal theory has exhibited important values and opened new research topics. Peters [1] put forward the fractal market hypothesis and employed R/S analysis method to prove the existence of fractal structure in financial market. Fractal market hypothesis based on the nonlinear dynamical systems explains multiple phenomena that cannot be achieved by the efficient markets hypothesis, such as the long memory, self-similarity, and scaling invariance of the stock returns. As an extension of the effective market, the fractal market has been widely accepted and provides a new theoretical environment. Under such market, various pricing theories and methods arise, and then the models close to the real market are constructed. The extension and application of the pricing models for financial derivative are still central issue for research scholars and experts.

Fractional Brownian motion with self-similarity and non-stationarity is a forceful tool in fractal market, in which Hurst exponent is a measure for the chaos and fractal character of financial market. The fractional Brownian motion was first introduced by Kolmogorov [2] in 1940, which is a pioneering work. In a fractal market, the fractional Black-Scholes models [3, 4] are deduced by replacing the standard Brownian motion involved in the classical model with fractional Brownian motion. Chen et al. [5] derived a mixed fractional-fractional version of the Black-Scholes model and gave simultaneously the corresponding Itô's formula and then obtained the option pricing formulas. Afterwards, Sun [6] presented the currency options model in the mixed fractional Brownian market and proved the reasonableness of the model by empirical studies. Ballestra et al. [7] priced the barrier options under the mixed fractional Brownian motion and they [8] gave a numerical method to compute the first-passage probability density function in a time-changed Brownian model. Further, stochastic interest rates and transaction costs are added to the fractional models. For the models with stochastic interest rates, Zhang et al. [9] obtained European option pricing model and the pricing formula in fractional Brownian motion. Xu [10] gave European option pricing formula using the mixed fractional Brownian motion assuming that the risk-free interest rate satisfies the Vasicek model. The existence of transaction costs will directly affect the hedging portfolios and the option price. The option pricing models containing transaction costs have sprung up rapidly, since transaction costs were introduced by Leland [11] in 1985. Under the fractional Brownian motion

environment, Wang [12] studied the problem of discrete time option pricing model with transaction costs and the series of achievements were made [13, 14]. Gu et al. [15] presented a fractional subdiffusive Black-Scholes model to handle the option problems. Liu et al. [16] gave an approximation to Hoggard-Whalley-Wilmott equation and then a pricing formula for the European option with transaction costs was obtained. Xiao et al. [17] used the subfractional Brownian motion to construct the warrants pricing model with transaction costs. Shokrollahi et al. [18] obtained a new formula for option pricing with transaction costs in a discrete time setting under fractional Brownian motion.

In the fractional market, the literatures that care simultaneously about stochastic interest rates and transaction costs are not so much. The work focuses on European option pricing with stochastic interest rates derived by fractional Vasicek model and transaction costs and tries to explain the models using data from the national debt reverse repurchase and European option. The paper is organized as follows. In Section 2, the new pricing models of zero-coupon bond and European option are established, and then the corresponding pricing formulas are derived. In Section 3, the nonlinear European option model is tested by the data in real market. Conclusions and discussions are presented in Section 4.

2. Fractional Black-Scholes Model

In a fractional Brownian market, the following assumptions are made in financial market with transaction costs and stochastic interest rates:

(I) The price S_t of the underlying asset follows the fractional exponential equation

$$\delta S_t = r_t S_t \delta t + \sigma_1 S_t \delta B_H^1(t), \tag{1}$$

where r_t denotes the risk-free rate of interest and σ_1 is the volatility of the asset price. H is the Hurst exponent. $B_H^1(t)$ is a fractional Brownian motion with Hurst exponent H and obeys the following proposition.

Proposition 1 (see [19]). *If $B_H(t)$ is a fractional Brownian motion with Hurst exponent $H \in (0,1)$, then, for any given $A > 0$, one has*

$$\lim_{h\longrightarrow 0} \sup_{0\le t\le A-h} \frac{\left|B_H(t+h) - B_H(t)\right|}{h^H \sqrt{2\log(h/A)^{-1}}} = 1. \tag{2}$$

(II) The risk-free interest rate r subjects to the fractional Vasicek equation

$$\delta r_t = a(b - r_t)\delta t + \sigma_2 \delta B_H^2(t), \tag{3}$$

where a, b, and σ_2 denote the speed of reversion, the long-term mean level, and the volatility of the interest rate. $B_H^2(t)$ is also a fractional Brownian motion with Hurst exponent H. And the correlation coefficient between $\{B_H^1(t), t \ge 0\}$ and $\{B_H^2(t), t \ge 0\}$ is ρ; namely,

$$\operatorname{cov}\left(\delta B_H^1(t), \delta B_H^2(t)\right) = \rho(\delta t)^{2H}. \tag{4}$$

(III) Transaction cost is the fixed proportion c of the trading amount for the underlying asset; namely,

$$Cost = cS_t\left|\nu_t\right|, \tag{5}$$

where ν_t denotes the shares of the underlying asset which are bought ($\nu_t > 0$) or sold ($\nu_t < 0$) at the price S_t.

