Research Article

Positive Periodic Solutions for an Generalized SIS Epidemic Model with Time-Varying Coefficients and Delays

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This paper mainly explores a generalized SIS epidemic model with time-varying coefficients and delays. By employing Lyapunov function method and differential inequality approach, a sufficient criterion to guarantee the existence and exponential stability of positive periodic solutions for the addressed model is obtained, which complements the earlier publications. Particularly, an example and its numerical simulations are given to demonstrate our theoretical results.

1. Introduction

In the study of infectious dynamics, under the assumption that there is no mortality due to illness, A. Iggridr, K. Niri b, and E. Ould Moulay Ely [1] proposed the following delayed SIS epidemic model:

$$\begin{aligned} i'(t) = & \beta [1 - i(t)] i(t) - \beta [1 - i(t - \omega)] i(t - \omega) e^{-d\omega} \\ & - di(t), \end{aligned} \quad (1)$$

where $i(t) = I(t)/N(t)$, $N(t) = S(t) + I(t)$, $S(t)$, $I(t)$, and $N(t)$ denote the susceptible numbers, infective numbers, and total numbers at time t , respectively. Here β is the transmission rate, $\omega > 0$ designates the average latent period of the disease, and d represents the natural death rate. A more detailed description of the model can be found in [2]. Recently, some criteria ensuring the global attractivity of the endemic equilibrium of (1) have been established in [3]. On the one hand, any biological or environmental parameters are naturally subject to fluctuation in time and it is more realistic to consider the model with time-varying coefficients and delays. One can easily find that (1) can be extended to the case with time-varying delay and time-varying coefficients when $d(t)$ has a special form. On the other hand, in the real world, the model coefficients are usually assumed to be periodic because the correlation coefficients are susceptible to the change of climate and other factors. In fact, periodic phenomena are found in the spread of many infectious diseases

such as influenza and chickenpox. Therefore it is worthwhile to investigate how periodic solutions arise and the stability of periodic solutions in an epidemiological model (see, for instance, [4–7]). However, to the best of our knowledge, there are no existing papers on positive periodic solutions of (1). According to the previous analysis, in this paper, our goal is to study the existence and the exponential stability of positive periodic solutions of the following generalized SIS model with time-varying delays and coefficients:

$$\begin{aligned} i'(t) = & \beta(t) [1 - i(t)] i(t) \\ & - \beta(t) [1 - i(t - \omega(t))] i(t - \omega(t)) e^{-d(t)\omega(t)} \\ & - d(t) i(t), \end{aligned} \quad (2)$$

where $\beta(t)$, $\omega(t)$, $d(t)$ are nonnegative bounded and continuous T -periodic functions.

For convenience, we introduce the following notations.

$$\begin{aligned} g^+ &= \sup_{t \in \mathbb{R}} |g(t)|, \\ g^- &= \inf_{t \in \mathbb{R}} |g(t)|, \\ C &= C([- \omega^+, 0], \mathbb{R}), \\ C_+ &= C([- \omega^+, 0], \mathbb{R}_+), \\ i_t(\theta) &= i(t + \theta). \end{aligned} \quad (3)$$

By virtue of the biological interpretation of model (2), one can find that all solutions of (2) should remain in the interval $[0, 1]$. Then, we introduce the initial value conditions of (2) as follows:

$$i_{t_0} = \varphi, \quad \varphi \in C([- \omega^+, 0], [0, 1]) \subset C_+ \text{ and } i(0) > 0. \quad (4)$$

In addition, define a continuous map $f: \mathbb{R} \times C_+ \rightarrow \mathbb{R}$ by

$$\begin{aligned} f(t, \varphi) &= \beta(t) \varphi(0) (1 - \varphi(0)) \\ &\quad - \beta(t) \varphi(-\omega(t)) (1 - \varphi(-\omega(t))) e^{-d(t)\omega(t)} \\ &\quad - d(t) \varphi(0). \end{aligned} \quad (5)$$

Clearly, the existence and uniqueness of the solution of (2) with the initial condition (4) are guaranteed by which f is a locally Lipschitz map with respect to $\varphi \in C_+$. Let $i(t; t_0, \varphi)$ be a solution of the initial value problem (2) and (4) and $[t_0, \eta(\varphi))$ be the maximal right-interval of the existence of $i(t; t_0, \varphi)$.