(IV) The portfolio is revised at the time δt, where δt is a small and fixed time-step.

(V) The expected return of the hedge portfolio is suggested to satisfy the equality

$$E(\delta\Pi_t) = r_t \Pi_t \delta t. \tag{6}$$

Based on assumptions $(I) - (V)$, the pricing problem of zero-coupon bond and European option is discussed in the next sections.

2.1. Pricing Zero-Coupon Bonds

Theorem 2. *Under the fractional Vasicek model, the zero-coupon bond obeys the following equation:*

$$\frac{\partial P}{\partial t} + [a(b-r) - \theta\sigma]\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2(\delta t)^{2H-1}\frac{\partial^2 P}{\partial r^2} - rP = 0. \tag{7}$$

Proof. Two different zero-coupon bonds are employed to construct the hedge portfolios:

$$\Pi_t = P_1 - \Delta P_2. \tag{8}$$

With Taylor's theorem, one can obtain

$$\begin{aligned}\delta P &= \frac{\partial P}{\partial t}\delta t + \frac{\partial P}{\partial r}\delta r + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\left(\delta r^2\right) + o\left((\delta t)^2\right)\\ &= \left(\frac{\partial P}{\partial t} + [a(b-r)]\frac{\partial P}{\partial r}\right)\delta t + \sigma\frac{\partial P}{\partial r}\delta B_H(t)\\ &\quad + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2}\left(\delta B_H(t)\right)^2 + O\left(\delta t\cdot\delta B_H(t)\right),\end{aligned} \tag{9}$$

where $\sigma = \sigma_2$, which is the volatility of the interest rate.

Then one has

$$\begin{aligned}\delta\Pi_t = \delta P_1 - \Delta\delta P_2 &= \left[\left(\frac{\partial P_1}{\partial t} + [a(b-r)]\frac{\partial P_1}{\partial r}\right)\delta t\right.\\ &\quad \left. + \sigma\frac{\partial P_1}{\partial r}\delta B_H(t) + \frac{1}{2}\sigma^2\frac{\partial^2 P_1}{\partial r^2}\left(\delta B_H(t)\right)^2\right]\\ &\quad - \Delta\left[\left(\frac{\partial P_2}{\partial t} + [a(b-r)]\frac{\partial P_2}{\partial r}\right)\delta t + \sigma\frac{\partial P_2}{\partial r}\delta B_H(t)\right.\\ &\quad \left. + \frac{1}{2}\sigma^2\frac{\partial^2 P_2}{\partial r^2}\left(\delta B_H(t)\right)^2\right] + O\left(\delta t\cdot\delta B_H(t)\right).\end{aligned} \tag{10}$$

Taking $\Delta = (\partial P_1/\partial r)/(\partial P_2/\partial r)$, one has

$$\delta\Pi_t = \left[\left(\frac{\partial P_1}{\partial t} + [a(b-r)]\frac{\partial P_1}{\partial r}\right)\delta t\right.$$

$$+\frac{1}{2}\sigma^2\frac{\partial^2 P_1}{\partial r^2}\left(\delta B_H(t)\right)^2\Bigg]$$
$$-\Delta\left[\left(\frac{\partial P_2}{\partial t}+[a(b-r)]\frac{\partial P_2}{\partial r}\right)\delta t\right.$$
$$\left.+\frac{1}{2}\sigma^2\frac{\partial^2 P_2}{\partial r^2}\left(\delta B_H(t)\right)^2\right]+O\left(\delta t\cdot\delta B_H(t)\right). \tag{11}$$

Based on [20], the nonarbitrage pricing principle tells that

$$E\left(\delta\Pi_t\right)=r\left(P_1-\Delta P_2\right)\delta t. \tag{12}$$

Hence, we get

$$\left[\left(\frac{\partial P_1}{\partial t}+[a(b-r)]\frac{\partial P_1}{\partial r}\right)\delta t+\frac{1}{2}\sigma^2\frac{\partial^2 P_1}{\partial r^2}(\delta t)^{2H}\right]$$
$$-\Delta\left[\left(\frac{\partial P_2}{\partial t}+[a(b-r)]\frac{\partial P_2}{\partial r}\right)\delta t+\frac{1}{2}\sigma^2\frac{\partial^2 P_2}{\partial r^2}(\delta t)^{2H}\right]$$
$$=r\left(P_1-\Delta P_2\right)\delta t. \tag{13}$$

Equation (13) can be rewritten as

$$\left(\frac{\partial P_1}{\partial t}+[a(b-r)]\frac{\partial P_1}{\partial r}+\frac{1}{2}\sigma^2\frac{\partial^2 P_1}{\partial r^2}(\delta t)^{2H-1}-rP_1\right)$$
$$-\Delta\left(\frac{\partial P_2}{\partial t}+[a(b-r)]\frac{\partial P_2}{\partial r}+\frac{1}{2}\sigma^2\frac{\partial^2 P_2}{\partial r^2}(\delta t)^{2H-1}\right. \tag{14}$$
$$\left.-rP_2\right)=0.$$

And it is equivalent to

$$\frac{\partial P_1/\partial t+[a(b-r)]\left(\partial P_1/\partial r\right)+(1/2)\sigma^2(\delta t)^{2H-1}\left(\partial^2 P_1/\partial r^2\right)-rP_1}{\partial P_1/\partial r}$$
$$=\frac{\partial P_2/\partial t+[a(b-r)]\left(\partial P_2/\partial r\right)+(1/2)\sigma^2(\delta t)^{2H-1}\left(\partial^2 P_2/\partial r^2\right)-rP_2}{\partial P_2/\partial r}. \tag{15}$$

Introducing the market price of risk θ and assuming that

$$\frac{\partial P/\partial t+[a(b-r)]\left(\partial P/\partial r\right)+(1/2)\sigma^2(\delta t)^{2H-1}\left(\partial^2 P/\partial r^2\right)-rP}{\partial P/\partial r}=\theta\sigma, \tag{16}$$

one can derive (7).

When $H=1/2$, the zero-coupon bond model in a standard Brownian market has been studied in [20]. Based on the standard pricing formula, one can obtain the following conclusion. □

Theorem 3. *The zero-coupon bond model with the terminal condition $P(r,T;T)=1$ can derive the following formula:*

$$P(r,t;T)=\exp\left(-rB(r,T)-A(t,T)\right), \tag{17}$$

where

$$A(t,T)=\left(b-\frac{\theta}{a}\right)\sigma(T-t)-\left(b-\frac{\theta}{a}\right)B(T-t)$$
$$-\frac{1}{2}\sigma^2(\delta t)^{2H-1}\int_t^T B^2(s,T)\,ds, \tag{18}$$
$$B(t,T)=\frac{1}{a}\left(1-\exp\left(-a(T-t)\right)\right).$$

2.2. Pricing European Option

Theorem 4. *In fractional Brownian markets, option pricing model with stochastic interest rates and transaction costs satisfies the following equation:*

$$\frac{\partial V}{\partial t}+\left[a(b-r)-\theta\sigma_2\right]\frac{\partial V}{\partial r}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma_1^2S^2\frac{\partial^2 V}{\partial S^2}(\delta t)^{2H-1}$$
$$+\frac{1}{2}\sigma_2^2\frac{\partial^2 V}{\partial r^2}(\delta t)^{2H-1}+\rho\sigma_1\sigma_2S\frac{\partial^2 V}{\partial S\partial r}(\delta t)^{2H-1}-rV$$
$$+cS(\delta t)^{H-1} \tag{19}$$
$$\times\sqrt{\frac{2}{\pi}}\sqrt{\sigma_1^2S^2\left(\frac{\partial^2 V}{\partial S^2}\right)^2+\sigma_2^2\left(\frac{\partial^2 V}{\partial S\partial r}\right)^2+2\rho\sigma_1\sigma_2S\frac{\partial^2 V}{\partial S^2}\frac{\partial^2 V}{\partial S\partial r}}$$
$$=0.$$

Proof. European option price $V_t=V(t,S_t)$ is replicated by the portfolio Π_t [11], which is constructed as

$$\Pi_t=\Delta_1S_t+\Delta_2P_t+X_tD_t, \tag{20}$$

where Δ_1, Δ_2, and X_t are the shares of S_t, P_t, and D_t, respectively.

After δt, the value change of the portfolio (20) in the absence of transaction costs can be rewritten as

$$\delta\Pi_t = \Delta_1\delta S_t + \Delta_2\delta P_t + X_t\delta D_t. \tag{21}$$

Similar to the zero-coupon bond discussed in [20], the return of D is set to the risk-free rate, the spot rate.

So, the value change of portfolio (20) in [t, t + δt] is

$$\delta\Pi_t = \Delta_1\delta S_t + \Delta_2\delta P_t + rX_tD_t\delta t - c\left|v_t\right|S_t. \tag{22}$$

Multivariable Taylor's series gives

$$\begin{aligned}
\delta V_t &= \frac{\partial V}{\partial t}\delta t + \frac{\partial V}{\partial r}\delta r + \frac{\partial V}{\partial S}\delta S \\
&\quad + \frac{1}{2}\left[\frac{\partial^2 V}{\partial r^2}(\delta r)^2 + \frac{\partial^2 V}{\partial S^2}(\delta S)^2 + 2\frac{\partial^2 V}{\partial S\partial r}(\delta S\cdot\delta r)\right] \\
&\quad + o\left((\delta t)^2\right) \\
&= \left(\frac{\partial V}{\partial t} + [a(b-r)]\frac{\partial V}{\partial r} + rS\frac{\partial V}{\partial S}\right)\delta t \\
&\quad + \sigma_1 S\frac{\partial V}{\partial S}\delta B_H^1(t) + \sigma_2\frac{\partial V}{\partial r}\delta B_H^2(t) \\
&\quad + \frac{1}{2}\sigma_1^2 S^2\frac{\partial^2 V}{\partial S^2}\left(\delta B_H^1(t)\right)^2 + \frac{1}{2}\sigma_2^2\frac{\partial^2 V}{\partial r^2}\left(\delta B_H^2(t)\right)^2 \\
&\quad + \sigma_1\sigma_2 S\frac{\partial^2 V}{\partial S\partial r}\left(\delta B_H^1(t)\cdot\delta B_H^2(t)\right) \\
&\quad + O\left(\delta t\cdot\delta B_H(t)\right).
\end{aligned} \tag{23}$$