Let $\theta(t) = \beta(t)((1 - e^{-d(t)\omega(t)})/d(t))$, $g(t, x) = \beta(t)[(1 - x) - (1/4x)e^{-d(t)\omega(t)}]/d(t) - 1$, $t \in \mathbb{R}$. In the following, we always

assume that $\theta(t) > 2$, for all $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$; it is not difficult to verify that $g(t, x)$ is increasing on $(0, (1/2)e^{-d(t)\omega(t)/2})$ and decreasing on $((1/2)e^{-d(t)\omega(t)/2}, +\infty)$ about the variable x . Note that

$$\begin{aligned} g\left(t, \frac{1}{2}e^{-d(t)\omega(t)/2}\right) &\geq g\left(t, \frac{1}{2}\right) \\ &= \frac{\beta(t)(1 - e^{-d(t)\omega(t)})}{2d(t)} - 1 \\ &= \frac{\theta(t)}{2} - 1 > 0, \quad \forall t \in \mathbb{R}. \end{aligned} \quad (6)$$

And

$$\begin{aligned} g(t, x) &\rightarrow -\infty \quad \text{as } x \rightarrow 0^+; \\ g(t, x) &\rightarrow -\infty \quad \text{as } x \rightarrow +\infty. \end{aligned} \quad (7)$$

For fixed $t \in \mathbb{R}$, we denote the unique zero solution of $g(t, x)$ on $(0, (1/2)e^{-d(t)\omega(t)/2})$ by $\kappa(t)$. For any $\kappa \in (\kappa^+, 1/2)$, denote $M = 1 - \kappa$, $C^\kappa = \{\varphi \in C \mid \varphi(\theta) \in (\kappa, M) \text{ for all } \theta \in [-\omega^+, 0]\}$. Furthermore, it is not hard to see that

$$g(t, \kappa) > 0, \quad \forall t \in \mathbb{R}. \quad (8)$$

2. Preliminary Results

The following lemmas are helpful to prove our main results in Section 3.

Lemma 1. Assume that $d^- > 0$, $\omega^- > 0$, and

$$2d(t) < \beta(t)(1 - e^{-d(t)\omega(t)}) < \frac{d(t)}{\kappa} \quad \text{for all } t \in \mathbb{R}. \quad (9)$$

Then, for $\varphi \in C^\kappa$,

$$\begin{aligned} \eta(\varphi) &= \infty \text{ and} \\ i(t; t_0, \varphi) &\in C^\kappa \quad \text{for } t \geq t_0. \end{aligned} \quad (10)$$

Proof. For the sake of simplicity of notations, let $i(t) = i(t; t_0, \varphi)$ for all $t \in [t_0 - \omega^+, \eta(\varphi))$. We first show that

$$i(t) > \kappa \quad \text{for all } t \in [t_0, \eta(\varphi)). \quad (11)$$

Arguing by contradiction, if this is not true, then there must exist $T_1 \in (t_0, \eta(\varphi))$ such that

$$\begin{aligned} i(T_1) &= \kappa, \\ i(t) &> \kappa \quad \text{for all } t \in (t_0 - \omega^+, T_1), \end{aligned} \quad (12)$$

which, with the help of (8), (9), and (12), entails that

$$\begin{aligned} i'(T_1) &= \beta(T_1)[1 - i(T_1)]i(T_1) - \beta(T_1) \\ &\quad \cdot [1 - i(T_1 - \omega(T_1))]i(T_1 - \omega(T_1))e^{-d(T_1)\omega(T_1)} \\ &\quad - d(T_1)i(T_1) \geq \beta(T_1)(1 - \kappa)\kappa - \frac{\beta(T_1)}{4} \\ &\quad \cdot e^{-d(T_1)\omega(T_1)} - d(T_1)\kappa = d(T_1) \\ &\quad \cdot \kappa \left\{ \frac{\beta(T_1)[(1 - \kappa) - (1/4\kappa)e^{-d(T_1)\omega(T_1)}]}{d(T_1)} - 1 \right\} \\ &> 0, \end{aligned} \quad (13)$$

which is contrary to the fact that $i'(T_1) \leq 0$. Hence (11) holds. Next, we demonstrate that

$$i(t) < M \quad \text{for } t \in [t_0, \eta(\varphi)). \quad (14)$$

If not, there must exist $T_2 \in (t_0, \eta(\varphi))$ such that

$$\begin{aligned} i(T_2) &= M, \\ i(t) &< M \quad \text{for all } t \in (t_0 - \omega^+, T_2), \end{aligned} \quad (15)$$