Then we have

$$\begin{aligned}
\delta V_t - \delta\Pi_t &= \delta V_t - \Delta_1\delta S_t - \Delta_2\delta P_t - rX_tD_t\delta t + c\left|v_t\right| \\
&\quad \cdot S_t = \left(\frac{\partial V}{\partial t} + [a(b-r)]\frac{\partial V}{\partial r} + rS\frac{\partial V}{\partial S}\right)\delta t \\
&\quad + \sigma_1 S\frac{\partial V}{\partial S}\delta B_H^1(t) + \sigma_2\frac{\partial V}{\partial r}\times\delta B_H^2(t) + \frac{1}{2} \\
&\quad \cdot \sigma_1^2 S^2\frac{\partial^2 V}{\partial S^2}\left(\delta B_H^1(t)\right)^2 + \frac{1}{2}\sigma_2^2\frac{\partial^2 V}{\partial r^2}\left(\delta B_H^2(t)\right)^2 \\
&\quad + \sigma_1\sigma_2\times S\frac{\partial^2 V}{\partial S\partial r}\left(\delta B_H^1(t)\cdot\delta B_H^2(t)\right) \\
&\quad - \Delta_1 S\sigma_1\delta B_H^1(t) - \Delta_1 rS\delta t - rX_tD_t\delta t \\
&\quad - \Delta_2\left[\frac{\partial P}{\partial t}\delta t + [a(b-r)]\frac{\partial P}{\partial r}\delta t + \sigma_2\frac{\partial P}{\partial r}\delta B_H^2(t)\right. \\
&\quad \left. + \frac{1}{2}\sigma_2^2\frac{\partial^2 P}{\partial r^2}\left(\delta B_H^2(t)\right)^2\right] + c\left|v_t\right|S + O(\delta t \\
&\quad \cdot \delta B_H(t)).
\end{aligned} \tag{24}$$

Taking $\Delta_1 = \partial V/\partial S$ and $\Delta_2 = (\partial V/\partial r)/(\partial P/\partial r)$, (24) can be reduced to

$$\begin{aligned}
&\delta V_t - \delta\Pi_t \\
&\quad = \left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S}\right)\delta t + \frac{1}{2}\sigma_1^2 S^2\frac{\partial^2 V}{\partial S^2}\left(\delta B_H^1(t)\right)^2 \\
&\quad + \frac{1}{2}\sigma_2^2\frac{\partial^2 V}{\partial r^2}\left(\delta B_H^2(t)\right)^2 \\
&\quad + \sigma_1\sigma_2 S\frac{\partial^2 V}{\partial S\partial r}\left(\delta B_H^1(t)\cdot\delta B_H^2(t)\right) - rS\frac{\partial V}{\partial S}\delta t \\
&\quad - rX_tD_t\delta t \\
&\quad - \frac{\partial V/\partial r}{\partial P/\partial r}\left[\frac{\partial P}{\partial t}\delta t + \frac{1}{2}\sigma_2^2\frac{\partial^2 P}{\partial r^2}\left(\delta B_H^2(t)\right)^2\right] \\
&\quad + c\left|v_t\right|S + O\left(\delta t\cdot\delta B_H(t)\right).
\end{aligned} \tag{25}$$

By (7), we know that

$$\begin{aligned}
E\left(\delta V_t - \delta\Pi_t\right) &= \frac{\partial V}{\partial t}\delta t + \left[a(b-r) - \theta\sigma_2\right]\frac{\partial V}{\partial r}\delta t \\
&\quad + rS\frac{\partial V}{\partial S}\delta t + \frac{1}{2}\sigma_1^2 S^2\frac{\partial^2 V}{\partial S^2}\times(\delta t)^{2H} \\
&\quad + \frac{1}{2}\sigma_2^2\frac{\partial^2 V}{\partial r^2}(\delta t)^{2H} \\
&\quad + \rho\sigma_1\sigma_2 S\frac{\partial^2 V}{\partial S\partial r}(\delta t)^{2H} + cE\left(\left|v_t\right|\right)S \\
&\quad - rV\delta t + O\left(\delta t\cdot\delta B_H(t)\right) = 0.
\end{aligned} \tag{26}$$

Equation (26) can be written as

$$\begin{aligned}
&\frac{\partial V}{\partial t}\delta t + \left[a(b-r) - \theta\sigma_2\right]\frac{\partial V}{\partial r}\delta t + rS\frac{\partial V}{\partial S}\delta t \\
&\quad + \frac{1}{2}\sigma_1^2 S^2\frac{\partial^2 V}{\partial S^2}(\delta t)^{2H} + \frac{1}{2}\sigma_2^2\frac{\partial^2 V}{\partial r^2}(\delta t)^{2H} \\
&\quad + \rho\sigma_1\sigma_2 S\frac{\partial^2 V}{\partial S\partial r}(\delta t)^{2H}cE\left(\left|v_t\right|\right)S - rV\delta t = 0.
\end{aligned} \tag{27}$$

Since

$$\begin{aligned}
v_t &= \Delta_1^{t+\delta t} - \Delta_1^t = \frac{\partial V}{\partial S}\left(S_{t+\delta t}, r+\delta r, t+\delta t\right) - \frac{\partial V}{\partial S}\left(S_t, r, t\right) \\
&= \sigma_1 S\frac{\partial^2 V}{\partial S^2}\delta B_H^1(t) + \sigma_2\frac{\partial^2 V}{\partial S\partial r}\delta B_H^2(t) + O(\delta t),
\end{aligned} \tag{28}$$

one knows that

$$\begin{aligned}
&E\left(v_t\right) = 0, \\
&E\left(v_t^2\right) = \sigma_1^2 S^2\left(\frac{\partial^2 V}{\partial S^2}\right)^2(\delta t)^{2H} + \sigma_2^2\left(\frac{\partial^2 V}{\partial S\partial r}\right)^2(\delta t)^{2H} \\
&\quad + 2\rho\sigma_1\sigma_2 S\frac{\partial^2 V}{\partial S^2}\frac{\partial^2 V}{\partial S\partial r}(\delta t)^{2H}.
\end{aligned} \tag{29}$$

If we take $E(v_t^2) = \beta^2$, then we can obtain

$$\begin{aligned} E(|v_t|) &= \int_{-\infty}^{+\infty} |v_t| \cdot \frac{1}{\sqrt{2\pi\beta^2}} e^{-v_t^2/2\beta^2} dv_t \\ &= \frac{2}{\sqrt{2\pi\beta^2}} \int_0^{+\infty} v_t e^{-v_t^2/2\beta^2} dv_t \\ &= \frac{2\beta^2}{\sqrt{2\pi\beta^2}} \int_0^{+\infty} e^{-(v_t/\sqrt{2}\beta)^2} d\left(\frac{v_t}{\sqrt{2}\beta}\right)^2 = \sqrt{\frac{2}{\pi}} |\beta| \\ &= \sqrt{\frac{2}{\pi}} (\delta t)^H \\ &\times \sqrt{\sigma_1^2 S^2 \left(\frac{\partial^2 V}{\partial S^2}\right)^2 + \sigma_2^2 \left(\frac{\partial^2 V}{\partial S \partial r}\right)^2 + 2\rho\sigma_1\sigma_2 S \frac{\partial^2 V}{\partial S^2} \frac{\partial^2 V}{\partial S \partial r}}. \end{aligned} \tag{30}$$

Substituting (30) into (27), one can derive (19).

When $H = 1/2$, (19) is just an option pricing model with transaction costs and stochastic interest rates in a standard Brownian market. For the case, [21] gave the answer using the hedge portfolio. □

Theorem 5. *Based on the fractional model (19), the price formulas of European call option and put option with exercise price K at exercise date T can be*

$$V_C(S, r, t) = SN(d_1) - KP(r, t, T) N(d_2) \tag{31}$$

and

$$V_P(S, r, t) = KP(r, t, T) N(-d_2) - SN(-d_1), \tag{32}$$

where

$$\begin{aligned} d_1 &= d_2 + \sqrt{2\int_t^T \left[\frac{1}{2}(\delta t)^{2H-1} \widetilde{\sigma^2} + c(\delta t)^{H-1} \sqrt{\frac{2}{\pi}} \widetilde{\sigma}\right] ds}, \\ d_2 &= \frac{\left[\ln(S/K) - \ln P - \int_t^T \left[(1/2)(\delta t)^{2H-1} \widetilde{\sigma^2} + c(\delta t)^{H-1} \sqrt{2/\pi}\widetilde{\sigma}\right] ds\right]}{\sqrt{2\int_t^T \left[(1/2)(\delta t)^{2H-1} \widetilde{\sigma^2} + c(\delta t)^{H-1} \sqrt{(2/\pi)}\widetilde{\sigma}\right] ds}}, \\ \widetilde{\sigma^2} &= \sigma_1^2 + \sigma_2^2 B^2 + 2\rho\sigma_1\sigma_2 B. \end{aligned} \tag{33}$$

The proof of *Theorem 5* can be found in the appendix.