which, together with (9) and (15), suggests that

$$\begin{aligned} i'(T_2) &= \beta(T_2)[1 - i(T_2)]i(T_2) - \beta(T_2) \\ &\quad \cdot [1 - i(T_2 - \omega(T_2))]i(T_2 - \omega(T_2))e^{-d(T_2)\omega(T_2)} \\ &\quad - d(T_2)i(T_2) \leq \beta(T_2)(1 - M)M - \beta(T_2) \\ &\quad \cdot (1 - M)Me^{-d(T_2)\omega(T_2)} - d(T_2)M \\ &= M[\beta(T_2)(1 - M)(1 - e^{-d(T_2)\omega(T_2)}) - d(T_2)] \\ &< 0, \end{aligned} \quad (16)$$

which is a contradiction with $i'(T_2) \geq 0$ and hence (14) is true. In view of (9), (11), and (14), we can show that

$$\kappa < i(t) < M \quad \text{for all } t \in [t_0 - \omega^+, \eta(\varphi)). \quad (17)$$

From Theorem 2.3.1 in [8], we can easily obtain $\eta(\varphi) = +\infty$. This completes the proof. $\square$

Lemma 2. Assuming that the conditions of Lemma 1 are established, we further assume that

$$d(t) > \beta(t)(1 - 2\kappa) \left(1 + e^{-d(t)\omega(t)}\right) \quad \text{for } t \in \mathbb{R}. \quad (18)$$

Then there exists a positive λ such that

$$|i(t; t_0, \varphi) - i(t; t_0, \varphi^*)| \leq K_{\varphi, \varphi^*} e^{-\lambda t} \quad (19)$$

for $\varphi, \varphi^* \in C^\kappa$ and $t \geq t_0$,

where $K_{\varphi, \varphi^*} = e^{\lambda t_0} (\max_{\theta \in [-\omega^+, 0]} |\varphi(\theta) - \varphi^*(\theta)| + 1)$.

Proof. Consider $\Gamma : [0, 1] \rightarrow \mathbb{R}$ defined by

$$\Gamma(u) = \sup_{t \in \mathbb{R}} \left\{ - \left[d(t) - (1 - 2\kappa) \beta(t) \left(1 + e^{-d(t)\omega(t)} e^{u\omega^+} \right) - u \right] \right\}, \quad (20)$$

for $u \in [0, 1]$.

Clearly, Γ is continuous. Note that $\Gamma(0) = \sup_{t \in \mathbb{R}} \{-[d(t) - (1 - 2\kappa)\beta(t)(1 + e^{-d(t)\omega(t)})]\} < 0$, which follows from the continuity and periodicity of $d(t)$, $\omega(t)$, and $\beta(t)$, then there are $\eta > 0$ and $1 > \lambda > 0$ such that $\Gamma(\lambda) < -\eta$ or

$$- \left[d(t) - (1 - 2\kappa) \beta(t) \left(1 + e^{-d(t)\omega(t)} e^{\lambda\omega^+} \right) - \lambda \right] < -\eta \quad (21)$$

$< 0 \quad \text{for } t \in \mathbb{R}.$

Let $\varphi, \varphi^* \in C^\kappa$. For simplicity, denote $i(t; t_0, \varphi)$ and $i(t; t_0, \varphi^*)$ by $i(t)$ and $i^*(t)$, respectively. From Lemma 1, we have

$$\begin{aligned} \kappa &< i(t), \\ i^*(t) &< M \end{aligned} \quad (22)$$

for all $t \in [t_0 - \omega^+, \infty)$.

Let $y(t) = i(t) - i^*(t)$ for $t \in [t_0 - \omega^+, \infty)$. Then, for $t \geq t_0$,

$$\begin{aligned} y'(t) &= -[\beta(t)(i(t) + i^*(t) - 1) + d(t)]y(t) + \beta(t) \\ &\quad \cdot e^{-d(t)\omega(t)} [i(t - \omega(t)) + i^*(t - \omega(t)) - 1] \\ &\quad \cdot y(t - \omega(t)). \end{aligned} \quad (23)$$

Consider the Lyapunov functional

$$V(t) = |y(t)| e^{\lambda t} \quad \text{for } t \geq t_0. \quad (24)$$

We claim that $V(t) \leq K_{\varphi, \varphi^*}$ for $t \geq t_0$. Otherwise, there exists $t_* > t_0$ such that

$$\begin{aligned} V(t_*) &= K_{\varphi, \varphi^*} \text{ and} \\ V(t) &< K_{\varphi, \varphi^*} \end{aligned} \quad (25)$$

for all $t \in [t_0 - \omega^+, t_*)$.