3. Application Analysis

In the next analysis, the parameters are estimated under standard Brownian motion circumstance for the sake of convenience, which does not affect explaining the nonlinear model. The following is to take the closing prices (the data came from the trading software of Essence Securities) of 50ETF and GC028 from 01/03/2016 to 11/22/2016 as a sample to estimate the model parameters.

Hurst Parameter and Volatility Estimation. By R/S analysis, the 50ETF data provides that $H = 0.6331866$. The historical volatility calculates that σ_1 and σ_2 are 0.137811229 and 0.025437944 for 50ETF and GC028.

Vasicek Parameter Estimation. For the estimations of the parameters a and b, the Vasicek model is reduced to the form under standard Brownian motion. $\{X_h, X_{2h}, \ldots, X_{Nh}\}$ are taken as a set of time series and $\{h, 2h, \ldots, Nh\}$ are a set of isometric time points. Phillips [22] proposed the approach model of the Vasicek model:

$$\begin{aligned} X_{ih} &= e^{-ah} X_{(i-1)h} + a\left(b - e^{-ah}\right) + \sigma_2 \sqrt{\frac{1 - e^{-2ah}}{2a}} \varepsilon_i, \\ \varepsilon_i &\sim N(0, 1), \end{aligned} \tag{34}$$

and one can know that

$$\begin{aligned} &X_{ih} | X_{(i-1)h} \\ &\sim N\left(e^{-ah} X_{(i-1)h} + a\left(b - e^{-ah}\right), \frac{{\sigma_2}^2\left(1 - e^{-2ah}\right)}{2a}\right). \end{aligned} \tag{35}$$

Reference [23] gave the likelihood function and the estimates of a and b as follows:

$$\begin{aligned} \widehat{b} &= \frac{r_y r_{xx} - r_x r_{xy}}{N\left(r_{xx} - r_{xy}\right) - \left({r_x}^2 - r_x r_y\right)}, \\ \widehat{a} &= -\frac{1}{h} \ln \frac{r_{xy} - b r_x - b r_y + N b^2}{r_{xx} - 2 b r_x + N b^2}, \end{aligned} \tag{36}$$

where $r_x = \sum_{i=1}^N r_{i-1}, r_y = \sum_{i=1}^N r_i, r_{xx} = \sum_{i=1}^N r_{i-1}^2, r_{yy} = \sum_{i=1}^N r_i^2$, and $r_{xy} = \sum_{i=1}^N r_{i-1} r_i$.

From (36), one can obtain

$$\begin{aligned} \widehat{a} &= 47.85086035, \\ \widehat{b} &= 0.02435795. \end{aligned} \tag{37}$$

θ Estimation. The following equation is used to calculate the market price of risk θ:

$$\theta = \frac{\mu - r}{\sigma}. \tag{38}$$

Reference [24] gave the estimate $\widehat{\mu}$ of μ as follows:

$$\widehat{\mu} = \frac{\sum_{t=1}^{n} (\ln (S_t) - \ln (S_{t-1}))}{n\Delta} + \frac{\widehat{\sigma}^2}{2}, \tag{39}$$

where Δ denotes the time interval and $\widehat{\sigma}$ is an estimate of the volatility.

Let r be 0.0225; the market price of risk θ is 1.88758899.

ρ Estimation. In standard Brownian motion market, one knows that

$$\Delta \ln S = \left(\mu - \frac{{\sigma_1}^2}{2}\right)\Delta t + \sigma_1 \varepsilon_1 \sqrt{\Delta t}. \tag{40}$$

Because of $E(\Delta \ln S) = (\mu - {\sigma_1}^2/2)\Delta t$, we can get

$$\sigma_1 dW_t^1 = \Delta \ln S - E(\Delta \ln S). \tag{41}$$

Similarly, we have

$$\Delta r = a(b - r_t)\Delta t + \sigma_2 \varepsilon_2 \sqrt{\Delta t}. \tag{42}$$

And

$$\sigma_2 dW_t^2 = \Delta r - a(b - r_t)\Delta t. \tag{43}$$

Since

$$\operatorname{cov}\left(\sigma_1 dW_t^1, \sigma_2 dW_t^2\right) = \Delta t \sigma_1 \sigma_2 \rho, \tag{44}$$

the closing prices of 50ETF and GC028 are used to estimate ρ and give $\rho = 0.0870841$.

δt Choice. The classical Black-Scholes formula for the call option gives

$$\begin{aligned} C_0 &= SN(d_1) - K\exp(-r(T-\tau))N(d_2), \\ d_1 &= d_2 + \sigma\sqrt{T-t}, \\ d_2 &= \frac{\ln(S/K) + \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned} \tag{45}$$

Comparing the classical Black-Scholes formula and the result in (33), we can suppose that

$$\begin{aligned} \widetilde{\sigma}^2 &= 2\left[\frac{1}{2}(\delta t)^{2H-1}\widetilde{\sigma^2} + c(\delta t)^{H-1}\sqrt{\frac{2}{\pi}}\widetilde{\sigma}\right] \\ &= \widetilde{\sigma^2}\left[(\delta t)^{2H-1} + 2\frac{c}{\widetilde{\sigma}}\sqrt{\frac{2}{\pi}}(\delta t)^{H-1}\right]. \end{aligned} \tag{46}$$

Taking $(\delta t)^{2H-1} = 2(c/\widetilde{\sigma})\sqrt{2/\pi}(\delta t)^{H-1}$, $\widetilde{\sigma}^2$ is minimal. Hence, we postulate that

$$\delta t = 2^{1/H}\left(\frac{c}{\sigma_1}\right)^{1/H}\left(\frac{2}{\pi}\right)^{1/2H}. \tag{47}$$

Taking c = 0.003 and 0.0003, δt are 0.004956344 and 0.000130666, respectively.

Table 1: c = 0.003.

Error	BS	FBS	Paper model
MSE	0.0755177	0.0773743	0.0703020
AVE	0.2694876	0.2726479	0.2601567
MAXE	0.3940891	0.3995409	0.3788573
MINE	0.1755372	0.1790838	0.1613028

Table 2: c = 0.0003.

Error	BS	FBS	Paper model
MSE	0.0755177	0.0773743	0.0687515
AVE	0.2694876	0.2726479	0.2571551
MAXE	0.3940891	0.3995409	0.3753519
MINE	0.1755372	0.1790838	0.1581340

Application of the Model. There is no zero-coupon bond that has the same existence time as SSE 50ETF option in Chinese finance market. Hence, we do a bold and probing work that employs the national debt reverse repurchase rate and have (17) to construct a zero-coupon bond. The closing prices of the SSE 50ETF call option from 09/01/2016 to 11/23/2016 are chosen as the real prices. Tables 1 and 2 ($MSE = (1/N)\sum_{i=1}^{N}[(Q_i - P_i)/Q_i]^2$, $AVE = (1/N)\sum_{i=1}^{N}|(Q_i - P_i)/Q_i|$, $MAXE = MAX|(Q_i - P_i)/Q_i|$, $MINE = MIN|(Q_i - P_i)/Q_i|$, where P_i, Q_i, and N indicate theoretical value, actual value, and number of samples) show four error indicators between our model, BS, FBS, and the real values.

From the tables, the paper model is better than BS and FBS in every error indicator. It could be that BS and FBS do not reflect transaction costs and stochastic interest rates that exist in reality. "c" is the proportion of transaction costs. In the above tables, we take c = 0.003 and 0.0003, respectively. It is evident that Table 2 is better than Table 1. This might be that $\widetilde{\sigma}^2$ become smaller when c = 0.0003. As a result, the empirical analysis is a little better.

4. Conclusions and Discussions

In the work, European option pricing with stochastic interest rates and transaction costs is reported and the closed-form formulas are given under fractional Brownian motion environment. Further, the result is introduced into the real market. By the examples in Section 3, one can find that the application of the mentioned model in the actual financial market is reliable and valuable. The following points out the conclusions and existing problems that are gained at the completing process of the work.

(i) In the aspect of stochastic interest rates, the work gets the zero-coupon bond pricing model by Taylor expansion, which is different from the one of [9], where the pricing problem is solved by Wick-Ito integral. For the option pricing model itself, a nonlinear partial differential equation satisfied by European option is derived using the replicating portfolio and Taylor expansion. It is clear that the expression has obvious differences from the known models and differs from the result of [21] under the standard Brownian motion.

Compared to [12], the nonlinear model cares for the practical matter and stochastic interest rates are added to the pricing problem.

(ii) The first step of the empirical study is to estimate parameters. The nonlinear model contains more parameters. If these parameters are made more accurate, one can obtain more better valuation. Their optimal determinations are the key to settle the problems. In addition, there exists the difficulties in finding data for the empirical study and it is possible to causes that most literatures did not analyze the results. In empirical analysis, this work uses theoretical values to replace the real prices due to the lack of data from the zero-coupon bond that has the same start-stop times as the considered option. There is no denying that can cause errors. If existing such a zero coupon bond, the empirical analysis is more rewarding. The work constructs the zero coupon bond in Section 3 and help to explain the actual utility of the nonlinear fractional model.

Appendix

In the work, the price $V(S,t)$ of European call option is suggested to obey

$$\begin{aligned}&\frac{\partial V}{\partial t}+\left[a\left(b-r\right)-\theta\sigma_2\right]\frac{\partial V}{\partial r}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma_1^2S^2\frac{\partial^2V}{\partial S^2}\left(\delta t\right)^{2H-1}\\&\quad+\frac{1}{2}\sigma_2^2\frac{\partial^2V}{\partial r^2}\left(\delta t\right)^{2H-1}+\rho\sigma_1\sigma_2S\frac{\partial^2V}{\partial S\partial r}\left(\delta t\right)^{2H-1}-rV\\&\quad+cS\left(\delta t\right)^{H-1}\\&\quad\cdot\sqrt{\frac{2}{\pi}}\sqrt{\sigma_1^2S_t^2\left(\frac{\partial^2V}{\partial S^2}\right)^2+\sigma_2^2\left(\frac{\partial^2V}{\partial S\partial r}\right)^2+2\rho\sigma_1\sigma_2S\frac{\partial^2V}{\partial S^2}\frac{\partial^2V}{\partial S\partial r}}\\&\quad=0,\\&V|_{t=T}=\left(S-K\right)^+.\end{aligned}\tag{A.1}$$