Then

$$\begin{aligned} 0 &\leq D^-(V(t_*)) \leq -[\beta(t_*)(i(t_*) + i^*(t_*) - 1) \\ &\quad + d(t_*)] |y(t_*)| e^{\lambda t_*} + \beta(t_*) e^{-d(t_*)\omega(t_*)} |i(t_* \\ &\quad - \omega(t_*)) + i^*(t_* - \omega(t_*)) - 1| |y(t_* - \omega(t_*))| \\ &\quad \cdot e^{\lambda t_*} + \lambda |y(t_*)| e^{\lambda t_*} = -[\beta(t_*)(i(t_*) + i^*(t_*) \\ &\quad - 1) + d(t_*) - \lambda] |y(t_*)| e^{\lambda t_*} + \beta(t_*) \\ &\quad \cdot e^{-d(t_*)\omega(t_*)} |i(t_* - \omega(t_*)) + i^*(t_* - \omega(t_*)) - 1| \\ &\quad \cdot |y(t_* - \omega(t_*))| e^{\lambda(t_* - \omega(t_*))} e^{\lambda\omega(t_*)} \leq -[\beta(t_*)(2\kappa \\ &\quad - 1) + d(t_*) - \lambda] |y(t_*)| e^{\lambda t_*} + (2M - 1) \beta(t_*) \\ &\quad \cdot e^{-d(t_*)\omega(t_*)} |y(t_* - \omega(t_*))| e^{\lambda(t_* - \omega(t_*))} e^{\lambda\omega(t_*)} \\ &\leq \left\{ -[d(t_*) + \beta(t_*)(2\kappa - 1) - \lambda] + (1 - 2\kappa) \right. \\ &\quad \cdot \beta(t_*) e^{-d(t_*)\omega(t_*)} e^{\lambda\omega^+} \left. \right\} K_{\varphi, \varphi^*} \leq \left\{ -[d(t_*) \right. \\ &\quad - \beta(t_*)(1 - 2\kappa) \left(1 + e^{-d(t_*)\omega(t_*)} e^{\lambda\omega^+} \right) - \lambda] \left. \right\} K_{\varphi, \varphi^*} \\ &< -\eta K_{\varphi, \varphi^*} < 0, \end{aligned} \quad (26)$$

a contradiction. Consequently, we infer that (19) holds. The proof is completed. $\square$

3. Main Results

Combined with Lemmas 1 and 2, we have the following theorem.

Theorem 3. Under the assumptions of Lemma 2, then system (2) has exactly one positive T -periodic solution which is exponentially stable.

Proof. Fix a $\varphi \in C^\kappa$; let $i(t) = i(t; t_0, \varphi)$ be a solution of system (2) and (4). According to Lemma 1, we have

$$\kappa < i(t; t_0, \varphi) < M \quad \text{for all } t \geq t_0 - \omega^+. \quad (27)$$

From the periodicity of coefficients and delays for system (2), for any nonnegative integer h , we get

$$\begin{aligned} i'(t + hT) &= \beta(t + hT) [1 - i(t + hT)] i(t + hT) \\ &\quad - \beta(t + hT) [1 - i(t + hT - \omega(t + hT))] \\ &\quad \cdot i(t + hT - \omega(t + hT)) e^{-d(t+hT)\omega(t+hT)} \\ &\quad - d(t + hT) i(t + hT) = \beta(t) [1 - i(t + hT)] \\ &\quad \cdot i(t + hT) - \beta(t) [1 - i(t + hT - \omega(t))] \\ &\quad \cdot i(t + hT - \omega(t)) e^{-d(t)\omega(t)} - d(t) i(t + hT) \end{aligned} \quad (28)$$

which implies that $i(t + hT; t_0, \varphi)$ is also a solution to system (2) on $[t_0 - \omega^+ - hT, \infty)$. Denote $\psi = i(t + T; t_0, \varphi)$. In view of

Lemma 2, for any nonnegative integer h and $t + hT \geq t_0$, we obtain

$$\begin{aligned} & |i(t + (h+1)T; t_0, \varphi) - i(t + hT; t_0, \varphi)| \\ &= |i(t + hT; t_0, \psi) - i(t + hT; t_0, \varphi)| \\ &\leq K_{\varphi, \psi} e^{-\lambda(t+hT)}, \end{aligned} \quad (29)$$

where $K_{\varphi, \psi} = e^{\lambda t_0} (\max_{\theta \in [-\omega^+, 0]} |\varphi(\theta) - \psi(\theta)| + 1)$.

Now, we claim that $i(t + qT; t_0, \varphi)$ is convergent on any compact interval as $q \rightarrow \infty$. For an arbitrary subset $[a, b] \subset \mathbb{R}$, we can pick a nonnegative integer q_0 satisfying $t + q_0T \geq t_0$ for $t \in [a, b]$. Then for $t \in [a, b]$ and $q > q_0$, one can see that

$$\begin{aligned} i(t + qT) &= i(t + q_0T) \\ &+ \sum_{h=q_0}^{q-1} [i(t + (h+1)T) - i(t + hT)] \end{aligned} \quad (30)$$

which implies that $\{i(t + qT)\}_q$ converges uniformly to a continuous function $i^*(t)$. Due to the arbitrariness of $[a, b]$, one can easily find that $i(t + qT) \rightarrow i^*(t)$ as $q \rightarrow \infty$ for $t \in \mathbb{R}$. Moreover, we get

$$\kappa \leq i^*(t) \leq M \quad \text{for all } t \in \mathbb{R}. \quad (31)$$

Next we show that i^* is a T -periodic solution of (2). The periodicity can be obtained immediately from the fact

$$\begin{aligned} i^*(t + T) &= \lim_{q \rightarrow \infty} i((t + T) + qT) \\ &= \lim_{q+1 \rightarrow \infty} i(t + (q+1)T) = i^*(t) \end{aligned} \quad (32)$$

for all $t \in \mathbb{R}$. Noting that $i(t + qT)$ is a solution to (2),

$$\begin{aligned} i(t + qT) - i(t_0 + qT) &= \int_{t_0}^t \{\beta(s) [1 - i(s + qT)] \\ &\cdot i(s + qT) - \beta(s) [1 - i(s + qT - \omega(s))] \\ &\cdot i(s + qT - \omega(s)) e^{-d(s)\omega(s)} - d(s) i(s + qT)\} ds \end{aligned} \quad (33)$$

for $t \geq t_0$. Letting $q \rightarrow \infty$ gives us

$$\begin{aligned} i^*(t) - i^*(t_0) &= \int_{t_0}^t \{\beta(s) [1 - i^*(s)] i^*(s) \\ &- \beta(s) [1 - i^*(s - \omega(s))] i^*(s - \omega(s)) e^{-d(s)\omega(s)} \\ &- d(s) i^*(s)\} ds \end{aligned} \quad (34)$$

for $t \geq t_0$; namely, i^* is a solution to (2) on $[t_0 - \omega^+, \infty)$. Finally, by the same method as that in the proof of Lemma 2, we can show that $i^*(t)$ is exponentially stable. This finishes the proof. $\square$

FIGURE 1: Numerical solutions $i(t)$ of model (35) for initial value $\varphi(t) = 0.4, 0.5, 0.6$, $t \in [-50, 0]$.

4. An Example

Example 1. Regard the following generalized SIS epidemic model with time-varying delays and coefficients:

$$\begin{aligned} i'(t) &= (0.32 + 0.08 \sin t) [1 - i(t)] i(t) \\ &- (0.32 + 0.08 \sin t) [1 - i(t - (40 + 10 \sin t))] \\ &i(t - (40 + 10 \sin t)) e^{-(0.08 + 0.02 \sin t)(40 + 10 \sin t)} \\ &- (0.08 + 0.02 \sin t) i(t). \end{aligned} \quad (35)$$

Then

$$e^{-4} \leq e^{-d(t)\omega(t)} \leq e^{-1.8} \approx 0.1653, \quad \text{for all } t \in \mathbb{R}, \quad (36)$$

and $\kappa^+ \approx 0.23025$. Choose $\kappa = 1/3$, then $1/4 < \kappa$. By a simple computation, we can easily check that all the conditions of Theorem 3 are satisfied. Therefore, system (35) with initial values in $\varphi \in C^\kappa$ has a unique positive 2π -periodic solution $i^*(t)$ which is exponentially stable. The numerical simulations in Figure 1 strongly support the conclusion.

Remark 5. To the author's knowledge, it is the first time to focus attention on the exponential stability of positive periodic solutions for (2), which is the generalization of (1). By constructing invariant sets ingeniously, under appropriate conditions, we represent that all solutions of the addressed model converge exponentially to the positive periodic solution. In addition, one can find that all the results in [1, 3–6] cannot be applicable to show the global exponential stability on the positive periodic solution of (35), which implies that the results of this paper are new and complement of previously known results. In future, we will consider the properties of almost periodic solutions and pseudo almost periodic solutions of the model.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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