Based on [9, 10], taking the transformation $y=S/P(r,t,T)$, $\widetilde{V}=V(S,r,t)/P(r,t,T)$ to the first equality of (A.1) can be derived:

$$\begin{aligned}&\frac{\partial\widetilde{V}}{\partial t}+\frac{1}{2}y^2\left(\delta t\right)^{2H-1}\\&\quad\cdot\left[\sigma_1^2+\sigma_2^2\frac{1}{P^2}\left(\frac{\partial P}{\partial r}\right)^2-2\rho\sigma_1\sigma_2\frac{1}{P}\frac{\partial P}{\partial r}\right]\left(\frac{\partial^2\widetilde{V}}{\partial y^2}\right)\\&\quad+cy^2\left(\delta t\right)^{H-1}\\&\quad\cdot\sqrt{\frac{2}{\pi}}\sqrt{\left[\sigma_1^2+\sigma_2^2\frac{1}{P^2}\left(\frac{\partial P}{\partial r}\right)^2-2\rho\sigma_1\sigma_2\frac{1}{P}\frac{\partial P}{\partial r}\right]\left|\frac{\partial^2\widetilde{V}}{\partial y^2}\right|}\\&\quad=0.\end{aligned}\tag{A.2}$$

In the real world, we usually consider $\partial^2V/\partial S^2>0$. So $\partial^2\widetilde{V}/\partial y^2$ of (A.2) is supposed to be greater than zero.

Equation (A.2) can be reduced to the following expression with the transformation $x=\ln y$:

$$\begin{aligned}&\frac{\partial\widetilde{V}}{\partial t}\\&\quad+\left[\frac{1}{2}\left(\delta t\right)^{2H-1}\widetilde{\sigma^2}+c\left(\delta t\right)^{H-1}\sqrt{\frac{2}{\pi}}\widetilde{\sigma}\right]\left(\frac{\partial^2\widetilde{V}}{\partial x^2}-\frac{\partial\widetilde{V}}{\partial x}\right)\\&\quad=0,\end{aligned}\tag{A.3}$$

where $\sigma_1^2+\sigma_2^2B^2+2\rho\sigma_1\sigma_2B=\widetilde{\sigma^2}$.

Further, we take $\widetilde{V(x,t)}=\mu(\eta,\tau)$, $\eta=x+\alpha(t)$, $\tau=\gamma(t)$. Note that $\alpha(T)=\gamma(T)=0$. Let $\alpha'(t)=(1/2)(\delta t)^{2H-1}\widetilde{\sigma^2}+c(\delta t)^{H-1}\sqrt{(2/\pi)}\widetilde{\sigma}$ and $\gamma'(t)=-[(1/2)(\delta t)^{2H-1}\widetilde{\sigma^2}+c(\delta t)^{H-1}\sqrt{(2/\pi)}\widetilde{\sigma}]$. Then, problem (A.1) is translated into the problem

$$\begin{aligned}&\frac{\partial\mu}{\partial\tau}-\frac{\partial^2\mu}{\partial\eta^2}=0,\\&\mu\left(\eta,T\right)=\left(e^{\eta}-K\right)^+.\end{aligned}\tag{A.4}$$

Equations (A.4) have the following solution by Fourier transform method:

$$\begin{aligned}\mu\left(\eta,\tau\right)&=\frac{1}{2\sqrt{\pi\tau}}\int_{\ln K}^{+\infty}e^{\xi}e^{-(\eta-\xi)^2/4\tau}d\xi\\&\quad-\frac{1}{2\sqrt{\pi\tau}}\int_{\ln K}^{+\infty}Ke^{-(\eta-\xi)^2/4\tau}d\xi.\end{aligned}\tag{A.5}$$

Taking $\xi=\eta-\sqrt{2\tau}w$, the result in (A.5) can be rewritten as

$$\begin{aligned}\mu\left(\eta,\tau\right)&=e^{\eta+\tau}N\left(\frac{\eta-\ln K+2\tau}{\sqrt{2\tau}}\right)\\&\quad-KN\left(\frac{\eta-\ln K}{\sqrt{2\tau}}\right).\end{aligned}\tag{A.6}$$

If we make the variable (η,τ) back to the original variable (S,t), we will get the pricing formulas (32) and (33). In a similar manner, we can obtain the pricing formula for European put option.

Data Availability

The data needed by the article came from the trading software of Essence Securities. In fact, the data can be obtained by any trading software. But the problem that some trading software lost the data if the option expired exists. If readers that need these data of the article cannot obtain these data, the corresponding author is willing to share them.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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